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DIMENSIONAL ANALYSIS: NATURAL UNITS

1. Dimensional analysis: why?

In Physics, we deal with formulas that explain a quantity in terms of other quantities. Each quantity has its own dimensions (aka units) and sometimes there are adimensional quantities of great interest. Examples of dimensional quantities are the temperature, force, etc ... And of adimensional quantities, to choose one, the Reynolds number that is present in fluid dynamics.

2. Natural units

Imagine a set of dimensional constants, like the speed of light, that play a basic role in several physical formulas. It is interesting to derive a 'unit' length, or a 'unit' mass, from them. It would not be surprising if they end up having a some fundamental role in understanding the most basic questions of physics.

For instance, let us bring the following set of three well-known constants:

- The speed of light in vacuum, c
- The gravitational constant, G
- The (reduced) Planck constant, $\hbar$ (which is just the Planck constant divided by 2π, $\hbar = h/2\pi$)

QUESTION 1)

Can we combine them in some way as to define a 'unit' length? Like a reference meter.

HINT: First write the dimensions or units of each constant. For instance, an acceleration has units of $\mathrm{m/s^2}$ or dimensions of L/T^2, where L refers to length and T to time.

I personally prefer the dimensions, because I do not need to refer specifically to the SI, but you may choose what is more convenient for you. They are equally useful in practice.

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SOLUTION:

There are two ways of solving it. Both require to recall the units of the constants (brackets denote dimensions, not caring about actual numerical values):

$$[c] = \frac{L}{T} \qquad [c] = \frac{\mathrm{m}}{\mathrm{s}} \tag{1}$$

$$[G] = \frac{L^3}{T^2 M} \qquad [G] = \frac{\mathrm{m}^3}{\mathrm{s}^2\mathrm{kg}} \tag{2}$$

$$[\hbar] = \frac{ML^2}{T} \qquad [\hbar] = \frac{\mathrm{kgm}^2}{\mathrm{s}} \tag{3}$$

One way, is to figure out which combination gives the same units as length. It is not easy and inefficient. For instance, the combination of the three previous constants that gives a length dimension is:

$$l = \sqrt{\frac{\hbar G}{c^3}} \tag{4}$$

You can check it works.

The second way is more general and can be used in other cases, or with other constants.

Write a dimensional equation! (I will only continue with the dimensions, rather than the units).

We want to solve this:

$$L = [\hbar]^x [G]^y [c]^z, \tag{5}$$

Or

$$L = M^x L^{2x} T^{-x} L^{3y} T^{-2y} M^{-y} L^z T^{-z} = M^{x-y} L^{2x+3y+z} T^{-x-2y-z}. \tag{6}$$

This implies:

$$\begin{cases} x - y = 0 \\ 2x + 3y + z = 1 \\ -x - 2y - z = 0 \end{cases} \tag{7}$$

The solution is:

$$x = 1/2, \qquad y = 1/2, \qquad z = -3/2 \tag{8}$$

So,

$$l = \sqrt{\frac{\hbar G}{c^3}} \tag{9}$$

This unit length is called 'Planck length'. Its value in SI is: 1.616199(97) $\times 10^{-35}$ meters << than the radius of the proton ($\sim 0.8 \times 10^{-15}$m)

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QUESTION 2)
Can we combine them in some way as to define a unit mass and a unit time?.

The answer is yes.

The general method can be re-used to set the consitions on x, y, z to have either a mass or a time:

(10) $$\text{MASS} = \begin{cases} x - y = 1 \\ 2x + 3y + z = 0 \\ -x - 2y - z = 0 \end{cases}$$

(11) $$\text{TIME} = \begin{cases} x - y = 0 \\ 2x + 3y + z = 0 \\ -x - 2y - z = 1 \end{cases}$$

Whose solutions are

(12) $$\text{MASS}: \quad x = 1/2, \quad y = -1/2, \quad z = 1/2$$

(13) $$\text{TIME}: \quad x = 1/2, \quad y = 1/2, \quad z = -5/2$$

(14)

In summary:

(15) $$l_P = \sqrt{\frac{\hbar G}{c^3}} \sim 1.616199(97) \times 10^{-35}\text{m}$$

(16) $$m_P = \sqrt{\frac{\hbar c}{G}} \sim 2.17651(13) \times 10^{-8}\text{kg}$$

(17) $$t_P = \sqrt{\frac{\hbar G}{c^5}} \sim 5.39106(32) \times 10^{-44}\text{s}$$

Notice that the Planck time could also be easily derived from the Planck length using the speed of light, c: $t_P = l_P/c$.

We think that these units set a limit in our understanding of space and time in the sense that at those small scales, short lapses of time an heavy masses (within that small space) the concepts of space and time do not hold anymore as we deal with them in the theory of (specia or general) relativity. No definite theory is found yet that consistently deals with those limits. More importantly, it is very difficult to eve test theory candidates experimentally (Inflation theory and its observation might?? be a possibility).

Notice that 2.17e-8 kg is equivalent to a rest mass of 1.21e+16 TeV/c^2 (~1e15 times the maximum energy on an Earth accelarator). The most energetic cosmic particle ever recorded is 3e+8 TeV/c2. Planck mass is 10,000,000 times more.

QUESTION 2)
Can we combine them in some way as to define a unit charge?.
NO

One needs to consider another fundamental constant. In this case, it is Coulomb constant, $K = 8.98 \times 10^9 \mathrm{Nm}^2/rmC^2$.

The result (you can repeat a similar exercise as before) is:

$$q_P = \sqrt{\frac{\hbar c}{K}} \sim 1.875545956(41) \times 10^{-18} C \tag{18}$$

Notice that now, the Planck charge is very close to the charges of the electron or proton: 1e=1.60217662×10^{-19} C. Just about 10 times larger.

Interesting to think about!

By the way, for the temperature, one needs the Boltzmann constant. We do not need to go through it, but just let me point out that is $T_P \sim 1.416833(85) \times 10^{32}$ K, which is an incredibly hight temperature, probably only reached during the first instant of the universe (Inflation). So, the charge remains the only case close to our daily experience.

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Max Planck himself thought that this would be the way to let potential alien civilizations know the characteristics of our world.

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