PH1a: Linear Momentum

$$\mathbf{p} = m\mathbf{v}.$$

$$\Sigma \mathbf{F} = \frac{d(m\mathbf{v})}{dt}.$$

DEFINITION OF CENTER OF MASS:

$$\bar{\mathbf{r}} = \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{M},$$

DYNAMICS OF CENTER OF MASS:

$$M\bar{\mathbf{a}} = \frac{d\mathbf{p}_{\text{tot}}}{dt} = \mathbf{F}_{\text{tot}}.$$

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Who wrote this thoughts?

And with respect to the general cause, it seems manifest to me that it is none other than God himself, who, in the beginning, created matter along with motion and rest, and now by his ordinary concourse alone preserves in the whole the same amount of motion and rest that he placed in it. For although motion is nothing in the matter moved but its mode, it has yet a certain and determinate quantity, which we easily see may remain always the same in the whole universe, although it changes in each of the parts of it.

?

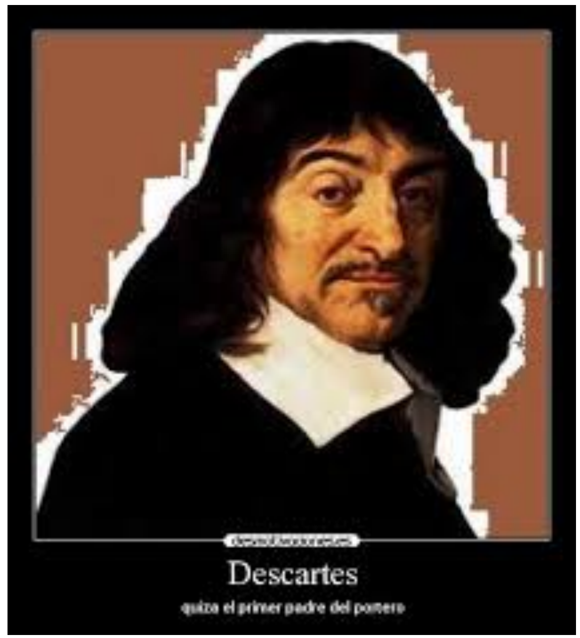
Newton?

Galileo?

...

Aristotle?

Advanced Area

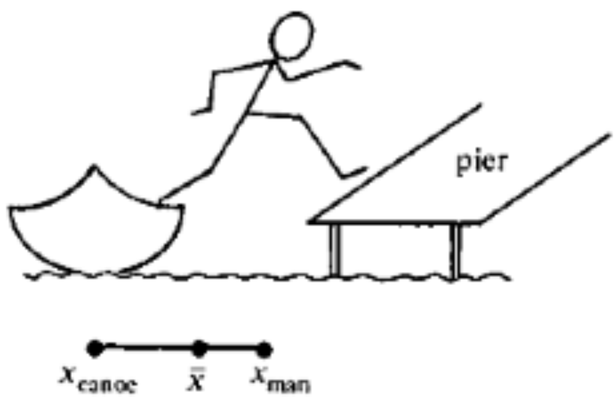


Said in 1644
(Newton was 2 years old ☺)

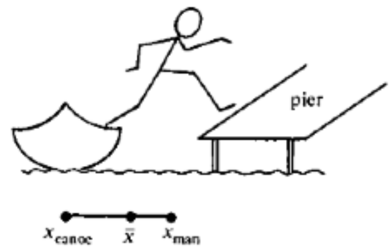
First example

Example 1

A 180-lb man tries to step out of a 90-lb canoe, initially at rest, onto a lakeside pier. What happens if he tries to step 2 ft laterally without holding on to the pier?



First Example



$(X_{CM})_{END} = (X_{CM})_{START} = 0$, say by choosing it as a reference origin.

$$(X_{CM})_{END} = \frac{m_{MAN} \cdot X_{MAN} + m_{CANOE} \cdot X_{CANOE}}{m_{MAN} + m_{CANOE}}$$

$$\Rightarrow \quad m_{MAN} X_{MAN} + m_{CANOE} X_{CANOE} = 0$$

$$\Rightarrow \quad \boxed{X_{CANOE} = -\frac{m_{MAN}}{m_{CANOE}} \cdot X_{MAN}}$$

Now, $m_{MAN} > m_{CANOE} \Rightarrow |X_{CANOE}| > |X_{MAN}|$! NOT GOOD!

NUMERICALLY:

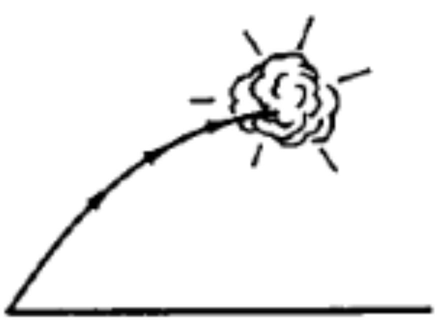
$$\boxed{X_{CANOE} = -\frac{180}{90} \cdot 2 = -4\,ft}$$ Farther from the pier

Second Example

If the influence of external forces is negligible

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + \cdots + m_n\mathbf{v}_n = \textbf{const.}$$

2. A projectile explodes while in flight. Fragments are blown in all directions as shown in the sketch. What can you say about the motion of the center of mass of the system after the explosion?



Where will the CM go?

Summary of Main Formulae: Collisions

$$\Sigma \vec{\boldsymbol{P}}_1 = \Sigma \vec{\boldsymbol{P}}_2$$

Elastic: $K_1 = K_2$ (One more equation)

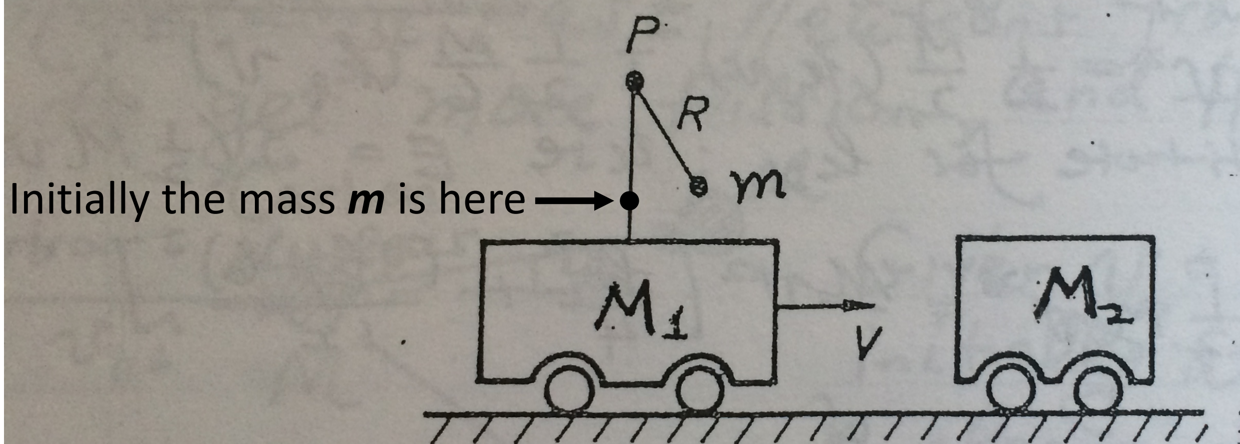
Inelastic: $K_1 > K_2$
If it is *totally inelastic*, the bodies end up together and we'll have one final velocity only)

INTERESTING FACT: In relativity, things will become more interesting due to $E=mc^2$!

Third Example

PROBLEM 2A

A cart of mass M_1 has a pole on it from which a ball of mass m hangs from a thin string of negligible mass and length R attached at point P, as shown in the figure. The cart and ball have initial velocity V (the ball is initially at rest with respect to the cart). The cart crashes into another cart of mass M_2 and sticks to it.



NB: for the collision, consider the mass of the pole is << M_1 and M_2.

[1 point] (a) Find the velocity of the two carts V' after the collision.

[2 points] (b) Find the smallest initial velocity V so that the ball will complete a circle around the point P after the collision.

7. Two carts and a pole

This problem is interesting, because it mixes linear momentum, forces and energy conservation. The first question requires to find the final speed of a totally inelastic collision:

$$V' = \frac{M_1}{M_1 + M_2} V. \tag{29}$$

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The second question is solved more easily in the inertial frame of the two carts after the collision. The reason is that, even though the mass of the pole m starts its circular motion with the speed V (the initial speed of the cart of mass M_1), it is also moving along with the carts, so it is in fact, the relative speed with respect to them that is available to turn.

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Moreover, let's remember that the basic equation for a pole to complete a circle is that the tension is 0 only at the highest point. If it were before that point, the mass m would not complete the circle. If it would be positive, it would complete it, but with more speed than necessary. Then

$$T(=0) - mg = -mv^2/R, \qquad \Rightarrow \qquad v_{\mathrm{UP}} = \sqrt{gR} \tag{30}$$

After the collision, conservation of energy can be used to find out the necessary initial speed to complete a circle:

$$E_{k,0} = \frac{1}{2}mv_{\mathrm{UP}}^2 + mg(2R) = \frac{5}{2}mgR. \tag{31}$$

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As mentioned before, the initial speed of the little ball available to turn around is

(32) $$v_0 = v - V' = M_2V/(M_1 + M_2)$$

So that

(33) $$E_{k,0} = \frac{1}{2}m\frac{V^2M_2^2}{(M_1 + M_2)^2} = \frac{5}{2}mgR, \Rightarrow V = \left(1 + \frac{M_1}{M_2}\right)\sqrt{5gR}.$$

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Two checks:

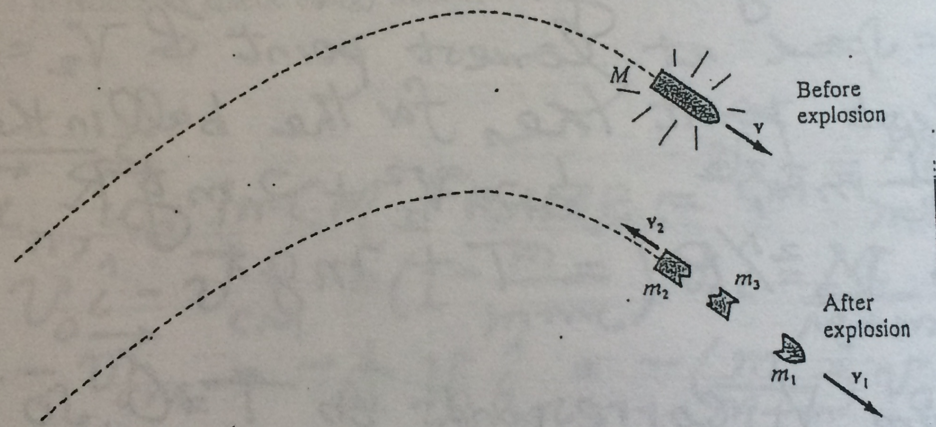
- dimensions make sense,
- the larger the ratio M_1/M_2 is, the higher the initial speed V needs to be.

That makes sense, because the larger M_1 is with respect to M_2, the least will it be slowed down, and the least impulse will have the mass of the pole to complete the circle.

Fourth Example

PROBLEM 2

A projectile of mass M initially traveling with speed v explodes in flight into three fragments (see the Figure). An energy E equal to 5 times the initial kinetic energy of the projectile is released in the explosion, and is transformed into additional kinetic energy of two of the projectiles. One fragment of mass $m_1 = M/2$ travels with speed $v_1 = k_1 v$ in the original direction of the projectile, while the second fragment of mass $m_2 = M/6$ travels in the opposite direction with speed $v_2 = -k_2 v$ and the third fragment of mass $m_3 = M/3$ is at rest the instant after the explosion



[2 point] (a) Write down equations expressing the conservation of momentum and energy in terms of M, k_1, k_2, v and E, immediately after the explosion.

[2 points] (b) Find the values of k_1 and k_2

The first question only requires to write down (v_2 is negative):

$$m_1\vec{v}_1 + m_2\vec{v}_2 = M\vec{v} \qquad \Rightarrow \qquad 3k_1 - k_2 = 6. \tag{34}$$

8. The broken rocket

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The second part is also a direct application of known formulas. The only detail is to add E:

$$E + \frac{1}{2}Mv^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{M}{4}v^2\left(k_1^2 + \frac{k_2^2}{3}\right) \tag{33}$$

Now, setting $E = 5/2Mv^2$: $12 = k_1^2 + k_2^2/3$. The equations to solve are:

$$6 = 3k_1 - k_2, \quad 12 = k_1 + \frac{k_2^2}{3} \Rightarrow k_1 = 3, k_2 = 3 \tag{34}$$

Fifth Example

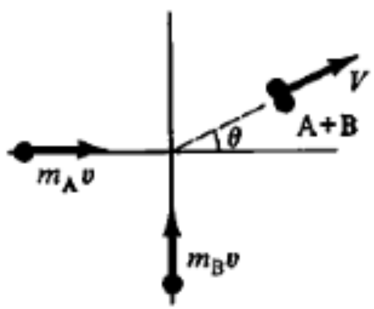
Example 7

A compact car with mass $m_A = 1300$ kg, and a sports car with mass $m_B = 1000$ kg approach an intersection, each traveling at 14 m/s. They collide and move off together at an angle θ as indicated in the diagram. Find

(a) the angle θ,

(b) the speed of the entangled cars after the collision,

(c) the amount of energy dissipated in the collision.



$$m_A v = (m_A + m_B)V \cos\theta$$

where both cars have the same speed after the collision. Because we have two unknowns, V and θ, we need another equation before we can solve for them. In the y direction, conservation of momentum implies

$$m_B v = (m_A + m_B)V \sin\theta.$$

Dividing our second equation by the first, we can solve for θ:

$$\frac{m_B}{m_A} = \frac{\sin\theta}{\cos\theta} = \tan\theta,$$

$$\tan\theta = 1000/1300,$$

which leads to $\theta = 37.6°$. As a check, note that $\theta \rightarrow 0$ in the limit $m_B/m_A \rightarrow 0$, and $\theta \rightarrow \pi/2$ in the limit $m_A/m_B \rightarrow 0$, consistent with the expectation that if one object has nearly all the mass, the combined system will continue to move in the same direction as the initial heavy object after the collision.

To find the speed of the cars after impact, we can use either momentum equation and solve for V; the result is $V = 10$ m/s. The difference between the initial kinetic energy and the final kinetic energy is the energy dissipated:

$$\text{energy dissipated} = (\tfrac{1}{2}m_A v^2 + \tfrac{1}{2}m_B v^2) - \tfrac{1}{2}(m_A + m_B)V^2,$$

which turns out to be 1.1×10^5 J.

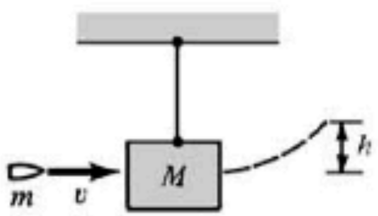
Sixth Example

14. A 0.03-kg mass traveling at 0.08 m/s collides head-on with a 0.05-kg mass which is initially at rest. If the collision is elastic, find the speed of each mass after the collision.

(SEE FINAL SLIDES ON THE ELASTIC COLLISION OF PARTICLES)

Seventh Example

7. Benjamin Thompson (Count Rumford) also used a method for determining the speed of a bullet or shell when it reaches the target. The bullet is fired horizontally into a block of wood mounted as a pendulum. The bullet stops in the wood, and the subsequent swing of the pendulum is measured. If the bullet has mass m and initial velocity v and the block has mass M,



(a) what is the horizontal velocity V of the pendulum just after impact?
(b) Show that kinetic energy is not conserved during the impact, and use this result to find the heat generated during the impact.
(c) Find v in terms of m, M, and the height h of the pendulum swing.

NO SOLUTION PROVIDED. YOU SHOULD BE ABLE TO SOLVE IT WITHOUT ANY BIG ISSUES

Collisions viewed from the Center of Mass reference frame

$$\frac{d\bar{\mathbf{r}}}{dt} = \bar{\mathbf{v}} = \frac{1}{M}\sum m_i \mathbf{v}_i .$$

$$\mathbf{p}_{tot} = 0 \longrightarrow \sum m_i \mathbf{v}_i' = 0$$

$$K = K' + \tfrac{1}{2} M \bar{v}^2 .$$

In other words, the kinetic energy that may be exchanged in a collision is not all, but the relative to the center of mass.

Eight Example

Example 9

A hydrogen atom of mass m_H and initial velocity v_0 in the laboratory collides with an electron that has mass m_e and is initially at rest. What fraction of the initial laboratory kinetic energy is available to increase the internal energy of the atom by an amount ΔE?

Final Test Time

In the center-of-mass frame, all of the energy is available. A conversion

$$K' = \Delta E$$

is possible. The kinetic energy in the laboratory frame is initially $\frac{1}{2}m_H v_0^2$ and is related to K' by Eq. (11.32):

$$\frac{1}{2}m_H v_0^2 = K' + \frac{1}{2}(m_H + m_e)\overline{v}^2.$$

According to Eq. (11.26), the center-of-mass velocity is

Final Test Time

25. In a nuclear fission reactor, neutrons emitted at high speed in the fission process must be slowed down by collisions with inert nuclei such as ^{12}C so that they may induce further fission events.
A fast neutron of initial velocity $v_0\hat{\mathbf{i}}$ collides elastically with a stationary ^{12}C nucleus.

(a) What is the initial speed of each particle in the center-of-mass frame?
(b) If the ^{12}C nucleus scatters into an angle θ in the center of mass, show that its final velocity in the laboratory frame is one-thirteenth of the vector

$$v_0(1 - \cos\theta)\hat{\mathbf{i}} + v_0 \sin\theta\hat{\mathbf{j}}.$$

$$\bar{v} = \frac{m_H v_0}{m_H + m_e} .$$

Eliminating $\bar{v}$ from the previous two equations, we find

$$\frac{m_H v_0^2}{2} = K' + \frac{m_H^2 v_0^2}{2(m_H + m_e)} ,$$

which, by rearrangement, becomes

$$K' = \frac{m_H v_0^2}{2}\left(1 - \frac{m_H}{m_H + m_e}\right)$$

$$= \frac{m_H m_e v_0^2}{2(m_H + m_e)} .$$

Thus only a fraction

$$\frac{K'}{K} = \frac{m_e}{m_H + m_e} \approx \frac{1}{1837}$$

of the laboratory-frame kinetic energy is available for conversion. Almost all of the initial kinetic energy is tied up in center-of-mass motion; a light electron cannot slow down a heavy proton very much.

If it is the hydrogen atom that is initially at rest, and the electron that has the initial momentum, the center of mass moves much more slowly than the electron, being dominated by the heavier proton. In this case a repeat of the above argument shows that $K'/K = m_H/(m_H + m_e)$; almost all of the laboratory-frame energy is available for conversion.

Final Test Time

(c) In such an elastic collision, what is the maximum fraction of its laboratory kinetic energy that the neutron can lose?

(d) If the average energy lost in such a collision is one-half of the maximum possible loss, what is the average number of collisions a neutron must undergo in order to reduce its kinetic energy from 1,000,000 eV to 1000 eV?

Let's use the following notation:

- m mass of the neutron, M mass of the ^{12}C.
- v for the neutron speed, e.g., v_0 initial speed of the neutron and V the speed of the ^{12}C after the collision.
- Prime quantities refer to the CM reference system.

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The first question is easy to solve, but it reminds us the steps to follow in several problems of collisions: go to the CM reference system: solve the collision and come back to the laboratory reference system. The solution is:

$$v_{\rm CM} = \frac{m v_0}{m+M}, \tag{1}$$

$$v' = v - v_{\rm CM} = \frac{M v_0}{m+M}, \quad V' = 0 - v_{\rm CM} = -\frac{m v_0}{M+m}. \tag{2}$$

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Notice that, in the reference system of CM, the total linear momentum is 0: $\vec{p'}_{\text{neutron}} + \vec{p'}_{^{12}\mathrm{C}} = \vec{0}$.

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Now, to go back to the laboratory reference system, we do:

$$\vec{V} = \vec{V}' + \vec{v}_{\rm CM} = (V'\cos\theta + v_{\rm CM})\,\vec{\imath} + V'\sin\theta\,\vec{\jmath}. \tag{4}$$

Substituting the expressions found so far, we obtain the expression shown in the problem.

$$\vec{V} = \frac{mv_0}{m+M}\left[(1-\cos\theta)\,\vec{i} + \sin\theta\,\vec{j}\right] \tag{5}$$

The factor 1/13 comes from $m/(m+M)$ with $M = 12m$.

For the third part of the problem, we *can not* use the theorem that says that the greatest amount of kinetic energy available to be extracted in a collision is the kinetic energy of the particles in the CM reference system. This is because this is an elastic collision, and that limit happens for an inelastic collision. Taking into account that the collision is elastic, we can write:

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We now notice that it is not necessary to derive v_1, the final velocity of the neutron in the laboratory reference system. In order to know the loss of kinetic energy of the neutron, we see that is is precisely the kinetic energy of the ^{12}C after the collision: $\frac{1}{2}mv_0^2 - \frac{1}{2}mv_1^2 = \frac{1}{2}MV^2$.

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The problem asks about the *maximum* possible loss. That will then happen when V is maximum. The magnitude of $\vec{V}$ is proportional to $(1-\cos\theta)^2 + \sin^2\theta = 2(1-\cos\theta)$. Therefore, V is maximum for $\theta = \pi$. For this case, we have that the loss of the neutron kinetic energy is:

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$$E_{k,\text{loss}} = \frac{1}{2}MV_{\max}^2 = \frac{1}{2}M\frac{4m^2}{(m+M)^2}v_0^2 = \frac{1}{2}mv_0^2\left[\frac{4mM}{(m+M)^2}\right]. \tag{7}$$

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Where I have written the loss in terms of the initial kinetic energy.

Notice that the factor *does not* depend on the initial velocity of the neutron. It is $4mM/(m+M)^2$. Therefore, it will be the same after the following collision, regardless of the final speed of the neutron after each collision. Moreover, notice that the factor becomes largest for $M=m$. This tells us that indeed to slow down neutrons, it's much more efficient to use materials with protons, like paraffin.

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The last question asks for the number of collisions that will be necessary to decrease the initial kinetic energy of the neutron by a factor of 1000. Therefore we have to solve an equation like:

$$\left[1 - 0.5 * \frac{4mM}{(m+M)^2}\right]^N = \frac{10^3}{10^6} = 10^{-3}, \tag{8}$$

where the right hand side shows the loss of going from 1,000,000 eV to 1,000 eV. And the **factor** 0.5 has been included to follow the instructions in the exercise. The solution is ($M = 12m$):

$$N = -\frac{3}{\log(1 - 24/169)} = 45.1 \Rightarrow 45. \tag{9}$$

45 collisions are necessary. In the case of paraffin, where $M = m$, N is much smaller, $N = 10$. The reaction might be stopped, rather than slowed down.