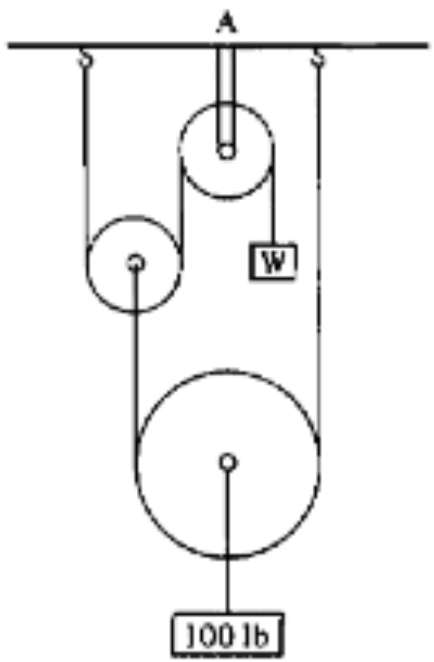
PH1a: Statics (Equilibrium)

$$\boxed{\sum \mathbf{F} = \mathbf{0}}$$

These notes cover **7** exercises of equilibrium in **full** detail

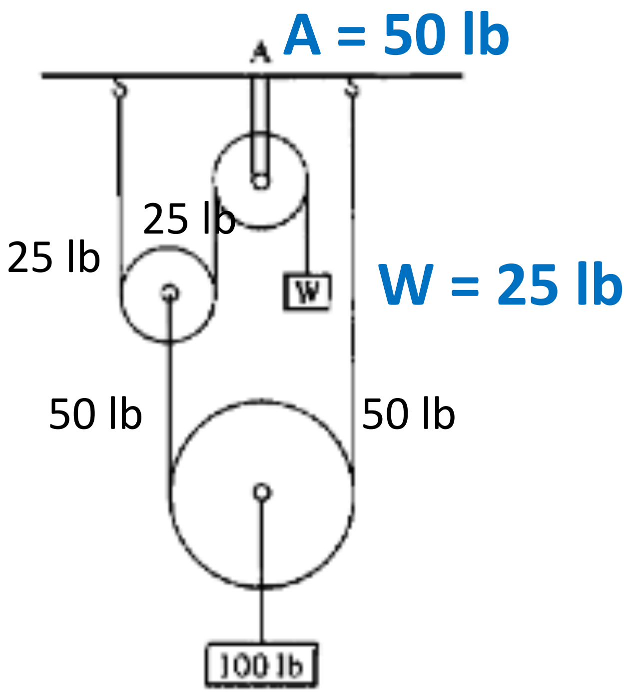
Problem 1. Let's practice

19. The pulleys in the picture below are frictionless and weightless. Find the weight W and the tension in each rope such that the system is in equilibrium, and find the downward pull of the support at A on the ceiling.

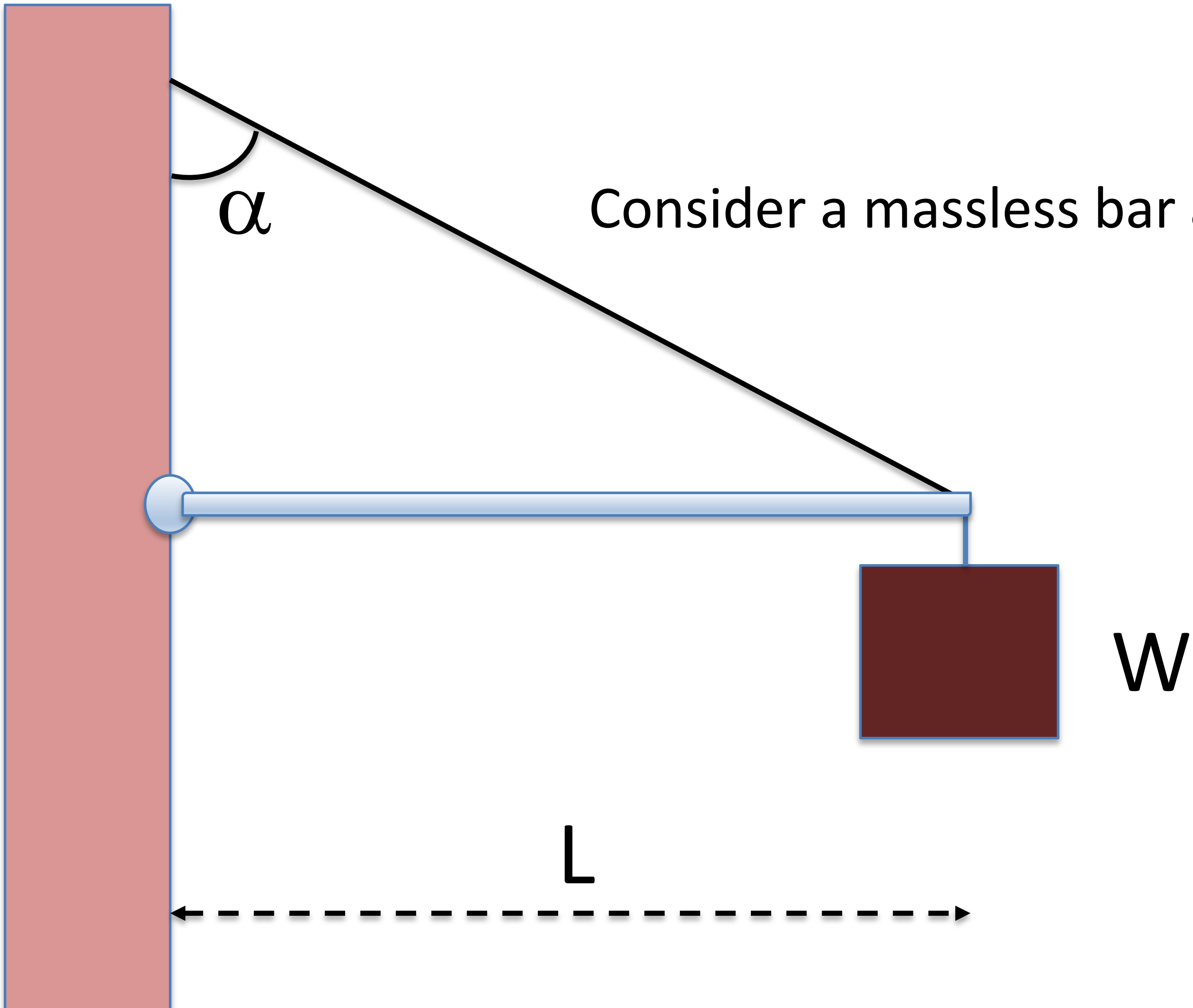


Problem 1. Solution

19. The pulleys in the picture below are frictionless and weightless. Find the weight W and the tension in each rope such that the system is in equilibrium, and find the downward pull of the support at A on the ceiling.



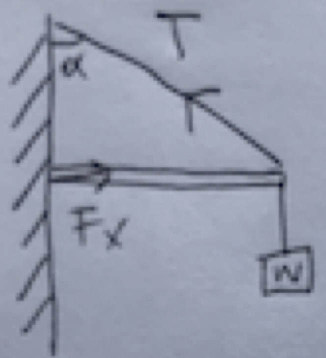
Problem 2



Consider a massless bar and string

Given **W**, **L,** and α find *all* the forces acting upon the bar

Problem 2. Solution



$$\sum \vec{F} = \vec{0} \Rightarrow T\cos\alpha = W \Rightarrow \boxed{T = \frac{W}{\cos\alpha}}$$

$$T\sin\alpha = F_x \Rightarrow \boxed{F_x = W\tan\alpha}$$

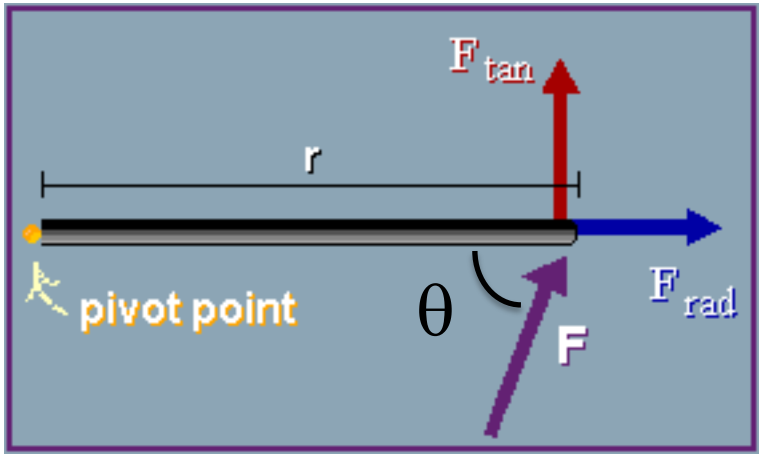
PS: for α close to 90^o it won't be possible

PH1a: Statics (Equilibrium with torques)

$$\boxed{\begin{aligned} \sum \mathbf{F} &= \mathbf{0}, \\ \sum \boldsymbol{\tau} &= \mathbf{0}. \end{aligned}}$$

Sergi Hildebrandt (srh@caltech.edu. Cahill 364)

PH1a: torque

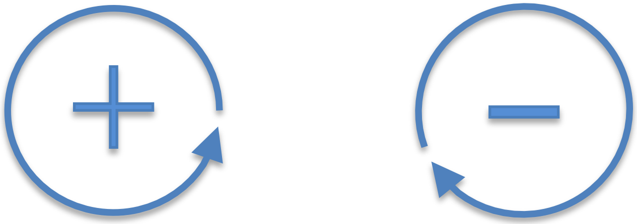


link

Magnitude: $\tau = r\,F\sin\theta = r\,F_{tan}$

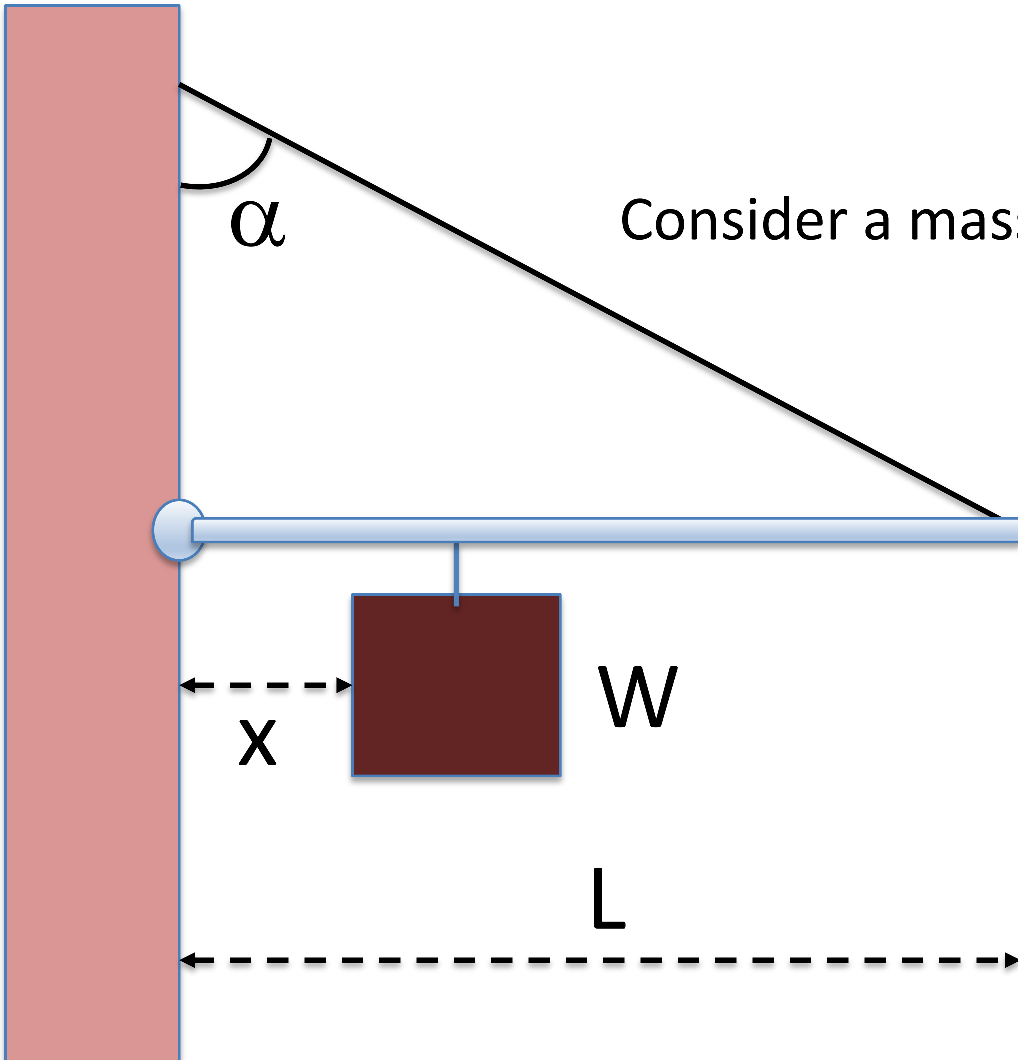
Choose the correct θ: maximum torque $\theta=\pi/2$

Choose the correct **sign**:



(or the opposite, but be consistent)

Problem 3: problem 2 with torques

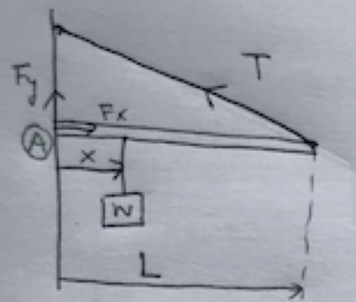


Consider a massless bar and string

1) Given **W**, **L,** α and **x** find *all* the forces acting upon the bar

2) Set x=L and compare with the previous case

Problem 3: solution



1/4

Notice the addition of F_y.

What happens if F_y is absent?

Then $\sum \vec{F} = \vec{0}$, produces the same solution as before. Namely:

$T = \frac{W}{\cos\alpha}$ & $F_x = W\tan\alpha$. So $\sum\vec{F} = \vec{0}$ is satisfied.

What's the issue then?

Torques! Let's choose A as the point (pivot) from which torques are computed:

$\sum \vec{\tau}_A \Rightarrow$ Using magnitude and signs: ⊕ ⊖

$\tau_{F_x} = 0$; $\tau_W = -W\cdot x$, $\tau_T = T\cdot L\cdot\cos\alpha$

$\Rightarrow \sum \tau_A = -Wx + TL\cos\alpha$, but $T = \frac{W}{\cos\alpha} \Rightarrow \sum \tau_A = W(L-x) \neq 0$.

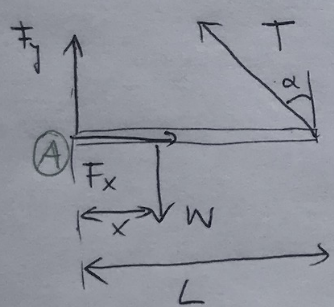
Indeed $\sum \tau_A > 0$, for any $x \leq L$, so without F_y, the $\sum\vec{F} = \vec{0}$

Problem 3: solution

Implies $\sum \vec{\tau} \neq \vec{0}$ and the system would rotate about its center of mass. This is twice problematic. On one side if it rotates with some torque, it will not be in equilibrium. On the other side, this system can't rotate about its CM if the bar is held at the wall. We must add F_y to the problem. Indeed, F_y should have been added in the simpler case seen before, and deduce that in that case (only) $F_y = 0$.

Let's continue:

$$\sum \vec{F} = \vec{0} \Rightarrow \begin{cases} T \sin\alpha = F_x & \text{(same as before)} \\ \cos\alpha\, T + F_y = W & \text{(new equation)} \end{cases}$$



$$\sum \vec{\tau}_A = \vec{0} \Rightarrow \tau_{F_x} = 0;\ \tau_{F_y} = 0;\ \tau_W = -xW;$$

$$\tau_T = TL\cos\alpha \Rightarrow$$

impose equilibrium.

$$\sum \tau_A = -xW + TL\cos\alpha = 0 \Rightarrow \boxed{T = \left(\frac{x}{L}\right)\frac{W}{\cos\alpha}}$$

Problem 3: solution

First, notice that for $x = L$ (previous problem), $T = \frac{W}{\cos\alpha}$, which is $\frac{3}{4}$ the same solution as before. Good. Furthermore, units are OK:

$$T = \underbrace{\left(\frac{x}{L}\right)}_{\text{adimensional}} \frac{W}{\cos\alpha}$$

↑ adimensional.

Next: $\boxed{F_x = T \sin\alpha = \frac{x}{L} W \tan\alpha}$ Similar comments as the ones for T hold here: $x = L$ & units give good results.

Finally: $\boxed{F_y = W - T\cos\alpha = W - \frac{x}{L} \cdot \frac{W}{\cancel{\cos\alpha}} \cdot \cancel{\cos\alpha} = W\left(1 - \frac{x}{L}\right)}$

So that, $F_y(x=L) = 0$, as we considered in the previous problem.

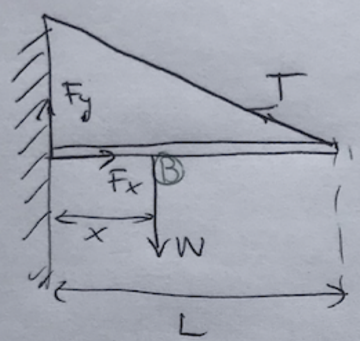
$\boxed{\text{IT IS SAFER TO ADD } F_x, F_y, \ldots \text{ AND SOLVE FOR THEM}}$

Sometimes intuition may not help.

Problem 3: solution

What if we change the pivot point for the calculation of the torques? <u>NOTHING</u> (up to you) $\frac{4}{4}$

Let's choose B, where the weigth is hung:



$\sum \vec{F} = \vec{0}$ does <u>not</u> change:

$$\begin{cases} F_y + T\cos\alpha = W \\ F_x = T\sin\alpha \end{cases}$$

$\sum \tau_B$: $\tau_{F_y} = -x F_y$; $\tau_{F_x} = 0$ (angle is π) & $\sin\pi = 0$; $\tau_W = 0$; $\tau_T = (L-x) \cdot T\cos\alpha$.

$+$ $-$

$$\sum \tau_B = \boxed{-xF_y + (L-x)T\cos\alpha = 0}$$

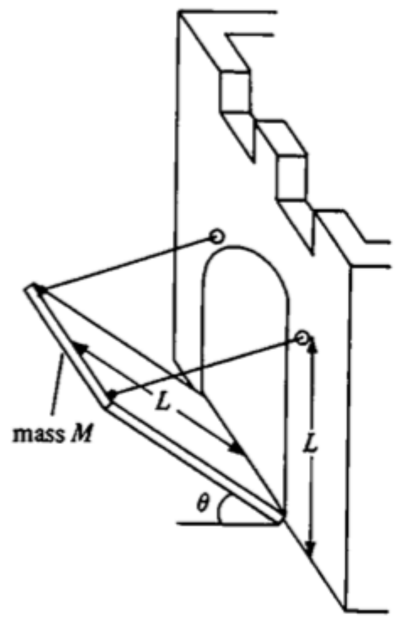
$$F_y = \frac{(L-x)}{x} T\cos\alpha \rightarrow \frac{(L-x)}{x} T\cos\alpha + T\cos\alpha = W$$

$$\Rightarrow \frac{L}{x} T\cos\alpha = W \Rightarrow \boxed{T = \left(\frac{x}{L}\right)\frac{W}{\cos\alpha}} \; ; \; \boxed{F_x = \frac{x}{L} W\tan\alpha}$$

and $\boxed{F_y = \frac{(L-x)}{x} T\cos\alpha = \frac{(L-x)}{\cancel{x}} \cdot \frac{\cancel{x}}{L} W = \left(1-\frac{x}{L}\right)W}$ <u>SAME SOLUTION</u>.

$\neq$ (More work...)

Problem 4: quiz level

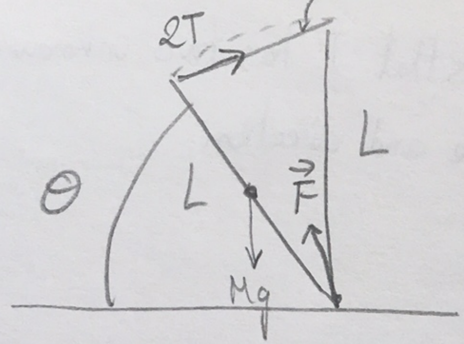


Mechanical Universe
Chapter 6. Example 7.

The gate is opened with constant speed.
Find the Tension on the chains and the Force by the hinge at the pivot

Discuss the solution for the limit cases: $\theta=0$ and $\theta=\pi/2$

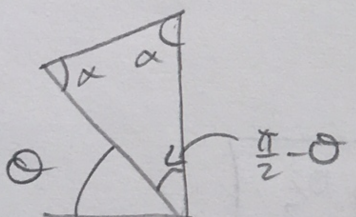
Example 7, chapter 6



2T, because there are <u>two</u> chains.
$\vec{F}$ is the reaction force along the hinge.

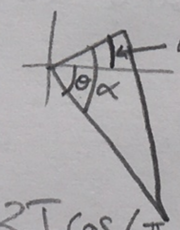
① Notice that $\vec{F}$ has <u>two</u> unknowns: magnitude and direction or (F_x, F_y).

② Also notice that the gate/chain/wall is an isosceles triangle:



$$\Rightarrow 2\alpha + \frac{\pi}{2} - \theta = \pi \Rightarrow \boxed{\alpha = \frac{\pi}{4} + \frac{\theta}{2}}$$

③ <u>Newton's law to each axis:</u> $\uparrow^{+} \ \rightarrow^{+}$



$\underline{\underline{\alpha - \theta}} = \frac{\pi}{4} - \theta/2$!

<u>x</u>: $2T\cos(\alpha-\theta) - F_x = 0 \Rightarrow F_x = 2T\cos(\frac{\pi}{4} - \theta/2)$

<u>y</u>: $2T\sin(\alpha-\theta) + F_y - Mg = 0 \Rightarrow F_y = Mg - 2T\sin(\frac{\pi}{4} - \frac{\theta}{2})$

<u>Now</u>: $\cos(\frac{\pi}{4} - \theta/2) = \cos\frac{\pi}{4}\cos\theta/2 + \sin\frac{\pi}{4}\sin\theta/2 = \frac{\sqrt{2}}{2}(\cos\frac{\theta}{2} + \sin\frac{\theta}{2})$

$\sin(\frac{\pi}{4} - \theta/2) = \sin\frac{\pi}{4}\cos\theta/2 - \sin\theta/2\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}(\cos\theta/2 - \sin\theta/2)$ $\longrightarrow$

$$\boxed{\begin{aligned} F_x &= \sqrt{2}T\left(\cos\theta/_2 + \sin\theta/_2\right) \\ F_y &= Mg - \sqrt{2}T\left(\cos\theta/_2 - \sin\theta/_2\right) \end{aligned}}$$

$\underline{\sum \vec{\tau} = \vec{0}}$: Let's choose the rotation point at the pivot

(+) (−) $\quad Mg\frac{L}{2}\cos\theta - 2TL\sin\alpha = 0 \Rightarrow$

$$Mg\frac{\cancel{L}}{2}\cos\theta = 2TL\sin\left(\tfrac{\pi}{4}+\theta/_2\right) = \sqrt{2}T\cancel{L}\left(\sin\theta/_2 + \cos\theta/_2\right)$$

$$T = \frac{Mg}{2\sqrt{2}}\,\frac{\cos\theta}{\sin\frac{\theta}{2}+\cos\theta/_2}\,;$$

Further: $\cos\theta = \cos\left(2\frac{\theta}{2}\right) = \cos^2\theta/_2 - \sin^2\theta/_2 = \left(\cos\theta/_2 - \sin\theta/_2\right)\cdot\left(\cos\theta/_2 + \sin\theta/_2\right)$

$$= \boxed{\frac{Mg}{2\sqrt{2}}\left(\cos\theta/_2 - \sin\theta/_2\right)}$$

$\underline{\text{Check:}}$ For $\theta = \frac{\pi}{2} \rightarrow T = 0$. Yes! (Door closed) and Max for $\underline{\theta = 0}$ Yes.

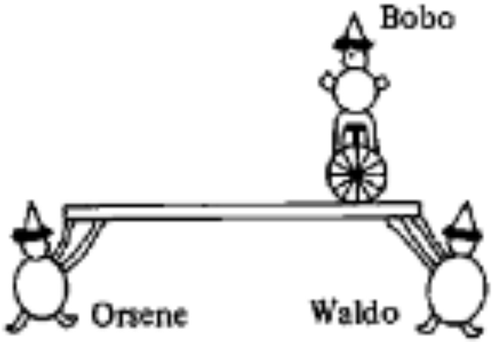
$\underline{\text{Now:}}$ $\boxed{F_x = \sqrt{2}T\left(\cos\theta/_2 + \sin\theta/_2\right) = \frac{Mg}{2}\cos\theta}$ Nice result

$$F_y = Mg - \sqrt{2}T\left(\cos\theta/_2 - \sin\theta/_2\right) = \ldots = \boxed{\frac{Mg}{2}\left(1+\sin\theta\right)}$$

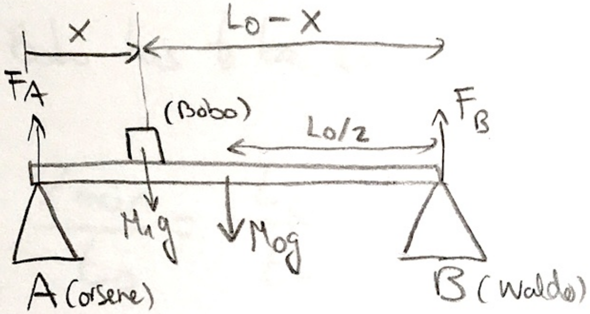
$(c-s)^2 = c^2+s^2-2cs = 1-2cs, \ldots$

Problem 5: difficult. Quiz level

25. Two clowns, Orsene and Waldo, support a 3-m-long, 10-kg plank while a third, Bobo, rides a unicycle back and forth between the two ends at a steady speed. Bobo and the unicycle together come to 55 kg. If Orsene can't hold masses over 40 kg for more than 5 s, how fast should Bobo ride?



Mechanical Universe
Chapter 6.



$\circlearrowleft +$ $\quad \circlearrowright -$

$\sum \vec{\tau} = \vec{0}$

Let's choose (B) to compute $\vec{\tau}$:

$$-F_A L_0 + M_0 g \frac{L_0}{2} + M_1 g (L_0 - x) = 0$$

$$\Rightarrow F_A = \underbrace{\frac{M_0 g}{2}}_{} + M_1 g \frac{(L_0 - x)}{L_0}$$

Logical, because the weight of the Plank is evenly shared between both clowns.

Now: $F_A \leq (F_A)_{max} \ (= 40\,\mathrm{Kg} \cdot g) \Rightarrow$

$$\frac{M_0 g}{2} + M_1 g \left(\frac{L_0 - x}{L_0}\right) \leq (F_A)_{max} \Rightarrow 1 - \frac{x}{L_0} \leq \frac{\left((F_A)_{max} - M_0 g/2\right)}{M_1 g}$$

$$\Rightarrow \boxed{x \geq L_0 \left[1 + \frac{M_0}{2M_1} - \frac{(F_A)_{max}}{M_1 g}\right]}$$

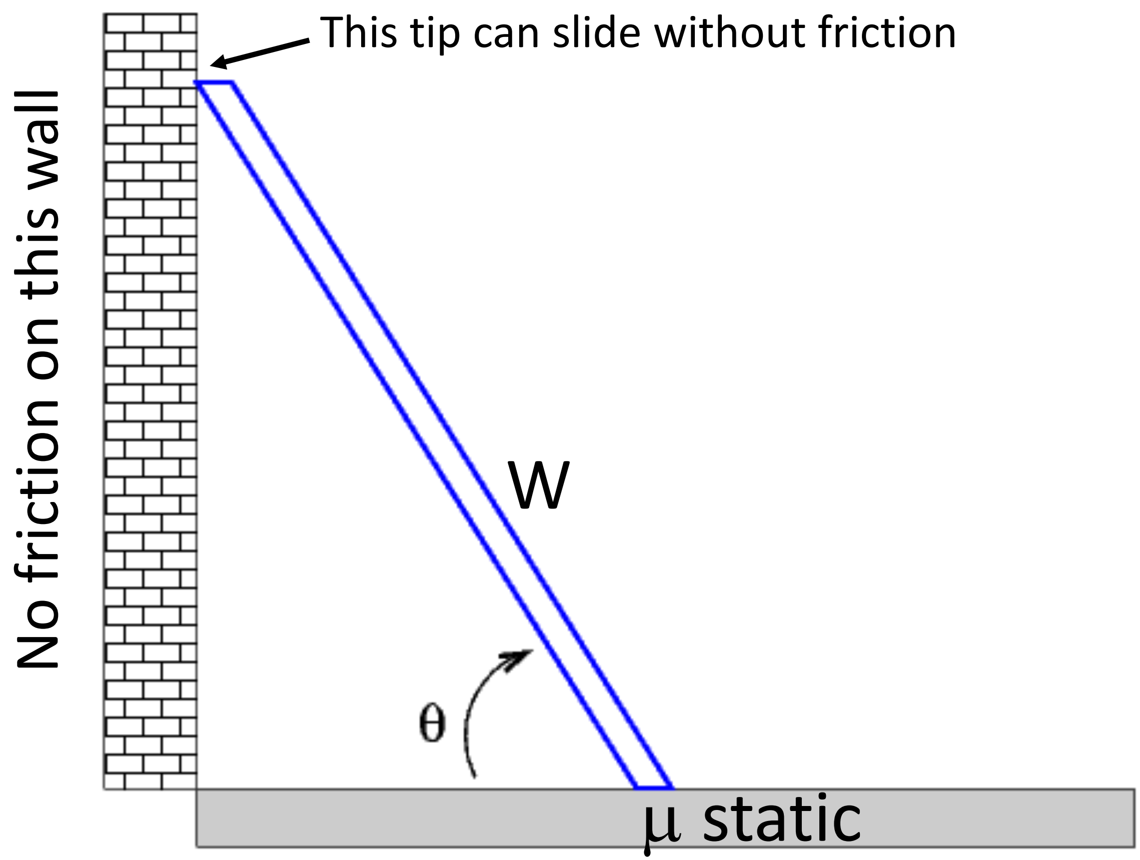
Minimum $x \Rightarrow$ (=)

The speed of Bobo has to be enough to cover $2x_{min}$ (back and forth) in 5 seconds (t_0)

$$v_{Bobo} = \frac{2x_{min}}{t_0} = \frac{2L_0}{t_0}\left[1 + \frac{M_0}{2M_1} - \frac{(F_1)_{max}}{M_1 g}\right]$$

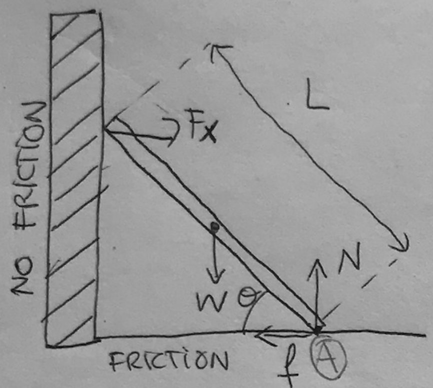
$$= \boxed{24/55 \ m/s}$$

Problem 6: more difficult

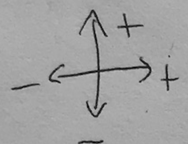


1) Given θ, μ, W find all the forces that keep the ladder in equilibrium
2) What is the limiting angle θ for a given W and μ βefore it slides?

PS: the problem changes if the vertical wall has friction



Notice that there is NO F_y on the wall because there is no friction (otherwise, we should add F_y)



$\sum \vec{F} = \vec{0}$
$$\begin{cases} N = W \\ F_x = f \le \mu_s N = \mu_s W \end{cases}$$
(Less or equal if it is static friction)

$\sum \vec{\tau} = \vec{0}$ Let's choose A: we have $\tau_f = 0$ and $\tau_N = 0$ then. Shorter.

(+) (−)

$$\tau_{F_x} = -L\, F_x \sin\theta + W \cdot \frac{L}{2} \cos\theta = 0 \Rightarrow \boxed{F_x = \frac{W}{2\tan\theta}}$$

ALWAYS THINK ABOUT sin/cos θ

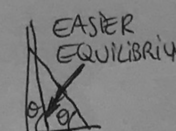
($\theta = 0 \Rightarrow$ No torque
$\theta = \frac{\pi}{2} \Rightarrow$ Max. torque in this problem)

$F_x = f \le \mu_s W$

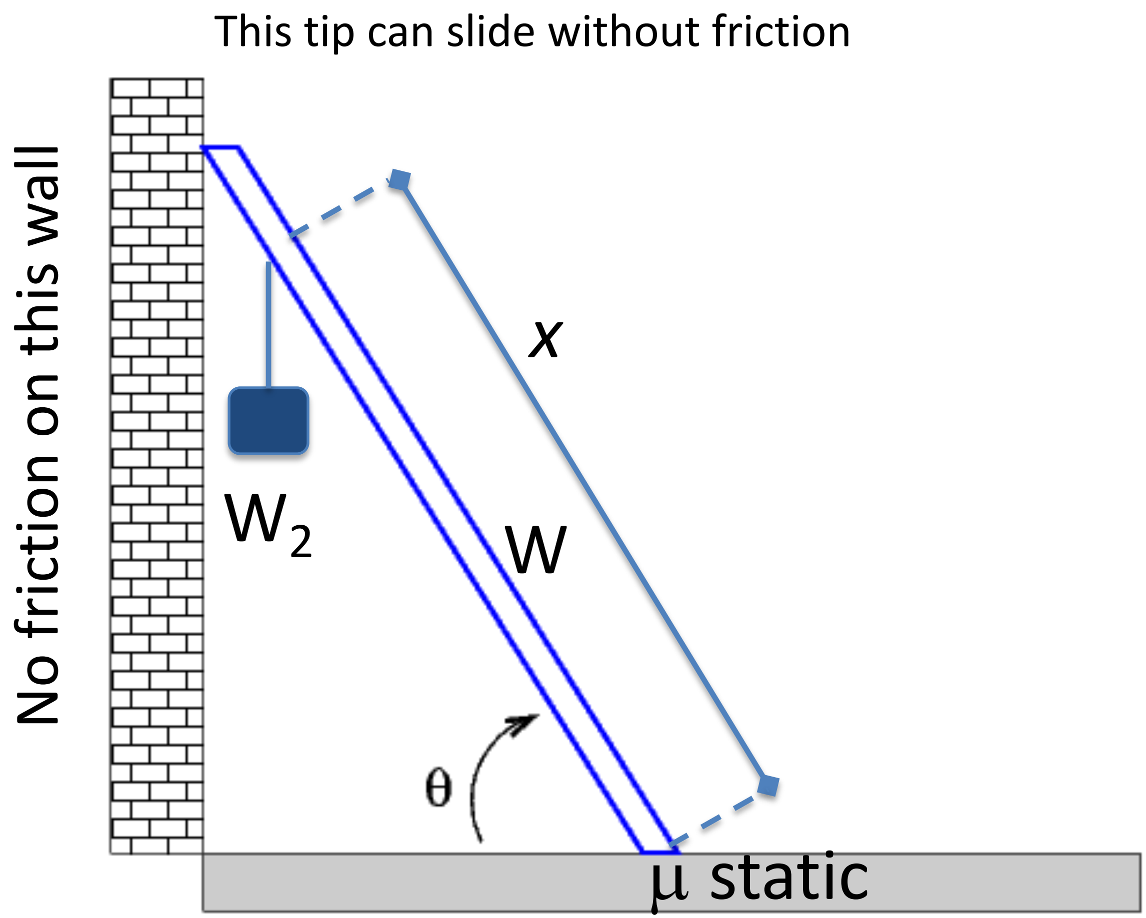
$$\Rightarrow \frac{W}{2\tan\theta} \le \mu_s W \Rightarrow \boxed{\tan\theta \ge \frac{1}{2\mu_s}}$$

Adimensional (OKAY!)

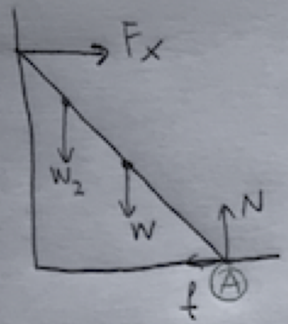
Makes sense. For a given limit: $\tan\theta_{limit} = \frac{1}{2\mu_s}$ & any $\theta > \theta_{limit}$ keeps the ladder in equilibrium:



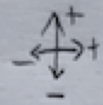
Final Challenge!



1) Given θ, μ, L, W and W_2 find *x* that keeps the ladder in equilibrium
2) Study the limits $W_2 \ll W$ and $W_2 \gg W$
3) What happens when *x*=L/2?



$\sum \vec{F} = \vec{0}$ $\begin{cases} N = W + W_2 \\ f \le \mu_s N \\ F_x = F \end{cases}$



TORQUES: From A $\quad \sum \tau_A = + W_2 x \underbrace{\cos\theta}_{\text{Always pay attention.}} + W\frac{L}{2}\cos\theta - F_x L \sin\theta = 0$

$$\Rightarrow \boxed{F_x = \left(W_2 \frac{x}{L} + \frac{W}{2}\right) \cdot \frac{1}{\tan\theta}} \qquad F_x \le \mu_s N = \mu_s (W + W_2)$$

Maximum static friction: $\mu_s (W + W_2) \Rightarrow$

$$\boxed{x = \Big(\underbrace{\frac{F_x \tan\theta}{W_2}}_{\text{adimensional}} - \underbrace{\frac{W}{2W_2}}_{\text{adimensional}}\Big) \underset{\uparrow \text{ length}}{L} \le \left[\mu_s\left(1 + \frac{W}{W_2}\right)\tan\theta - \frac{W}{2W_2}\right] L}$$

UNITS: **OK.**

$$\boxed{x_{max} = \left[\mu_s \tan\theta + \left(\mu_s \tan\theta - \frac{1}{2}\right)\frac{W}{W_2}\right] L} \quad \ldots \rightarrow$$

If $x > x_{max}$, the ladder will SLIDE.

Limits:

1) If $W_2 << W$, we expect W_2 to not add any force nor torque.

$$\lim_{\frac{W}{W_2}\to\infty} x_{max} = \left(\mu_s \tan\theta - \frac{1}{2}\right) \cdot L \cdot \infty .$$ If the ladder is in equilibrium

with W, $\mu_s \tan\theta - \frac{1}{2} > 0 \Rightarrow \lim_{\frac{W}{W_2}\to\infty} x_{max} \to +\infty$. It can be anywhere,

no issues.

2) If $W_2 >> W$, we expect the weight of the ladder to be irrelevant.

$$\lim_{\frac{W}{W_2}\to 0} x_{max} = \mu_s \tan\theta \cdot L$$

3) If $x = L/2$, we expect that W_2 plays the same role as W in the previous problem:

$$x_{max} = \frac{L}{2} = L\left[\mu_s \tan\theta_{max} + \left(\mu_s \tan\theta_{max} - \frac{1}{2}\right)\frac{W}{W_2}\right] \Rightarrow$$

$$\frac{1}{2} = \mu_s \tan\theta_{max} + \left(\mu_s \tan\theta_{max} - \frac{1}{2}\right)\frac{W}{W_2}$$

$$\Rightarrow \quad W_2 = 2W_2\mu_s \tan\theta_{max} + 2\mu_s \tan\theta_{max} \cdot W - W$$

$$\Rightarrow \quad W + W_2 = 2(W+W_2)\mu_s \tan\theta_{max} \Rightarrow$$

$$1 = 2\mu_s \tan\theta_{max} \Rightarrow \boxed{\tan\theta_{max} = \frac{1}{2\mu_s}}$$

As in the case of only W. GOOD.