PH1a: kinematics

- **Constant** acceleration: g
- **Time dependent** acceleration: a(t)
- **Parabolic motion**: constant speed and g in 2 dimensions

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Kinematics: summary of Main Formulae

Constant acceleration

$$y = v_{y0}t - \frac{1}{2}gt^2.$$

$$h = \text{maximum } y = \frac{v_{y0}^2}{2g}. \qquad R = \frac{2v_{x0}v_{y0}}{g}. \qquad t = \frac{2v_{y0}}{g}.$$

Law of Inertia (Galileo)

A body will remain at rest or continue to move with a constant speed in a straight line unless acted upon by an outside force.

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FROM CHAPTER 6, P13 (reworded):

A zoo keeper wants to feed her favorite pet. The monkey is seating on a tree. The monkey has learned that if the the banana is thrown aiming at her and she lets herself fall from the branch at the same time, she'll always catch it, regardless of the initial speed of the banana, as far as it is enough speed to get to the vertical of where she is. Prove that's true.

Disregard air friction and treat the banana as a point like object.

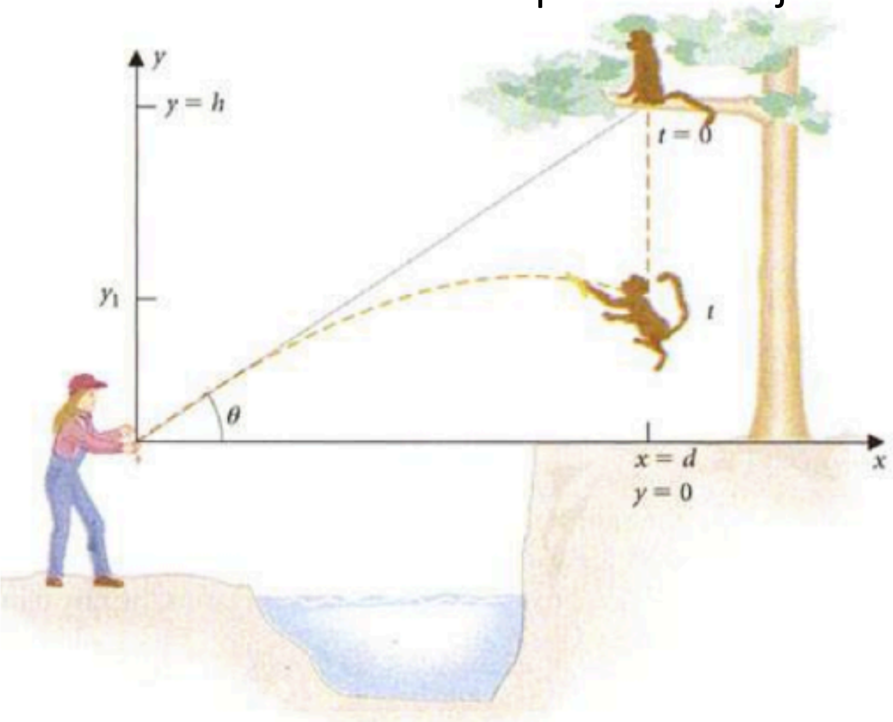
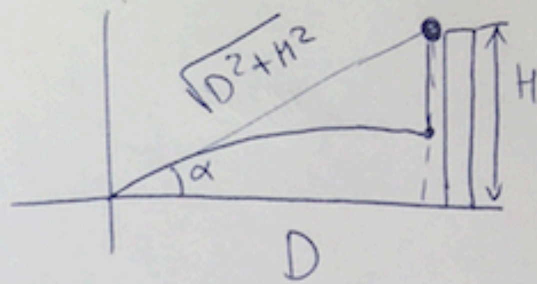


Image source: Wired

Monkey free fall:



$$\cos\alpha = \frac{D}{\sqrt{D^2+H^2}}$$

$$\sin\alpha = \frac{H}{\sqrt{D^2+H^2}}$$

$$y_M = H - \frac{1}{2} g t^2$$

$$y_D = v_{0y} t - \frac{1}{2} g t^2$$

Common (time is the same)

$\Rightarrow$ Is $H = v_{0y} t$? when the dart reaches $x = D$?

$$x_D = v_{0x} t \Rightarrow D = v_0 \cos\alpha \, t \Rightarrow t = \frac{D}{v_0 \cos\alpha} = \frac{D\sqrt{D^2+H^2}}{v_0 D}$$

$$\Rightarrow t = \frac{\sqrt{D^2+H^2}}{v_0} \rightarrow v_{0y} \cdot \frac{\sqrt{D^2+H^2}}{v_0} = \frac{H}{\sqrt{D^2+H^2}} \cdot \sqrt{D^2+H^2} = H$$ -> yes

$v_{0y} = v_0 \sin\alpha$

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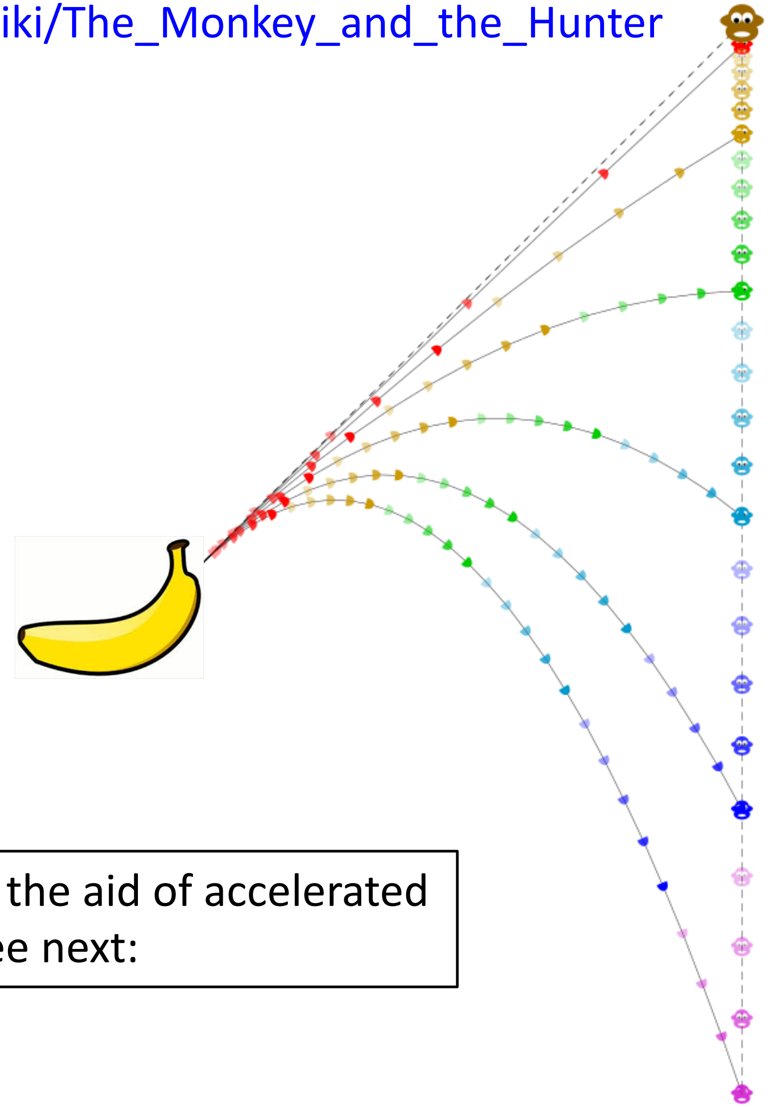
See also https://en.wikipedia.org/wiki/The_Monkey_and_the_Hunter

From Wikipedia:



PS: this problem can be solved with the aid of accelerated reference frames (non-inertial) … See next:

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(B) Reference frames:

From the monkey's viewpoint, there's no gravity: Earth accelerates towards him.
The dart is, in fact, also in free fall ⇒ Relative motion is just constant velocity. The Monkey waits for the dart to hit him. She sees the dart approaching her in a straight line.

$v_0 \cdot t_c = D \Rightarrow t_c$ = same time for the collision.
Same answer.

See also problem 4.9 from the Mechanical Universe for a good example of reference frames

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Time dependent acceleration

An object at rest starts to accelerate with $a(t)=a_0e^{-\lambda t}$, with λ a positive constant and *t* is the time:

- Find the speed after some time *t*
- Find the distance reached by the object after some time *t*

The object's acceleration is $a(t) = a_0 e^{-\lambda t}$, with λ a constant. The units of λ must be "time^{-1}" (s^{-1} or T^{-1}) so that (λt) is dimensionless.

Now, by definition (I use "$\tilde{t}$" inside the integral because "t" is used in the limit)

$$v(t) - v_0 = \int_0^t a(\tilde{t})d\tilde{t} = \frac{a_0}{(-\lambda)} e^{-\lambda \tilde{t}}\Big|_0^t = \frac{a_0}{(-\lambda)}(e^{-\lambda t} - 1) = \frac{a_0}{\lambda}(1 - e^{-\lambda t})$$

$v_0 = 0$ and $\boxed{v(t) = \frac{a_0}{\lambda}(1 - e^{-\lambda t})}$ UNITS CHECK: $\left(\frac{a_0}{\lambda}\right) = \frac{L/T^2}{T^{-1}} = \frac{LT}{T^2} = \boxed{\frac{L}{T}}$ GOOD

Notice that for $(\lambda t) \gg 1$ ("long times"), $v(t) \to \frac{a_0}{\lambda}$ = constant.

Next, by definition:

$$x(t) - x_0 = \int_0^t v(\tilde{t})d\tilde{t} = \frac{a_0}{\lambda}\int_0^t (1 - e^{-\lambda \tilde{t}})d\tilde{t} = \frac{a_0}{\lambda}\left[\tilde{t} + \frac{e^{-\lambda \tilde{t}}}{\lambda}\right]_0^t.$$

Therefore ($x_0 = 0$)

$$\boxed{x(t) = \left(\frac{a_0}{\lambda}\right)\left[t + \frac{1}{\lambda}(e^{-\lambda t} - 1)\right] = \left(\frac{a_0}{\lambda}\right)t + \frac{a_0}{\lambda^2}(e^{-\lambda t} - 1)}$$

UNITS CHECK: $\left(\frac{a_0}{\lambda}\right)t = \frac{L/T^2}{T^{-1}} T = \frac{LT^2}{T^2} = L$; GOOD. And $\left(\frac{a_0}{\lambda^2}\right) = \frac{L/T^2}{T^{-2}} = \frac{LT^2}{T^2} = L$, GOOD.

Notice that for $(\lambda t) \gg 1$, $x(t) \to \left(\frac{a_0}{\lambda}\right)t$. That is "constant speed x time". This is consistent with our findings before with $v(t)$.

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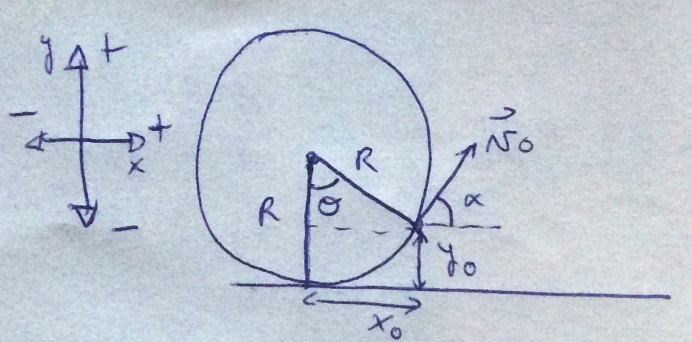
14. A rock is swung on a string in a circle. When the string is θ° from the vertical, it is cut. What trajectory does the rock follow? If the rock is initially moving at $\mathbf{v_0}$

m/s, how far does it fly horizontally before falling back to the same height it had when the string was cut?

How far away from *O* is the rock when it reaches the floor?



$$\begin{cases} y_0 = R(1-\cos\theta) \\ x_0 = R\sin\theta \\ \alpha = \theta \end{cases}$$

$$\Rightarrow \begin{cases} y = R(1-\cos\theta) + (v_0 \sin\theta)\Delta t - \frac{1}{2} g \Delta t^2 \quad (1) \\ x = R\sin\theta + (v_0 \cos\theta)\Delta t \end{cases}$$

$$y = 0 \Rightarrow (1) \times -2 : \quad g\Delta t^2 - (2 v_0 \sin\theta)\Delta t - 2R(1-\cos\theta) = 0$$

$$\Delta t = \frac{2 v_0 \sin\theta \pm \sqrt{4 v_0^2 \sin^2\theta + 8 g R (1-\cos\theta)}}{2g}$$

$$\Delta t = \frac{v_0 \sin\theta}{g} \pm \sqrt{\frac{v_0^2 \sin^2\theta}{g^2} + \frac{2R}{g}(1-\cos\theta)}$$

$(\theta \neq 0)$

$$\Delta t = \frac{v_0 \sin\theta}{g}\left[1 \oplus \sqrt{1 + \frac{2Rg}{v_0^2}\frac{(1-\cos\theta)}{\sin^2\theta}}\right]$$

UNITS: $\left[\frac{v_0}{g}\right] = T$, ok. & $\left[\frac{Rg}{v_0^2}\right] = \frac{L \cdot L/T^2}{L^2/T^2} = 1$, Adimensional, OK.

Recall: $x = R\sin\theta + v_0 \cos\theta\, \Delta t$

$$= R\sin\theta + \frac{v_0^2 \cos\theta \sin\theta}{g}\left(1 + \sqrt{1 + \frac{2Rg}{v_0^2}\frac{(1-\cos\theta)}{\sin^2\theta}}\right)$$

LIMITS: $R \to 0 \Rightarrow x \to \frac{2v_0^2 \sin\theta\cos\theta}{g} = \frac{v_0^2}{g}\sin 2\theta$, OKAY ($\theta$ free) (let θ free)

$v_0 \to 0 \Rightarrow x \to R\sin\theta$. OKAY (no motion)

$\theta \to \frac{\pi}{2} \Rightarrow x \to R\sin\theta$. OKAY (only vertical motion)

$\theta \to 0 \Rightarrow x \to R\cdot 0 = 0$. OKAY (starts / finishes at the same place)

NICER EXPRESSION FOR PLOTTING (OPTIONAL):

Define: $x_{MAX}^{STD} = \frac{v_0^2}{g}$, the range of a standard (STD) parabolic motion.

& $\lambda = \frac{2Rg}{v_0^2} = \frac{Rg}{(v_0^2/2)}$ = ratio $\frac{\text{Potential energy } (\theta = \frac{\pi}{2})}{\text{Kinetic energy (start)}}$

$$\Rightarrow x = x_{MAX}^{STD}\left[\frac{\lambda}{2} + \cos\theta\left(1 + \sqrt{1 + \frac{\lambda(1-\cos\theta)}{\sin^2\theta}}\right)\right]\sin\theta$$

OR:

$$\boxed{\frac{x}{x_{MAX}^{STD}} = \left[\frac{\lambda}{2} + \cos\theta\left(1 + \sqrt{1 + \frac{\lambda(1-\cos\theta)}{\sin^2\theta}}\right)\right]\sin\theta}$$

$\lambda > 0$

λ adimensional

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Extra

- Verify that for λ=0, one retrieves the well-known result of the range of a parabolic shot.

- Derive the result for the first correction in λ (λ<<1). Taylor expansion. Beware of θ.

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Let's **plot** it and study the angle that corresponds to the largest horizontal distance.

Use for instance: **DESMOS GRAPHING CALCULATOR** (free, flexible and nice plots)

https://www.desmos.com/calculator

CODE THAT I WILL SHARE BY EMAIL(MATLAB, MANY OTHER LANGUAGES ARE FINE)

```
function [ x y ] = parabolic_range_ph1a( opt )
% Solution for the range of a parabolic shot from a string with an arbitrary starting height
% run it as:
% opt.v_0_m_s = 10 ; opt.height_m = 5 ; parabolic_range_ph1a( opt ) ;
% For an all angle plot do:
% figure(1 ) ; clf ; hold all ; theta_deg = { '40', '45', '50', '55', '60' } ; for i = 1 : numel( theta_deg ), opt.v_0_m_s = 10 ; opt.height_m = 10 * ( 1 - cosd( str2num( theta_deg{ i } ) ) ) ; [ x y ] = parabolic_range_ph1a( op
t ) ; h_plot( i ) = plot( x, y, 'LineWidth', 2 ) ; end ; hold off ; grid ; grid minor ; xlabel('X (meters)', 'FontSize', 14 ) ; ylabel( 'Y (meters)', 'FontSize', 14 ) ; title( 'PARABOLIC MOTION WITH INITIAL HEIGHT' ) ; %legend(
 h_plot, theta_deg )


% General debugging mode
dbstop if error

% Preliminary
  if ~exist( 'opt', 'var' )
  opt = [ ] ;
  end

% Constants
% Gravitational acceleration close to Earth' surface
g = 9.8 ; % m/s^2

% Default values
% The length of the string
  if ~isfield( opt, 'string_length_m' )
  opt.string_length_m = 10 ;
  end
% The initial height
  if ~isfield( opt, 'height_m' )
  opt.height_m = 0 ;
  end

% Check of consistency
  if ( 2 * opt.height_m > opt.string_length_m )
  disp( sprintf( 'The starting height of the object (%2.2f m) is larger than twice the length of the string (%2.2f m). Impossible. Choose a smaller initial height or make the string longer: opt.height_m, opt.string_length_m', op
t.height_m, opt.string_length_m ) )
  return
  end

  if ( opt.height_m < 0 )
  disp( sprintf( 'The value of the height is negative: %2.2f m. It can not be underground. Choose a positive value.', opt.height_m ) )
  return
  end

% Initial speed of the object
  if ~isfield( opt, 'v_0_m_s' )
  opt.v_0_m_s = 0 ; % No horizontal motion
  end

% Avoiding redundancy
  if ( opt.v_0_m_s < 0 )
  opt.v_0_m_s = abs( opt.v_0_m_s ) ;
  disp( sprintf( 'Considering a positive definite initial speed of %2.2f m/s. The negative value is redundant, due to symmetry in the problem', opt.v_0_m_s ) )
  end


% Do some work
```

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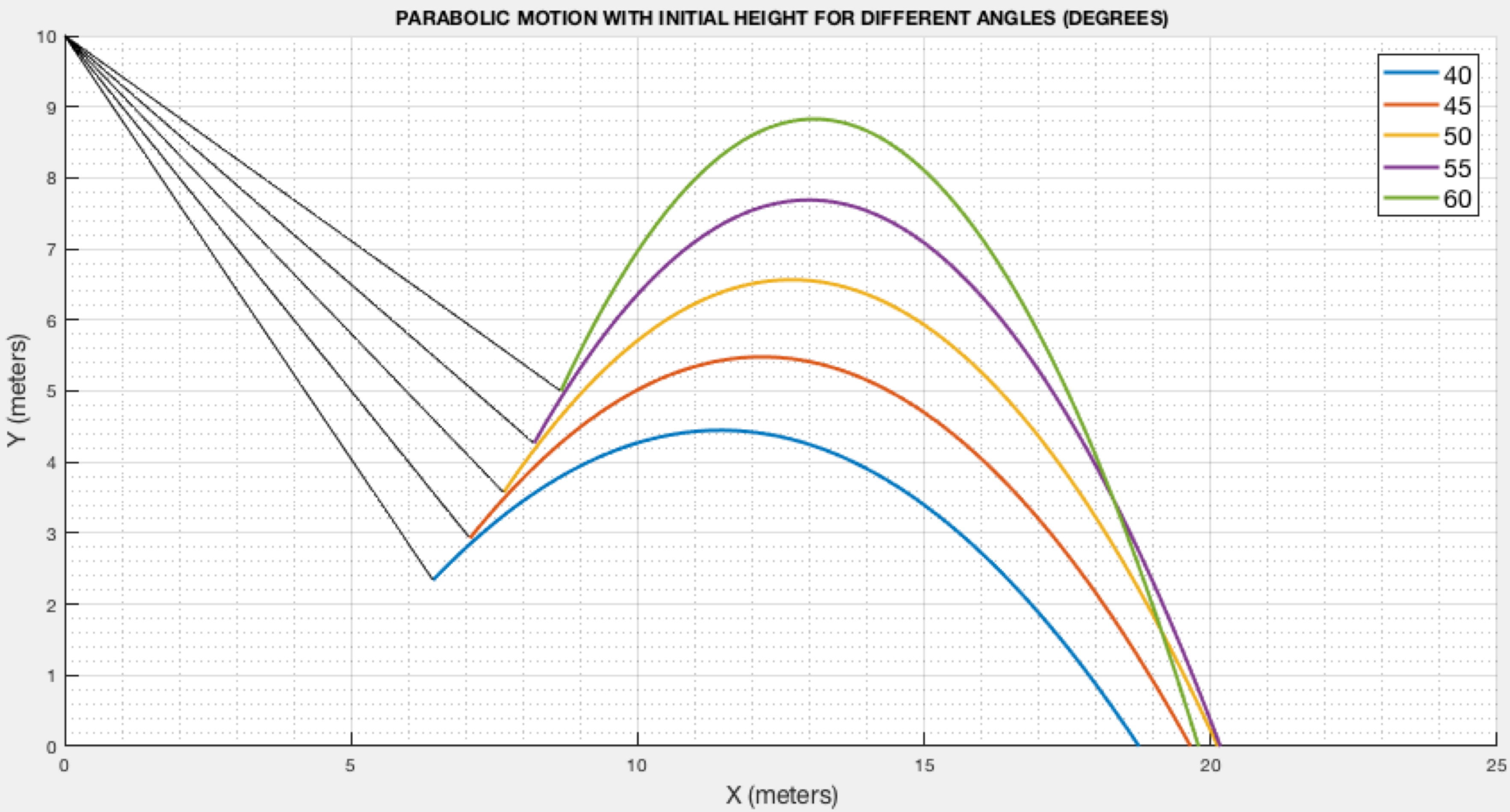

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We usually consider ***g***, the gravitational acceleration, **constant**.

INTERESTING QUESTION

Is it really constant? How much does it relatively change for a height ***h*** over Earth' surface (Earth's radius ***R***)

We'll deal with the full variation of ***g*** at the end of the term!

PH1a: free fall

Figure 1.1 Nicolaus Copernicus (1473–1573).

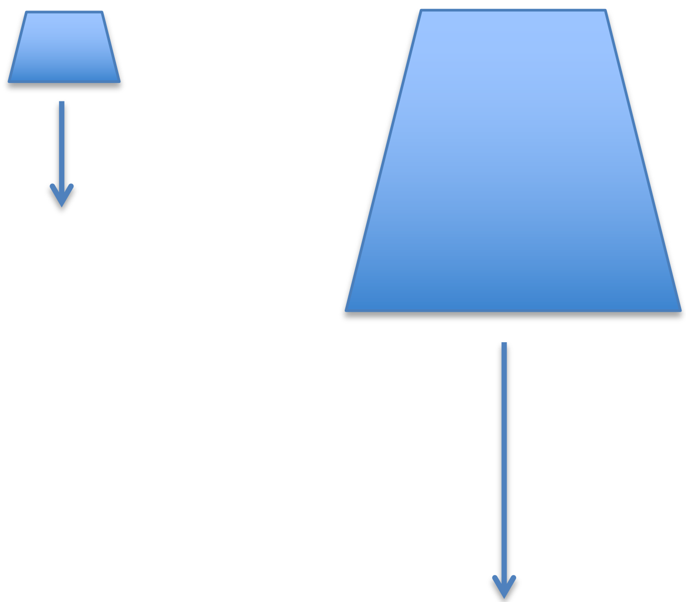
De revolutionibus orbium coelestium
(On the Revolutions of the Celestial Spheres)

Reference frames: *the first revolution!*

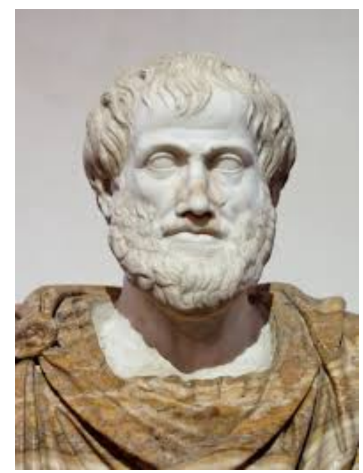
Heavier faster than lighter?

No!

Figure 1.4 Galileo Galilei (1564–1642).



Yes!



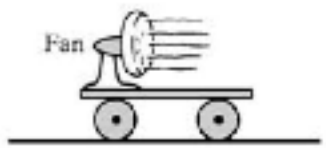
Aristotle: 384 BC-322 BC

Women in Physics

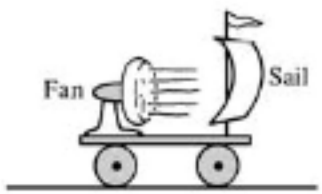
Hypatia of Alexandria (c.360 – 415 AD)
Influential philosopher, astronomer and mathematician. Unfortunately, she was killed for her political opinions.

Next time

6. A fan is mounted on a cart as shown below. If the fan is turned on, does the cart move? If so, in which direction?



Suppose a sail were added to the cart. What would be the motion of the cart if the fan were now turned on?



Watch also:

https://www.youtube.com/watch?v=KKLmAYiyd3M

https://www.youtube.com/watch?v=Zy7JQ0lwj1w