PH1a: 1 dimensional elastic collisions

In many occasions, we have a collision between two objects where the masses of the objects are the same at the end than at the beginning and everything happens on a line (1-dimensional). If the collision is elastic, one uses conservation of linear momentum and kinetic energy to derive the final speeds. It always takes some algebra to get the final solution. After some algebra, one can prove that the final speeds for this common situation are given by the following formula (at least, it may help you check that your work was correct)

Here v_{1F}, v_{2F} are the final speeds of the two masses, m_1 and m_2, respectively, and v_{10}, v_{20} are the initial speeds.

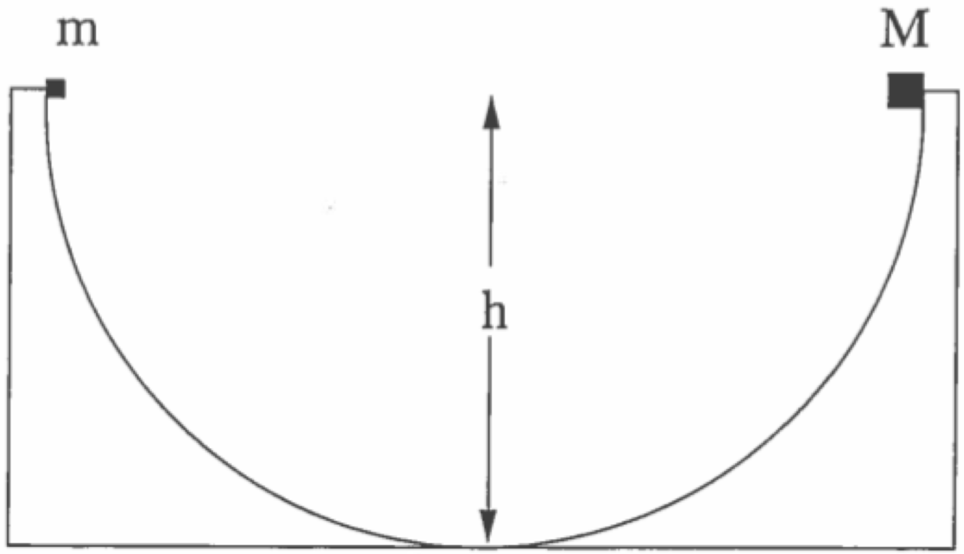
$$\mathrm{v}_{1\mathrm{F}} = \frac{(m_1 - m_2)}{(m_1 + m_2)} v_{10} + \frac{2m_2}{(m_1 + m_2)} v_{20}$$

$$\mathrm{v}_{2\mathrm{F}} = \frac{(m_2 - m_1)}{(m_1 + m_2)} v_{20} + \frac{2m_1}{(m_1 + m_2)} v_{10}$$

Collision and energy

PROBLEM 1

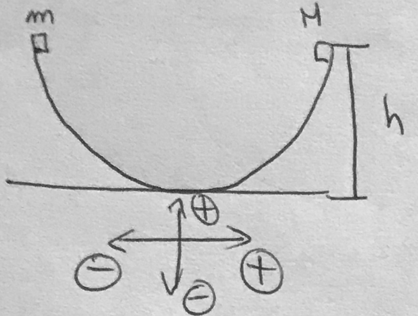
Two masses (M, m) slide without friction (under the influence of a uniform gravitational field g) down the sides of a hemisperical bowl. They each start with zero velocity at the lip of the bowl, which is a height h above the bottom.



(1 point) (a) What is the total kinetic energy when the masses first touch (assume the size of the masses is much less than h)?

(2 points) (b) How high will the masses move if the collision is completely inelastic (i.e. they stick together)? Can it be greater than h?

(2 points) (c) Assuming that the collision between the masses is elastic, what is the maximum height that mass m will acheive? Can it be greater than h?



a) The speed at the bottom is:

$$\frac{1}{2} m v_B^2 = mgh \Rightarrow v_B = \sqrt{2gh}.$$

Similarly for M:

$$v_B = -\sqrt{2gh}$$. Notice the <u>sign</u> (left)

$$\boxed{K_{\text{Before colision}} = \frac{1}{2} m v_B^2 + \frac{1}{2} M v_B^2 = \frac{1}{2}(m+M)2gh = (M+m)gh}$$

<u>Or</u>: $K_{\text{Before Colision}}$ is the potential (gravitational) energy at the beginning (no friction) $\boxed{K = (M+m)gh}$

b) Totally inelastic:

$$m\sqrt{2gh} - M\sqrt{2gh} = (m+M) v_F \Rightarrow$$

$$v_F = \frac{(m-M)}{(m+M)}\sqrt{2gh}.$$ (Notice the ⊖ sign)

Maximum height:

$$mg\, h_{max} = K_{\text{after colission}} \Rightarrow \boxed{h_{max} = \frac{\frac{1}{2} m v_F^2}{mg} = \frac{v_F^2}{2g} = \frac{(m-M)^2}{(M+m)^2} \cdot h}$$

If $h_{max} > h \Rightarrow \left(\frac{m-M}{m+M}\right)^2 > 1 \Rightarrow \frac{m-M}{m+M} > 1$ or $\frac{m-M}{m+M} < -1 \Rightarrow$

$m - M > m + M \Rightarrow M < 0$ IMPOSSIBLE, or $m - M < -m - M \Rightarrow m < 0$ IMPOSSIBLE.

(masses < 0 would allow for $K < 0$ which would allow to violation of basic conservation laws in physics).

So, not possible to have $h > h_{max}$ if it is (completely) inelastic.

The reason is that Kinetic energy is NOT conserved, is less and therefore there's less to be converted into gravitational potential energy.

c) Elastic collision: Remember the general solution:

$$\left.\begin{aligned} v_1' &= \frac{(m_1 - m_2)}{(m_1 + m_2)} v_1 + \frac{2m_2}{(m_1+m_2)} v_2 \\ v_2' &= \frac{(m_2 - m_1)}{(m_1+m_2)} v_2 + \frac{2m_1}{(m_1+m_2)} v_1 \end{aligned}\right\} \xrightarrow[m_1 = m,\ m_2 = M]{v_1 = \sqrt{2gh},\ v_2 = -\sqrt{2gh}} \left\{\begin{aligned} \boxed{v_1'} &= \frac{(m-M)}{(m+M)}\sqrt{2gh} - \frac{2M}{(m+M)}\sqrt{2gh} = \\ &\boxed{\left(\frac{m-3M}{m+M}\right)\sqrt{2gh}} \end{aligned}\right.$$

$$\boxed{v_2' = \dots = -\frac{(M-3m)}{(m+M)}\sqrt{2gh}}$$

Given the speed after the collision, one can derive the maximum height by energy conservation, and the answer is:

$$h_{max} = \frac{v^2}{2g} \quad , \quad \text{so that:}$$

$$\boxed{\left(h_{max}\right)_{\text{mass "}m\text{"}} = \frac{(m-3M)^2}{(m+M)^2} \frac{\left(\sqrt{2gh}\right)^2}{2g} = \left(\frac{m-3M}{m+M}\right)^2 h}$$

$$h_{max} > h \Rightarrow \left(\frac{m-3M}{m+M}\right) > 1 \quad \text{OR} \quad \left(\frac{m-3M}{m+M}\right) < -1 \text{, so that:}$$

Either: $\cancel{m} - 3M > \cancel{m} + M \Rightarrow M < 0$ (<u>NOT</u>) or $m - 3M < -m - M \Rightarrow$
$2m < 2M \Rightarrow \underline{\underline{M > m}}$

Therefore, if $\underline{\underline{M > m}}$, m will end up at a <u>higher</u> place ($h_{max} > h$) than initially.

This means that 'm' will end up with <u>more</u> gravitational potential energy that it had at the beginning. But energy is conserved. Let's look at the maximum height of the mass "M"!

$$(h_{max})_{mass\ "M"} = \frac{v_M^2}{2g} = \frac{(M-3m)^2}{(m+M)^2}\, h,$$ where we have used our previous result: $v_M (v_2') = -\frac{(M-3m)}{(M+m)}\sqrt{2gh}$. Now let's realize that:

$$-1 < \frac{M-3m}{m+M} \Rightarrow -(M+m) < M-3m \Rightarrow -M-m < M-3m \Rightarrow 2m < 2M.$$

$$m < M.$$

So that if $M > m$, $(h_{max})_{"m"} > h$ but $(h_{max})_{"M"} < h$.

The heavier body will end up at a <u>lower</u> height than initially.

In fact, let's compute the final potential energy:

$$\boxed{mg(h_F)_m + Mg\,(h_F)_M} = \frac{\left[m(m-3M)^2 + M(M-3m)^2\right]}{(m+M)^2}\, gh$$

$$= \frac{m(m^2 - 6mM + 9M^2) + M(M^2 - 6Mm + 9m^2)}{(m+M)^2}\, gh =$$

$$= \frac{m^3 + 3m^2M + 3mM^2 + M^3}{(m+M)^2}\, gh = \frac{(m+M)^3}{(m+M)^2}\, gh = \boxed{(m+M)gh}; \quad \text{GREAT!}$$