PH1a: Forces

Newton's second law:

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

$$\vec{p} = \sum m\vec{v}$$

These notes cover **11** exercises of dynamics in great detail

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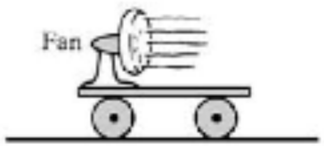
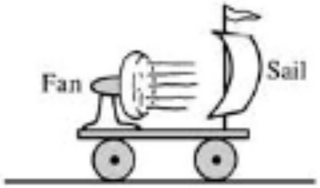
PH1a: Forces

6. A fan is mounted on a cart as shown below. If the fan is turned on, does the cart move? If so, in which direction?



Suppose a sail were added to the cart. What would be the motion of the cart if the fan were now turned on?



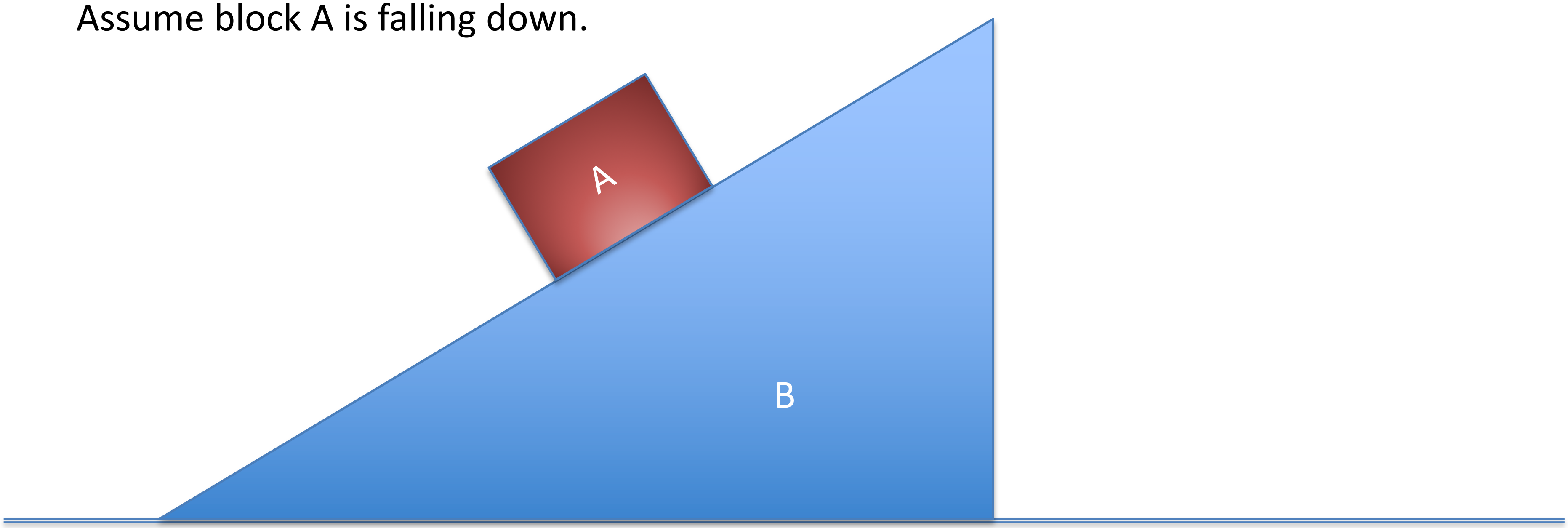
Watch also:

Video 1

Video 2

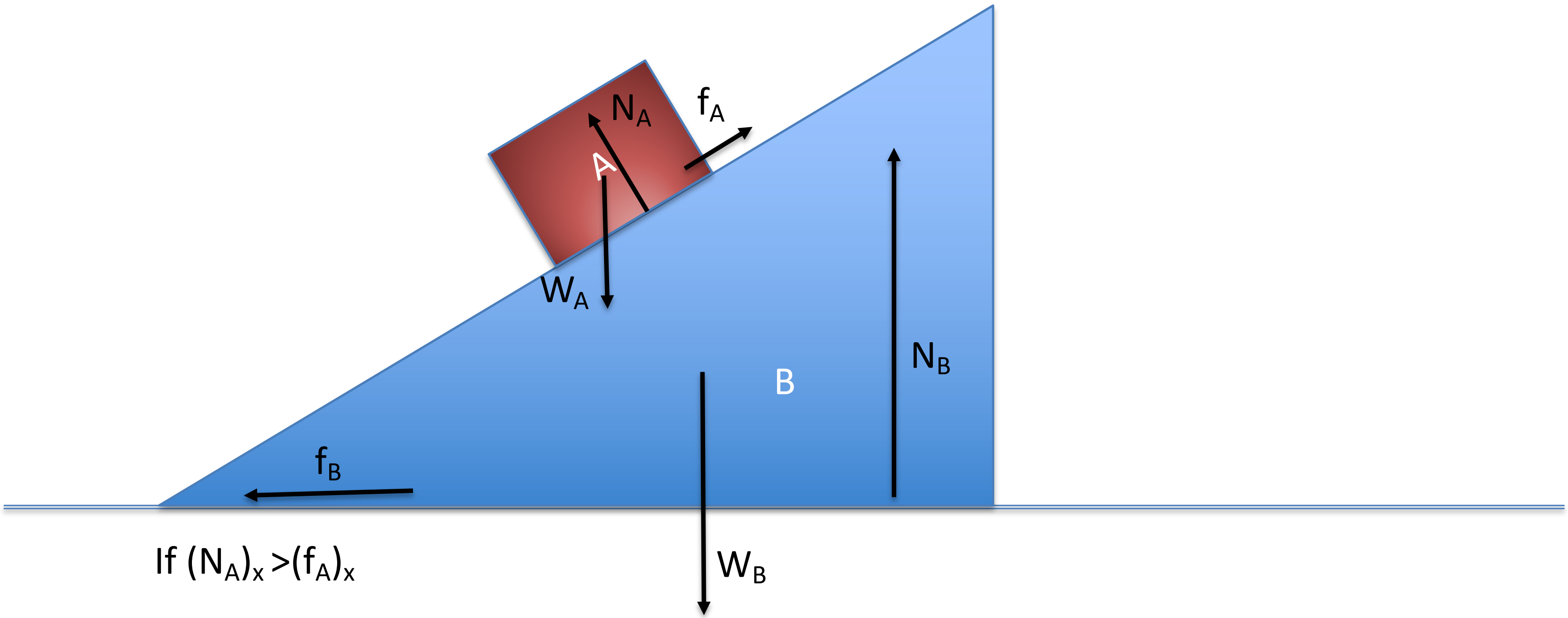
Problem 1: Action and Reaction in Full Detail

Newton's Third Law: Draw all the forces on this diagram. There's friction on all surfaces. Assume block A is falling down.



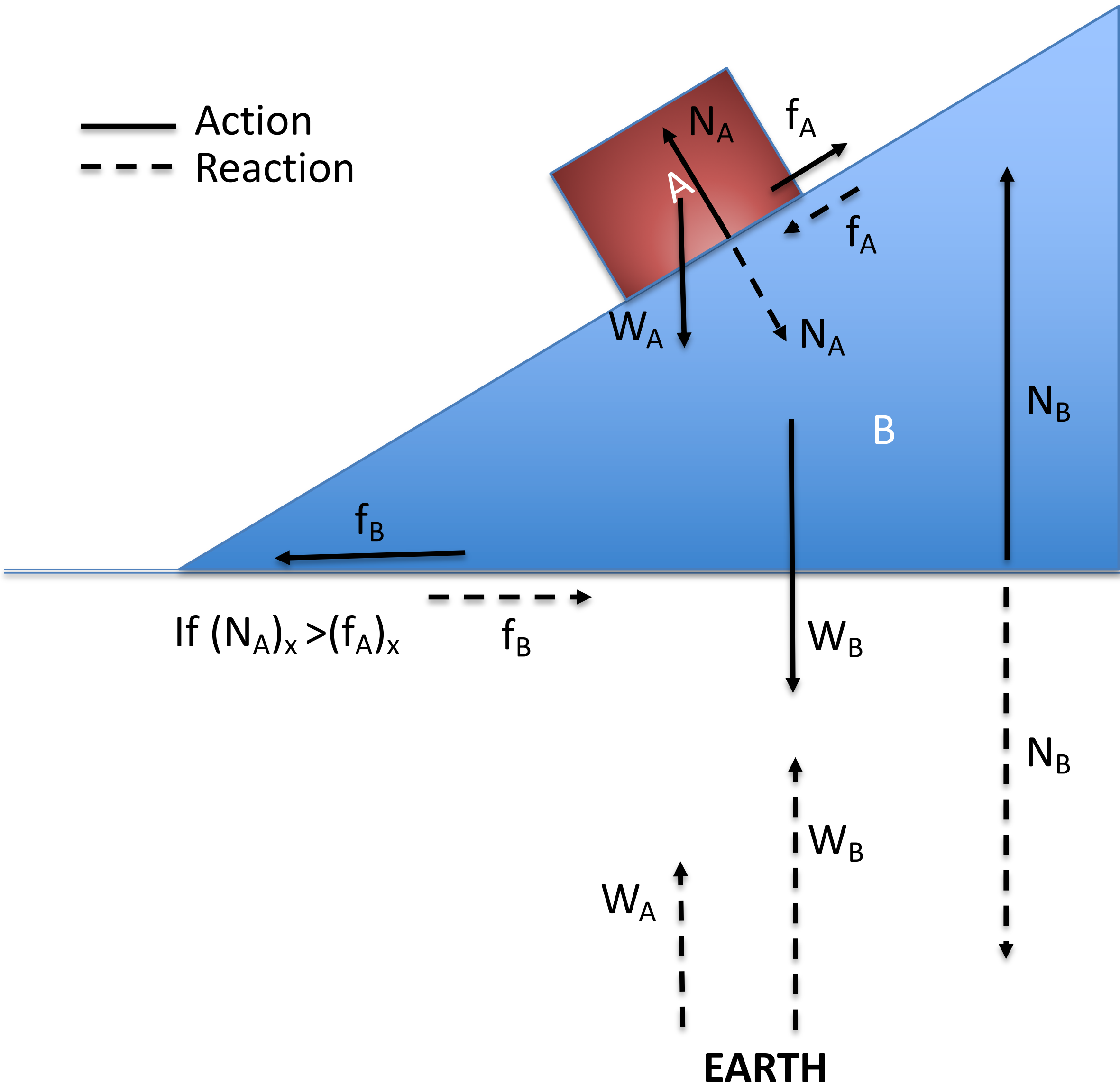
Problem 1: Action and Reaction in Full Detail

These would be the forces without drawing the reactions



If $(N_A)_x > (f_A)_x$

Problem 1: Action and Reaction in Full Detail



$\mathbf{N_B}$ has to support $\mathbf{W_B}$, part of $\mathbf{W_A}$, and part of $\mathbf{f_A}$!

If you apply Newton's law *F=ma* to each block, remember all these forces.

If the incline is not fixed to the ground, it could certainly move!

PH1a: Forces of friction

Static friction:

$$f_s \leq \mu_s N$$

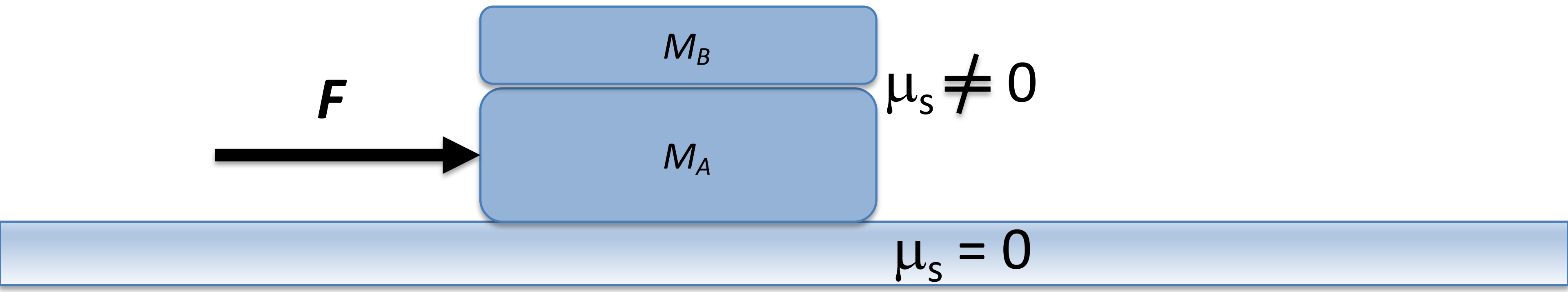
Static friction is *not* always *equal* to μN

Kinetic friction:

$$f_k = \mu_k N$$

Kinetic/sliding friction *is* always *equal* to μN

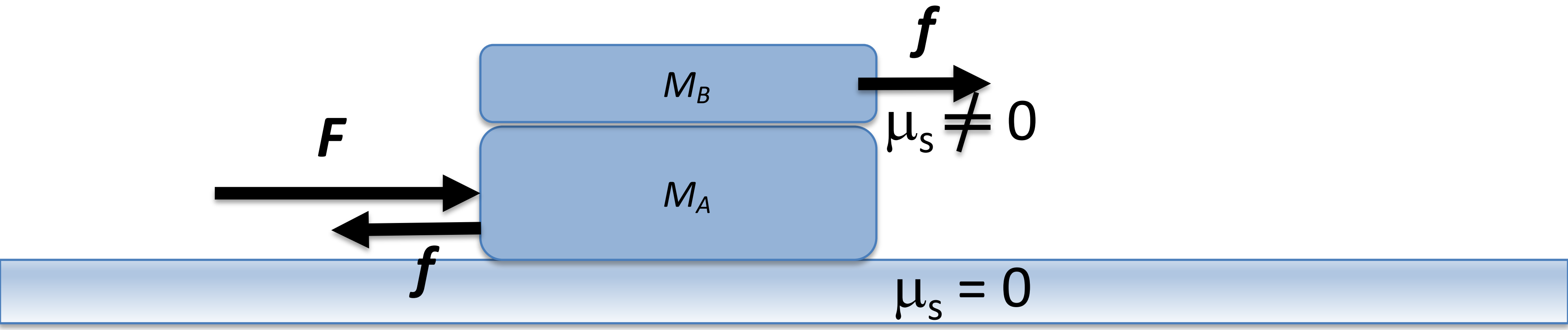
Problem 2: Let's practice



For M_A, M_B and μ_s, find the maximum force, ***F***, that can be applied so that M_B remains on M_A

Problem 2: Solution

The most interesting thing to learn from this problem is the **action/reactio**n pair due to friction between body ***A*** and ***B*** and their direction: to the right in ***B (in the sense of motion)*** and to the left in ***A***



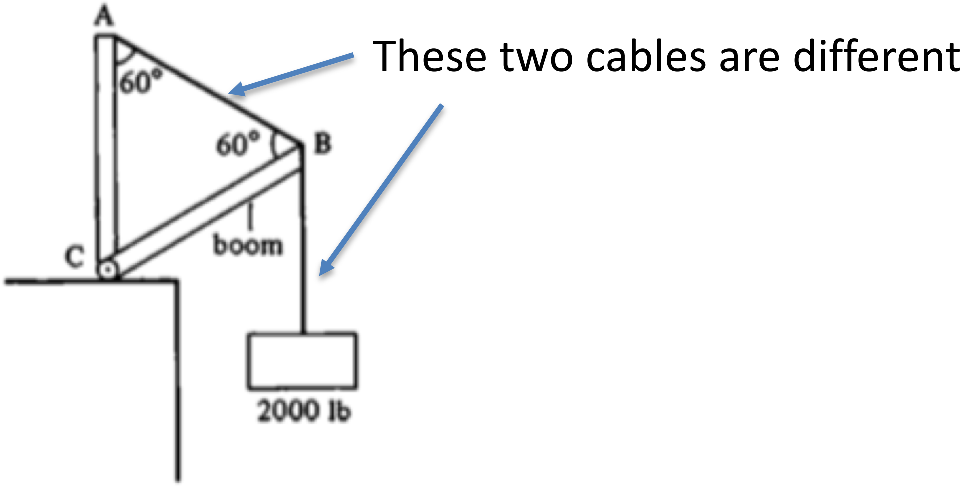
$$f = M_B\, a_B \quad \text{and} \quad F - f = M_A\, a_A$$

We want to impose $a_A = a_B = a$. Summing up get: $F = (M_A + M_B)a$. On the other hand, the static friction is at most $f = \mu_s N_B = \mu_s M_B g$. Therefore, the maximum acceleration is $a = f/M_B = \mu_s g$ and:

$$F_{max} = (M_A + M_B)a_{max} = (M_A + M_B)\mu_s g$$

Problem 3: Drawing all forces

24. The weight of the 400-lb boom in the derrick pictured below is uniformly distributed along its length. By considering the equilibrium conditions for the boom, find the tension in the cable running from A to B (which need not be the same as the tension in the cable below B) and the horizontal and vertical forces exerted on the boom at point C by the hinge of the derrick. By considering the equilibrium conditions for point B, calculate the force exerted by the boom on the cable at point B.



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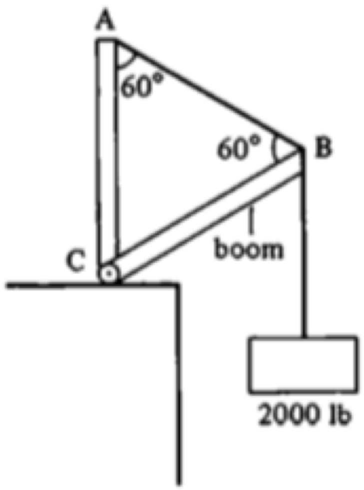
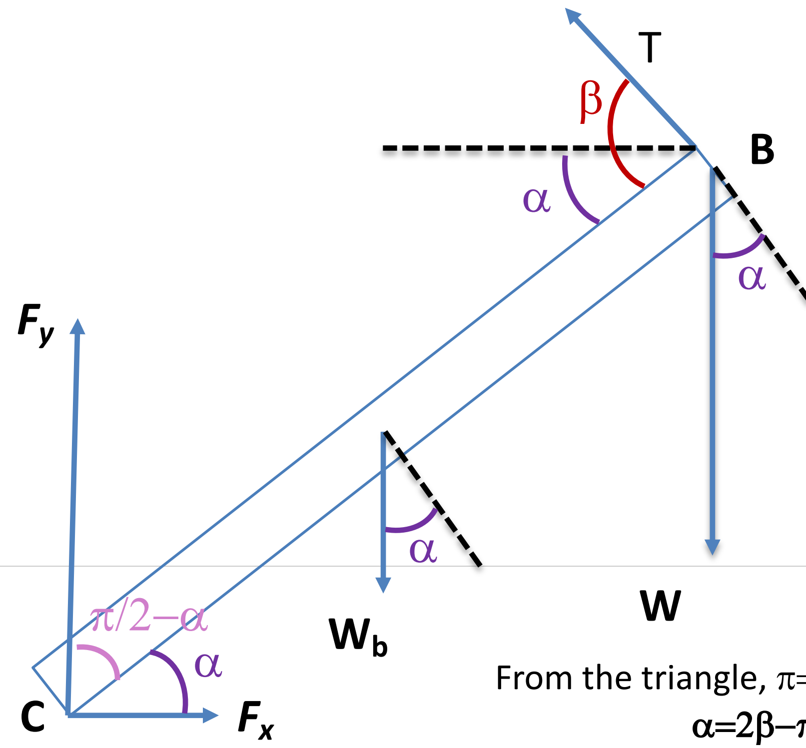




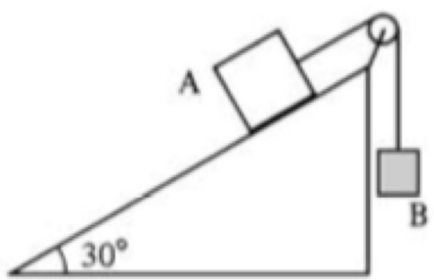
From the triangle, $\pi=\beta+\beta+\pi/2-\alpha$, or $\alpha=2\beta-\pi/2$

PS: you should get with torques about **C**: $\boldsymbol{T=(2W+W_b)\cos\beta=2{,}200}$ **lb**

Problem 4: Drawing all forces

Chapter 8

18. Two blocks are connected over a massless, frictionless pulley as shown. The mass of A is 8.0 kg and the coefficient of kinetic friction is 0.20. If block A slides down the plane with constant speed, what is the mass of B?



Problem 4: Drawing all forces

We will consider the incline fixed to the ground!

Chapter 8

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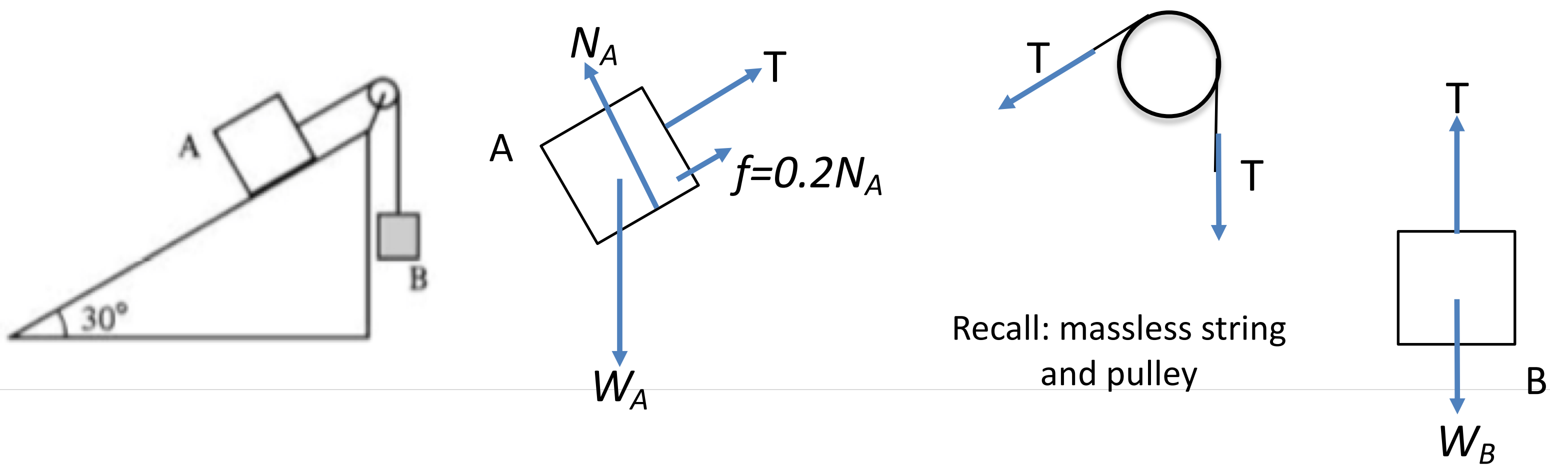
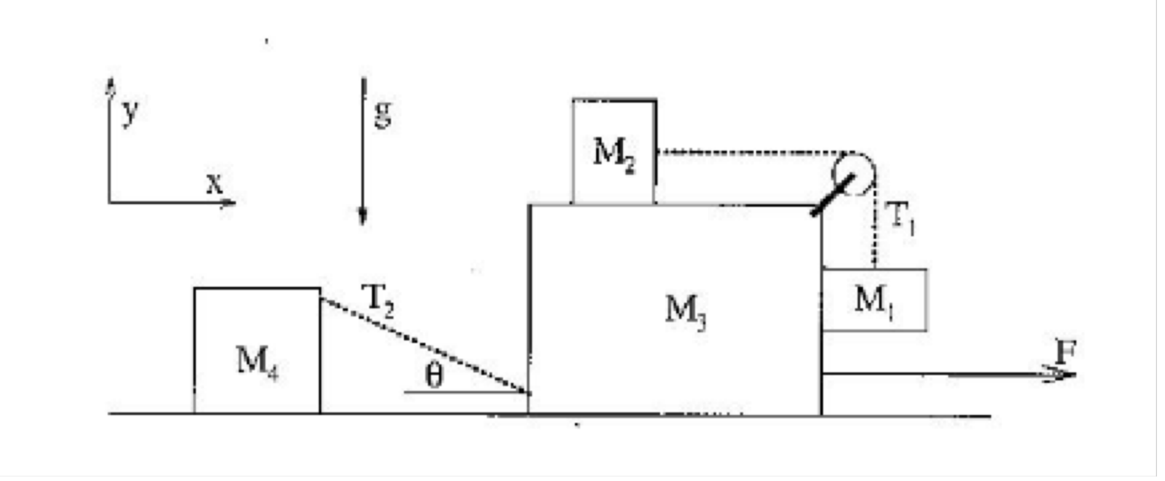


PS: you should arrive to this equation, $\boldsymbol{W_A \sin 30 - 0.2\, W_A \cos 30 = W_B}$

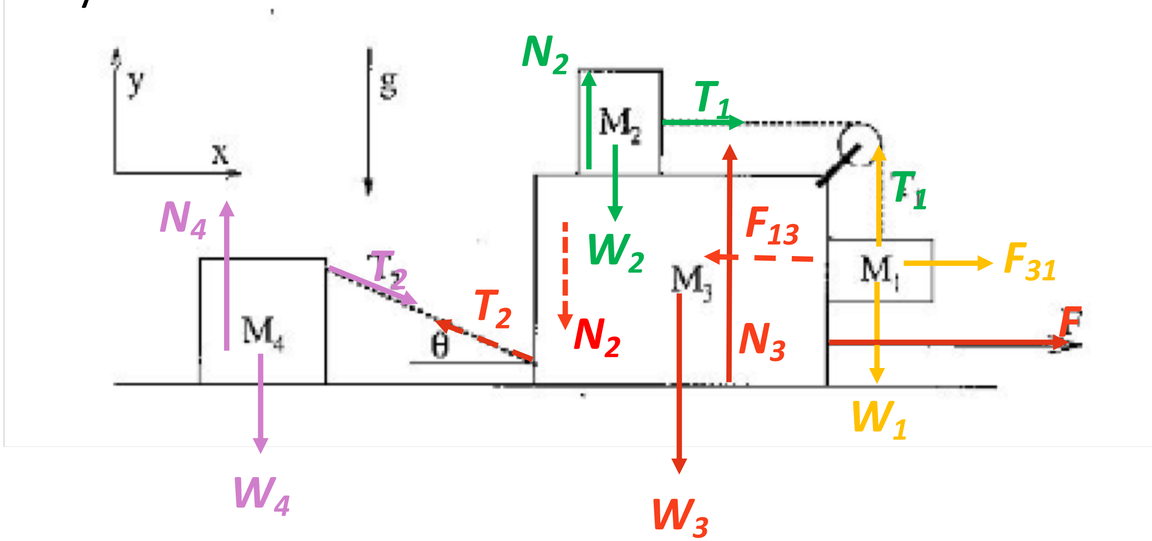
Problem 5: Drawing all forces

QP4



Problem 5: Drawing all forces

3 forces upon body #**1**
3 forces upon body #**2**
6 forces upon body #**3**
3 forces upon body #**4**



PS: magnitude of $\boldsymbol{F_{31}}$ is the same as the magnitude of $\boldsymbol{F_{13}}$. On the **pulley**, there are 2 equally opposed tensions with same magnitude as $\boldsymbol{T_1}$. The reaction of the **weights** *all* go in the Earth. $\boldsymbol{N_3}$ and $\boldsymbol{N_4}$ also go in the Earth.

What is the relationship between F, M_1, M_2, M_3 and M_4?

- Easiest answer: we know all blocks move together (same acceleration). Therefore:

$$\Sigma F = M_{TOTAL} \cdot a = (M_1 + M_2 + M_3 + M_4).$$

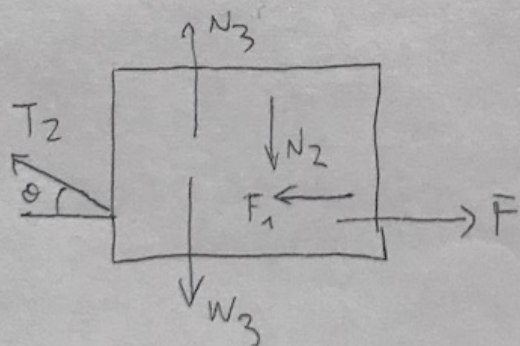
$\Sigma F = F$, since all the other forces are just action/reaction (internal)

$$\Rightarrow \quad F = (M_1 + M_2 + M_3 + M_4)\, a$$

The "a" can be found using the condition between M_2 and M_1. Namely,

$M_2 a = T_1 = M_1 g \Rightarrow a = M_1 g / M_2$.

- What if we just look at body #3?



Along x axis: $\Sigma F_x = M_3 \cdot a$

$\Sigma F_{x,\,body3} = -T_{2x} - F_1 + F$; (Some algebra...)

$T_{2x} = M_4 a$ & $F_1 = M_1 a$

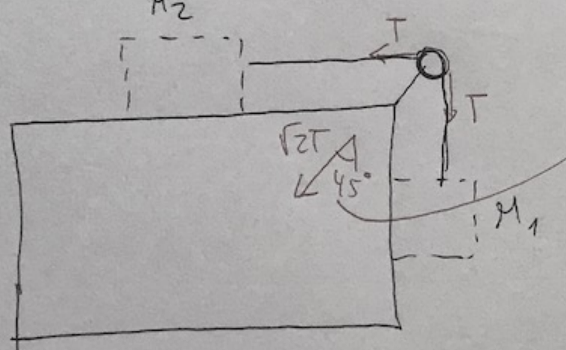
$\Rightarrow \quad -M_4 a - M_1 a + F = M_3 a \Rightarrow$

$$F = (M_1 + M_3 + M_4)\, a \quad \text{(wrong!)}$$

Where's M_2 gone?

SOLUTION: M_2 moves thanks to M_1 & thanks to the pulley!

We need to include the pulley to understand the motion of M_3:



The pulley pushes down and to the left with a net force of $\sqrt{2}T$ and 45°.

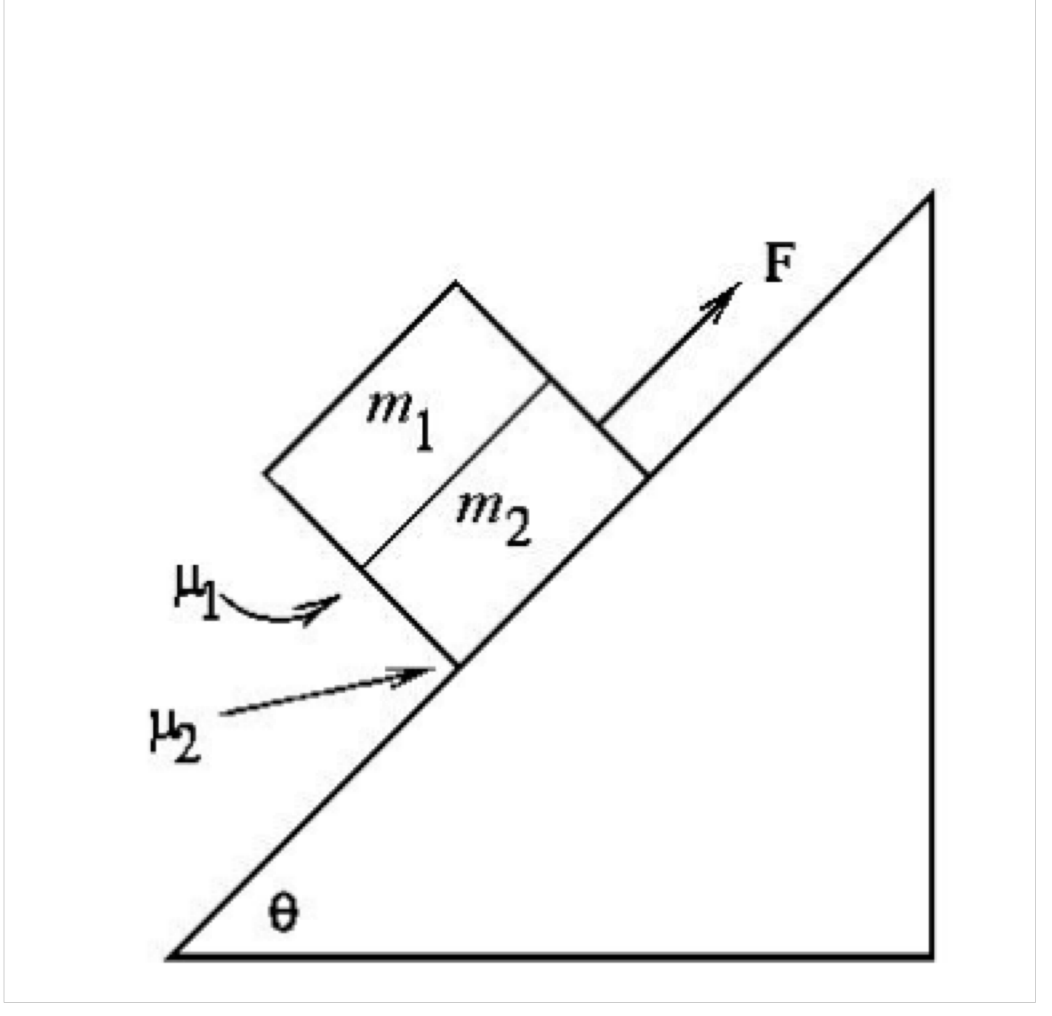
$$\Rightarrow \sum_{\text{body 3}} F_x = -T_{2x} - F_1 - T + F = M_3 a; \quad T = M_2 a \text{ (recall)}$$

($T = \sqrt{2}T \cdot \cos 45°$)

$$\Rightarrow \boxed{F = (m_1 + m_2 + m_3 + m_4)\, a}$$

CORRECT!

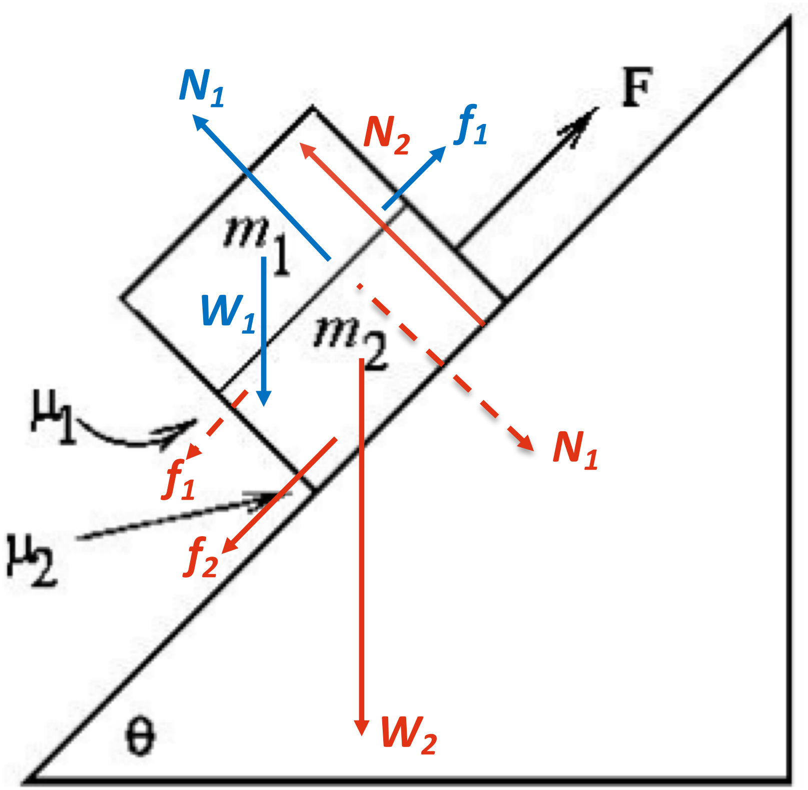
Problem 6: Drawing all forces



Problem 6: Drawing all forces

We will consider the incline fixed to the ground! Thus, I'm not drawing the action/reaction forces on the incline.

3 forces upon body #**1**
6 forces upon body #**2**

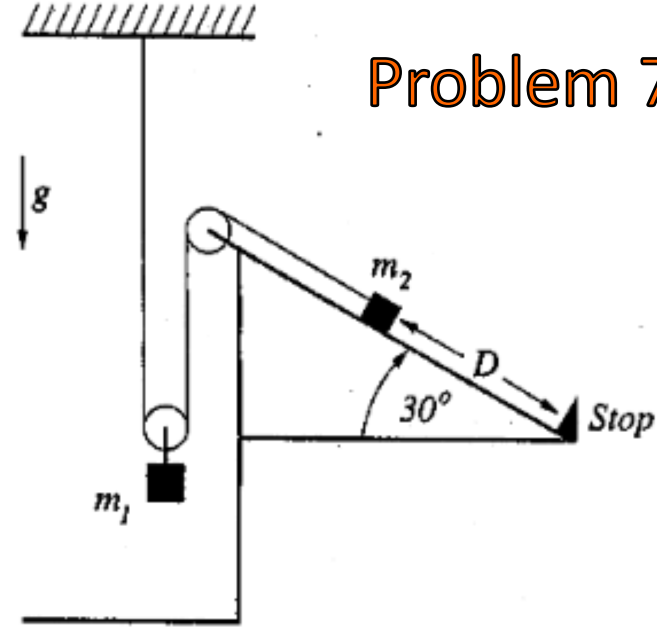


Notice ***N***$_1$ acts upon body #**2**

PS: the reaction of f_2 would be in the incline. Likewise for the reaction of N_2. And the reactions of W_1 and W_2 are in the Earth.

Problem 7: Pulleys

FP9 from the Course webpage



Two masses are connected by a string as shown. m_2 slides without friction on a fixed incline at an angle of 30° with respect to the horizontal. Neglect the mass and friction of the pulleys, and the mass of the string.

(a) **(2 points)** Find the ratio of the masses m_2/m_1 such that the masses will remain stationary, if they are initially at rest.

(b) **(1 points)** If the mass m_2 moves a small distance ΔD_2 along the incline, find the distance ΔD_1 that the mass m_1 moves.

(c) **(3 points)** If $m_2 = 2m_1$, and the masses are initially at rest as shown, find the acceleration of m_2.

(d) **(3 points)** If m_2 slides a distance D down the incline before encountering the stop at the bottom, what are the speeds of m_2 and m_1 just before encountering the stop.

(e) **(1 point)** When the moments of inertia of the pulleys are taken into account, do the speeds of the masses in part (d) increase, decrease, or remain the same?

3. FP9

a)

$$
\begin{aligned}
T_1 - W_2 \sin\theta &= 0 \\
2T_1 &= W_1 \\
\Rightarrow \frac{m_2}{m_1} &= \frac{1}{2\sin\theta}
\end{aligned}
$$

Notice that the limit $\theta \to \pi/2$, gives $m_2/m_1 = 1/2$ as expected, and the limit $\theta \to 0$ gives $m_2/m_1 \to \infty$, also as expected, since a horizontal chute cannot hold (without friction) the weight hanging on the pulley.

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b) The tip is to consider the total length of the rope, which is constant. Portion by portion:

$$L_{\text{total}} = l_a + \pi R + l_b + \pi R + l_c,$$

$$\Delta l_c = d \Rightarrow \Delta l_a = \Delta l_b = -d/2.$$

b) The tip is to consider the total length of the rope, which is constant. Portion by portion:

$$L_{\text{total}} = l_a + \pi R + l_b + \pi R + l_c,$$
$$\Delta l_c = d \Rightarrow \Delta l_a = \Delta l_b = -d/2.$$

c) Taking second derivatives:

$$\frac{d^2 l_a}{dt^2} = \frac{d^2 l_b}{dt^2}, \quad \frac{d^2 l_c}{dt^2} + \frac{d^2 l_a}{dt^2} + \frac{d^2 l_b}{dt^2} = 0,$$
$$\Rightarrow \frac{d^2 l_a}{dt^2} = \frac{d^2 l_b}{dt^2} = -\frac{1}{2}\frac{d^2 l_c}{dt^2}$$

That is, $a_1 = a_2/2$. Now, the equations of the forces are:

$$2T_1 - W_1 = m_1 a_1, \quad W_2 \sin\theta - T_1 = m_2 a_2,$$

1

c) Taking second derivatives:

$$\begin{aligned}\frac{d^2l_a}{dt^2} &= \frac{d^2l_b}{dt^2}, \quad \frac{d^2l_c}{dt^2} + \frac{d^2l_a}{dt^2} + \frac{d^2l_b}{dt^2} = 0, \\ \Rightarrow \frac{d^2l_a}{dt^2} &= \frac{d^2l_b}{dt^2} = -\frac{1}{2}\frac{d^2l_c}{dt^2}\end{aligned}$$

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$$2T_1 - W_1 = m_1a_1, \quad W_2 \sin\theta - T_1 = m_2a_2,$$

That is, $T_1 = W_2 \sin\theta - 2m_2a_1$, $2W_2 \sin\theta - 4m_2a_1 - W_1 = m_1a_1$, $2W_2 \sin\theta - W_1 = (m_1 + 4m_2)a_1$, and, finally,

$$\begin{aligned}a_1 &= \frac{2W_2 \sin\theta - W_1}{m_1 + 4m_2} = \left(\frac{2m_2 \sin\theta - m_1}{m_1 + 4m_2}\right) g. \\ a_2 &= 2a_1, \quad a_2 = 2\left(\frac{2m_2 \sin\theta - m_1}{m_1 + 4m_2}\right) g.\end{aligned}$$

That is, $T_1 = W_2 \sin\theta - 2m_2 a_1$, $2W_2 \sin\theta - 4m_2 a_1 - W_1 = m_1 a_1$, $2W_2 \sin\theta - W_1 = (m_1 + 4m_2)a_1$, and, finally,

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For $\theta = 30^\circ$, $m_2 = 2m_1$, $a_2 = 2g/9 \sim 2.2\mathrm{m/s}^2$.

d) Recall distance in terms of acceleration for the case of constant acceleration:

$$D = \frac{1}{2}a_2t^2 \Rightarrow t = \sqrt{\frac{2D}{a_2}},$$

Also, $v_2 = a_2t$, so that

$$v_2 = \sqrt{2a_2D} = \sqrt{2 * 2.2 * D} \sim 2.1\sqrt{D}, \qquad v_1 = v_2/2.$$

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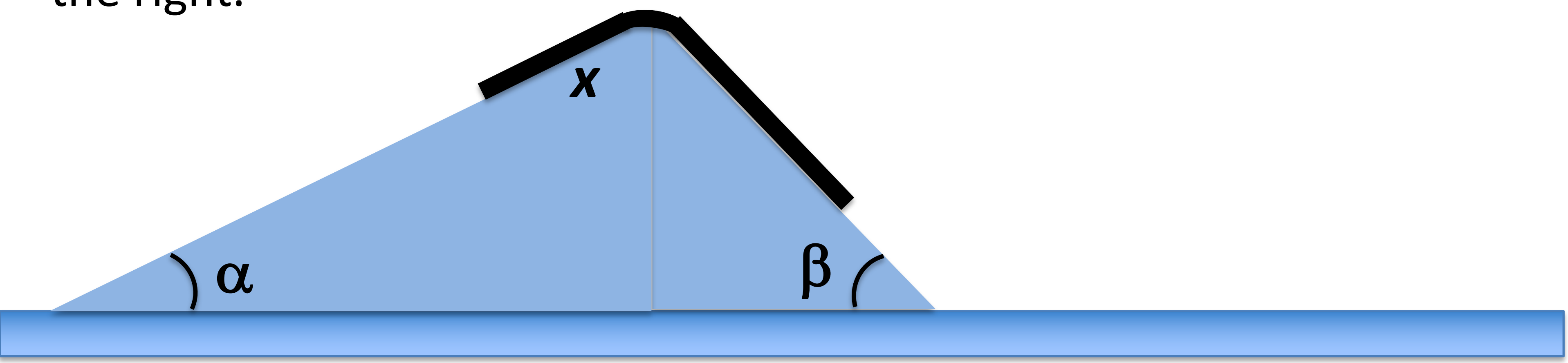
$$v_2 = \sqrt{2a_2 D} = \sqrt{2 * 2.2 * D} \sim 2.1\sqrt{D}, \qquad v_1 = v_2/2.$$

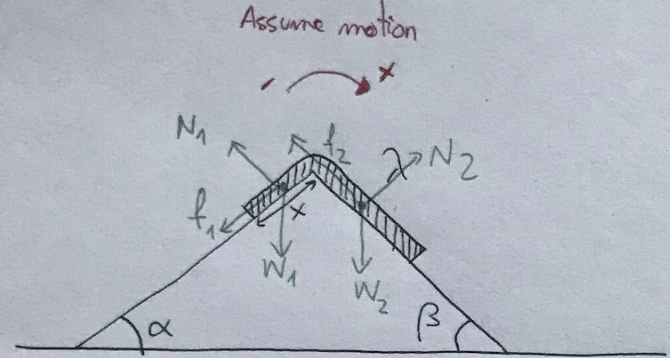
e) Not yet. Later on in the course, when we deal with angular momentum

Problem 8: Sliding chain

Consider a flexible chain of length L and linear density λ ($M=\lambda L$) lying on an incline as shown in the figure. There's friction all along the incline. The chain is initially at $x=x_0$ and at rest and is given a slight nudge that sets the chain in motion to the right.

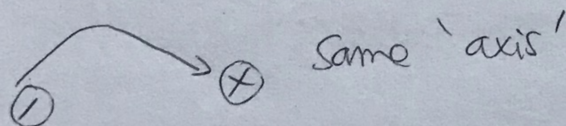


1) Find the acceleration of the chain in terms of x and any other relevant variable of the problem
2) Find the speed when $x=0$



The important thing to notice first is that the whole chain is in motion. The mass in motion is constant (the mass of the chain) so that we can safely use $\sum \vec{F} = m\vec{a}$.

We have an incline, but the chain is flexible (as a rope) and so we have to consider the forces along the chain. That is the 'axis' is the chain:



Forces along the chain: $W_2 \sin\beta - f_2 - f_1 - W_1 \sin\alpha = m_{chain} \cdot a_{chain}$.

Forces perpendicular to the chain: $N_2 - W_2 \cos\beta = 0$ & $N_1 - W_1 \cos\alpha = 0$;

It is moving, so that the friction force is kinematic: μ_K

$$f_1 = \mu_K \cdot N_1 = \mu_K W_1 \cos\alpha \quad \& \quad f_2 = \mu_K N_2 = \mu_K W_2 \cos\beta.$$

Going back to the equation along the chain:

$$W_2 \sin\beta - \mu_K W_2 \cos\beta - \mu_K W_1 \cos\alpha - W_1 \sin\alpha = m_{chain}\, a_{chain}.$$

Now:

$$W_2 = \lambda (L-x) g \; ; \quad W_1 = \lambda x g \, , \quad m_{chain} = \lambda \cdot L$$

$$\left[(L-x)\sin\beta - \mu_K (L-x)\cos\beta - \mu_K x \cos\alpha - x \sin\alpha \right] g = L\, a_{chain},$$

where λ has been canceled out.

We can re-arrange the terms as:

$$a_{chain} = -\underbrace{\left(\frac{x}{L}\right)}_{\text{adimensional}} \underbrace{\left[(\sin\alpha + \sin\beta) + \mu_K (\cos\alpha - \cos\beta) \right]}_{\text{adimensional}} g + \underbrace{(\sin\beta - \mu_K \cos\beta)}_{\text{adimensional}} g$$

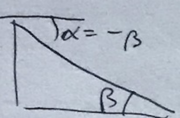
UNITS are OK. Also, if $x = 0 \Rightarrow a_{chain} = (\sin\beta - \mu_K \cos\beta) g$, which is the solution for a block sliding down an incline.

Next: Notice that when x decreases, the chain moves to the right, which is our ⊕ sign. So:

$$a_{chain} = \frac{d^2}{dt^2}(L-x) = -\frac{d^2x}{dt^2} \Rightarrow$$

$$\boxed{\frac{dx^2}{dt^2} = \left(\frac{x}{L}\right)\left[(\sin\alpha + \sin\beta) + \mu_K(\cos\alpha - \cos\beta)\right]g - (\sin\beta - \mu_K\cos\beta)g}$$

Check of consistency: (not easy) if $\alpha = -\beta$, it is as 1 single incline:



and, indeed, $\frac{d^2x}{dt^2} = -(\sin\beta - \mu_K\cos\beta)g$.

So we are good.

Let's go back to the main equation for x, and let's write it in a more useful way:

$$\frac{d^2x}{dt^2} = Ax + B,$$ with A & B constants.

How do we solve this equation??

It's a differential equation. We cannot do:

$$\frac{d^2x}{Ax+B} = dt^2$$ and same $\int\int$. NOPE!

Here's an important change of variables in physics:

$$\boxed{\frac{d}{dt} = \frac{dx}{dt}\cdot\frac{d}{dx} = v\frac{d}{dx}}$$

So, for instance:

$$\boxed{a = \frac{dv}{dt} = v\frac{dv}{dx}}.$$

Now we can write

$$\frac{d^2x}{dt^2} = a = v\frac{dv}{dx} = Ax+B \Rightarrow \boxed{v\,dv = (Ax+B)dx}$$

$$\Rightarrow \frac{v^2}{2} = \frac{A}{2}x^2 + Bx + C'$$ and since $v=0$ when $x=x_0 \Rightarrow$

$G = -\frac{A}{2}x_0^2 - Bx_0$ and:

$$v^2 = A(x^2 - x_0^2) + 2B(x - x_0) = (x - x_0)[A(x + x_0) + 2B]$$

or $v^2 = (x - x_0)(Ax + 2B + Ax_0)$

$$\boxed{v = \sqrt{(x - x_0)(Ax + 2B + Ax_0)}}$$

1) UNITS: $[v] = \frac{L}{T}$ and $\sqrt{L\cdot([A]\cdot L + [B])} = \sqrt{L\cdot\frac{L}{T^2}} = \sqrt{\frac{L^2}{T^2}} = L/T$ ok.

$\uparrow$
$[A] = \frac{1}{L}\cdot\frac{L}{T^2} = \frac{1}{T^2}$; $[B] = L/T^2$

2) $x = x_0$, $v = 0$, as we expect.

Finally: when $x = 0 \Rightarrow v = \sqrt{(-x_0)(2B + Ax_0)}$

It may seem we have an issue $\sqrt{-|number|}$, but recall $B = -(\sin\beta - \mu_K\cos\beta)g$; so, as far as the chain is indeed moving to the right, the term $2B + Ax_0$ is < 0 and $\sqrt{+}$.

ADVANCED CHALLENGE

1) Recover from this solution, the corresponding solution to QP20.

Answer: set $L=2\ell$, $x_0=\ell$, $\alpha=\beta=\frac{\pi}{2}$ and $\mu_K=0$. Then:

$B=-g$, $A=2g\cdot\frac{1}{2\ell}=g/\ell$ and $2B+Ax_0=-2g+\frac{g}{\ell}\cdot\ell=-g$.

Therefore: $v=\sqrt{(-\ell)(-g)}=\sqrt{g\ell}$, which is the correct answer.

2) How would you obtain $x=x(t)$, $v=v(t)$ and $a=a(t)$?

Answer: From $v=\sqrt{(x-x_0)(2B+Ax_0+Ax)}$, $\frac{dx}{dt}=\sqrt{\ldots}$ $\Rightarrow$

$dt=\frac{dx}{\sqrt{\ldots}}$ $\Rightarrow$ $t=\int\frac{dx}{\sqrt{\ldots}}$ and initial condition

is that for $t=0$, $x=x_0$. The $\int$ is complicated (involves cosh).

Once $x(t)$ is obtained, then $v(t)=\frac{dx(t)}{dt}$ and $a=\frac{dv(t)}{dt}$; also

$a=Ax+B$, so $a(t)=Ax(t)+B$, once $x(t)$ is known.

PH1a: Forces

Second Law: *The change of motion of an object is proportional to the force impressed; and is made in the direction of the straight line in which the force is impressed.*

Write it down!

PH1a: Forces

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NOPE!

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PH1a: Forces

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If m = constant, then $\boldsymbol{F} = m\mathbf{a}$

PH1a: Forces

$$\boldsymbol{F} = m\mathbf{a}$$

Who wrote this equation?

Probably the most used equation in physics!

PH1a: Forces

?

PH1a: Forces

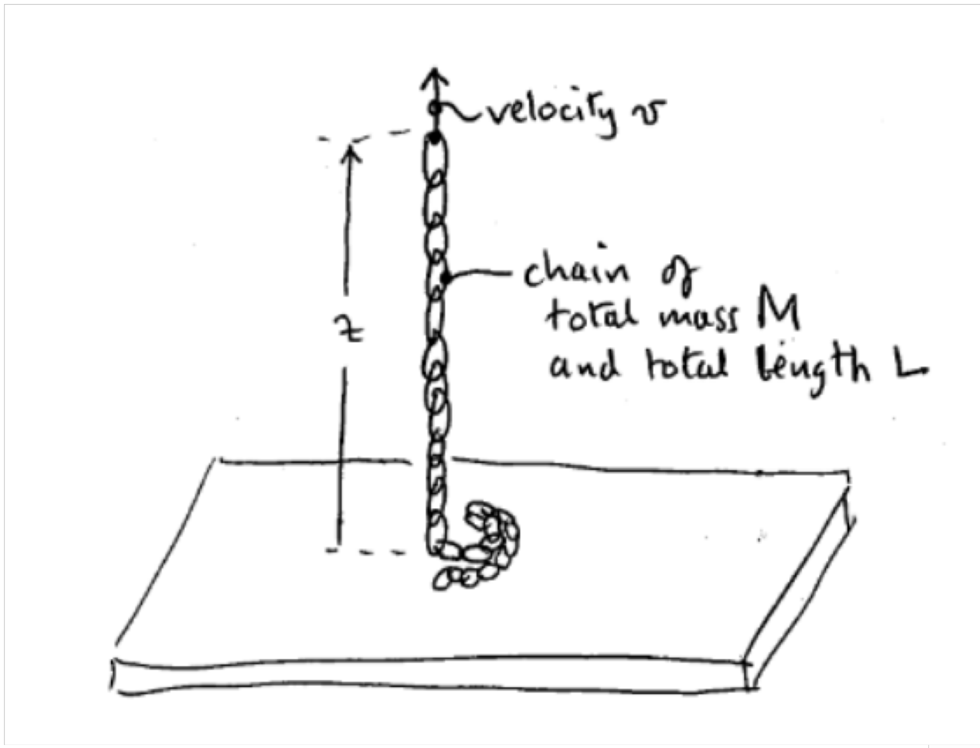
Leonhard Euler

1707, Basel, Switzerland
1783, Saint Petersburg, Russia

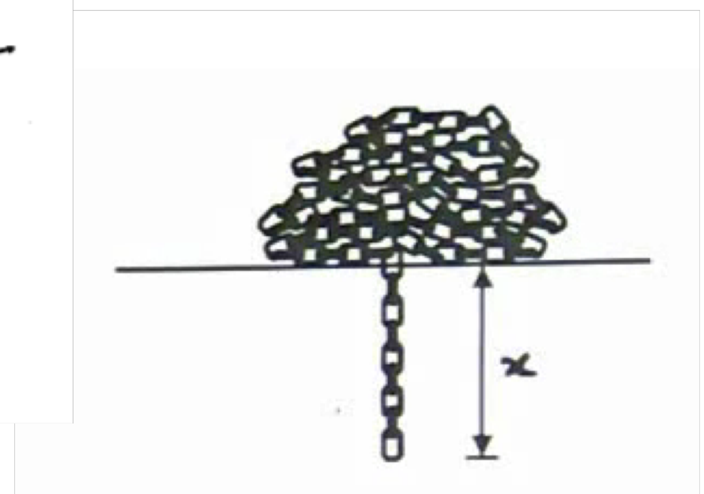
65 years after Newton published it!

Problems 9 & 10: *F* is not just "*ma*"

Two different examples



Lifting up a chain

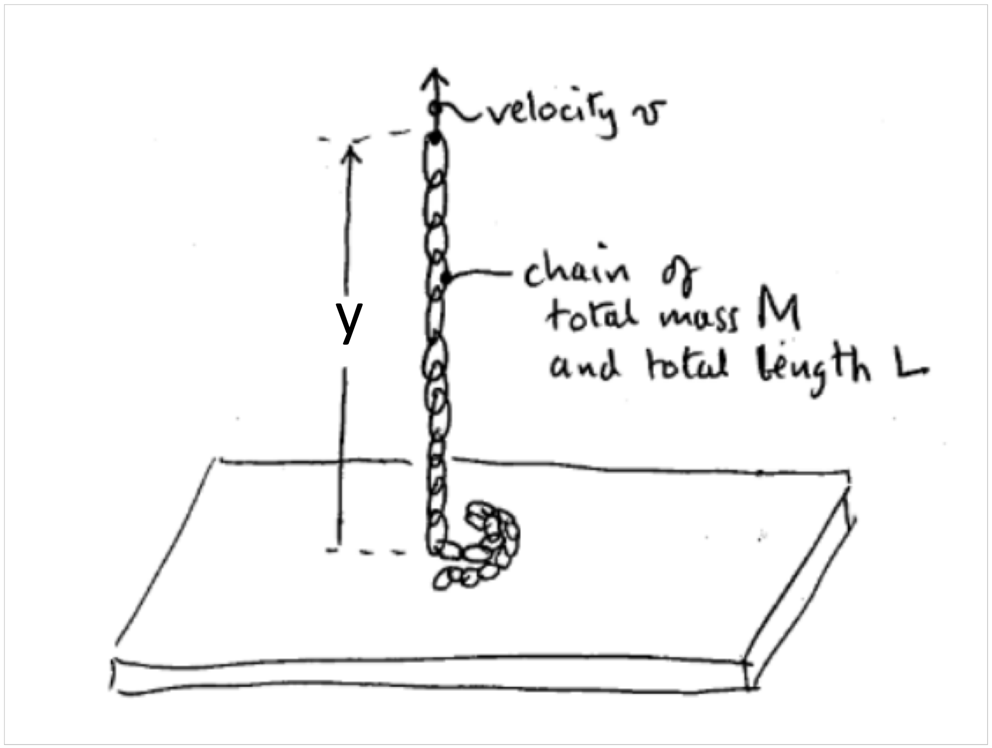


Chain falling through a hole

PH1a: Problem 9

2. Chain lifted vertically

The end of a chain, of mass per unit length η, is at rest on a table top at $t = 0$, is lifted vertically at a constant speed v. Evaluate the upward lifting force as a function of time.



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It is clear that as the chain is lifted, more mass is on the air, and we have to pull with a greater force.

There are two forces acting on the piece of chain off the table, the gravitational force and the force that lifts the chain, say F. The basic equation reads then:

$$F - m(t)g = \frac{dp}{dt} = v\,\frac{dm(t)}{dt}, \tag{10}$$

where we have taken into account that the speed is constant ($dv/dt = 0$) and that the mass being pulled is increasing in time.

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We can still work out the solution further, because the mass off the table is proportional to the height of the chain. That is $m(t) = \eta y$, where y is the height from the table. Moreover, since the chain is lifted with constant speed we also have that the height is a linear function of time, $y = vt$. Therefore:

$$F = m(t)g + v\,\frac{dm(t)}{dt} = \eta v t g + v \eta v = \eta v\,(v + gt). \tag{11}$$

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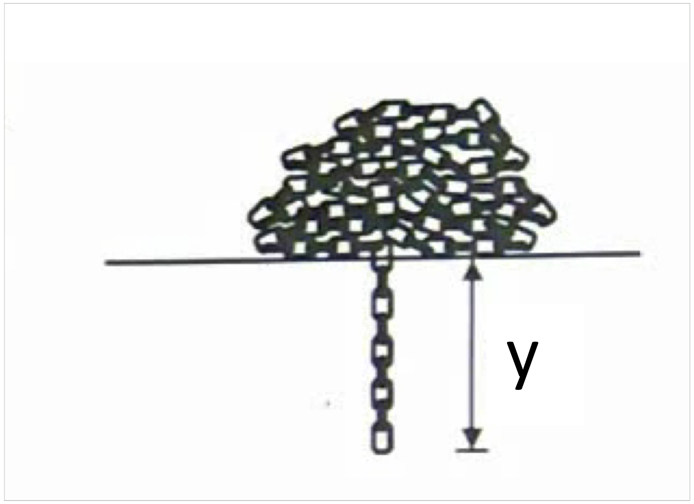
At some point, all the chain is on the air. Then, $dm/dt = 0$, and the total linear momentum does not change anymore. The pulling force is equal to the weight of the chain.

PH1a: Problem 10

A chain of length L, and mass per unit length η, is at rest on a table. The table has a hole in the middle. One end of the chain is pulled a little way through the hole and then released. Friction is negligible and, as a result, the chain runs smoothly through the hole with increasing speed. How long does it take for both ends of the chain to reach the floor?



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This problem reminds us again that the change of linear momentum is equal to the sum of the external forces. Contrary to the case of the lifted chain, there is no condition on the speed of the chain, so it may have an acceleration. In fact, if we imagine the chain falling, we know it speeds up as more of the chain has run through the hole.

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Forces?

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The only external force on the hole chain is the gravitational force of the part that has run through the hole (the remainder on the table is in equilibrium with some normal force from the table surface).

The basic equation is:

(12) $$mg = \frac{dp}{dt} = ma + v\frac{dm}{dt}.$$

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The change of the mass in time is related to the height of the chain pulled down by the gravitational force. We can write: $m(t) = \eta y(t)$. We know there will be some acceleration, so we do not impose any equation on $y(t)$ yet. But, obviously, $dy(t)/dt = v(t)$. The basic equation becomes then:

$$\eta g y(t) = \eta y(t) a + \eta v^2(t), \quad \Rightarrow a = g - \frac{v^2}{y}. \tag{13}$$

PH1a: Problem 10

The curious fact of this problem is that $a(t) =$ constant works. To see this, let's write $a(t) = a$, $v(t) = at$ and $y(t) = \frac{1}{2}at^2$ and substitute back in the previous equation:

$$a = g - 2a, \Rightarrow a = g/3. \tag{14}$$

Quite a nice result.

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To find out the time it takes the chain to touch the ground we should know that while there is some chain on the table, $a = g/3$, but when all of it has gone through the hole, the acceleration is g. Knowing the height of the table, H, one can work it out for different situations $H > L$ or $H \leq L$. The time for the whole chain to be out of the table is simply $t = \sqrt{2L/a} = \sqrt{6l/g}$. Then, one could add any more time if $H > L$, as $t_2 = \sqrt{2(H-L)/g}$.

PH1a: Problem 11

DR. NO'S ANTIGRAVITY MACHINE

SERGI R HILDEBRANDT

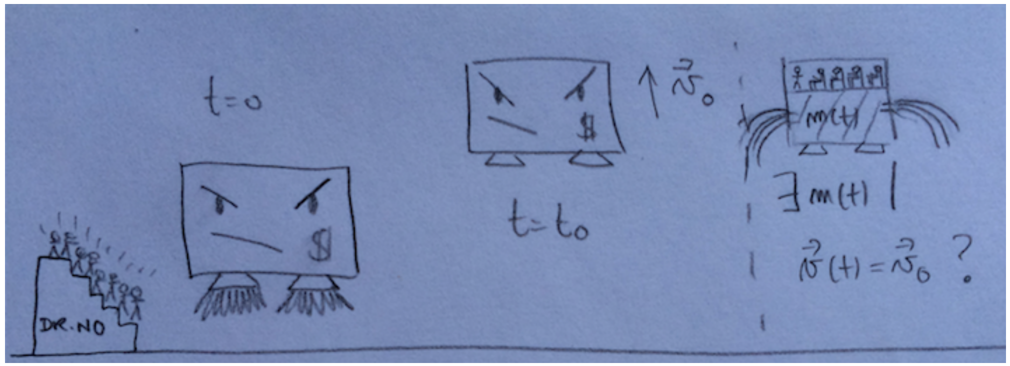

Figure 1

Dr. No wants to get big money from some CEO and Army Generals by selling the idea that he has discovered the secret for antigrativity. He plans to launch his Dr.No-Box1 and once it has attained a certain speeed, v_0, he'll turn off the engines as he pushes the button of what he claims is the antigravity engine. He will show to his audience that the spacecraft will continue to move upwards with constant velocity, despite the gravitational attraction of the Earth. After some demonstration time, he'll turn off the antigravity engine and come back to the ground, landing with some parachutes. Please, answer the following questions:

- a) How is it possible?
- b) The picture on the right hand side, shows what his fooling plan looks like: he will eject mass from the original spacecraft, after he pushes the antigravity button and during the demonstration time. The mass and speed with which the mass is ejected on both sides of the spacecraft is the same, so there is no horizontal force or motion. Find the relationship between v_0, g (the Earth's

gravitational acceleration), the initial mass of the whole system, M_0 and the time of the antigravity demonstration, Δt.

- c) How much mass should be ejected if $v_0 = 10$ m/s, $\Delta t = 1$ min? Does it make sense in practice? (Hint: assume that the minimum mass at the end has to be the mass of the enclosure of the spacecraft, plus engines, people, etc ... say about 10,000 kg).
- d) Dr. No's engines cannot speed up Dr.No-Box1 faster than the speed of sound (he also wants to keep his tests 'silent'). What would be the minimum initial weight of the spacecraft be if the antigravity demonstration is a minute long?

1. Solutions

- a) Yes. Force is equal to $d(m\vec{v})/dt$, so that velocity may be constant if the mass varies. The spacecraft can only lose mass. Therefore, $dm/dt < 0$ and that implies that the direction of the velocity has to be the contrary of that of the force. The trick may work when moving upwards, but not when descending.

2. Solutions

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- b) $\vec{F} = \vec{v}_0 dm/dt$, with $\vec{v}_0$ the velocity when Dr. No stops the engines of his spacecraft. On the other hand, $\vec{W} = m\vec{g}$ ($\vec{g}$ is pointing downwards). Consequently, i) $\vec{v}$ and $\vec{g}$ must be parallel. One can leave the sign of the direction free and check that $+$ would imply increase of mass, and $-$ decrease. Let's choose, as explained in a) the negative sign and re-arrange $\vec{W} = \vec{v}_0 dm/dt$: $-g/v_0 dt = dm/m$, which is a simple differential equation.

  Integrating, with the initial condition of $m(t = t_0) = m_0$:

$$m(t) = m_0 e^{-g/v_0 \Delta t}, \quad \vec{v}_0 = -(v_0/g)\vec{g} = v_0\hat{k}, \tag{1}$$

where Δt is the time of the demonstration ($\Delta = t_f - t_0$, if one wishes). It makes sense dimensionally, and $\vec{W} = d(m\vec{v}_0)/dt$ is satisfied.

3. Solutions

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- c) The final mass will be: $m_F = m_0 e^{-58.8} = 3\,10^{-26} m_0$!! No way in practice of having m_0. Recall $m_F = 10,000$ kg. However:
- d) The speed of sound is approximately 342 m/s at sea level and at 20 °C. Now, $m_F = m_0 e^{-1.72} = 0.18 m_0$. This sets the minimum mass of the spacecraft. Less will not provide enough mass to be ejected. If the final mass is 10,000 kg, then the minimum initial mass of Dr.No-Box1 has to be: $m_0 = 10,000/0.18 = 55,806$ kg. Seems okay! Let's do it.

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