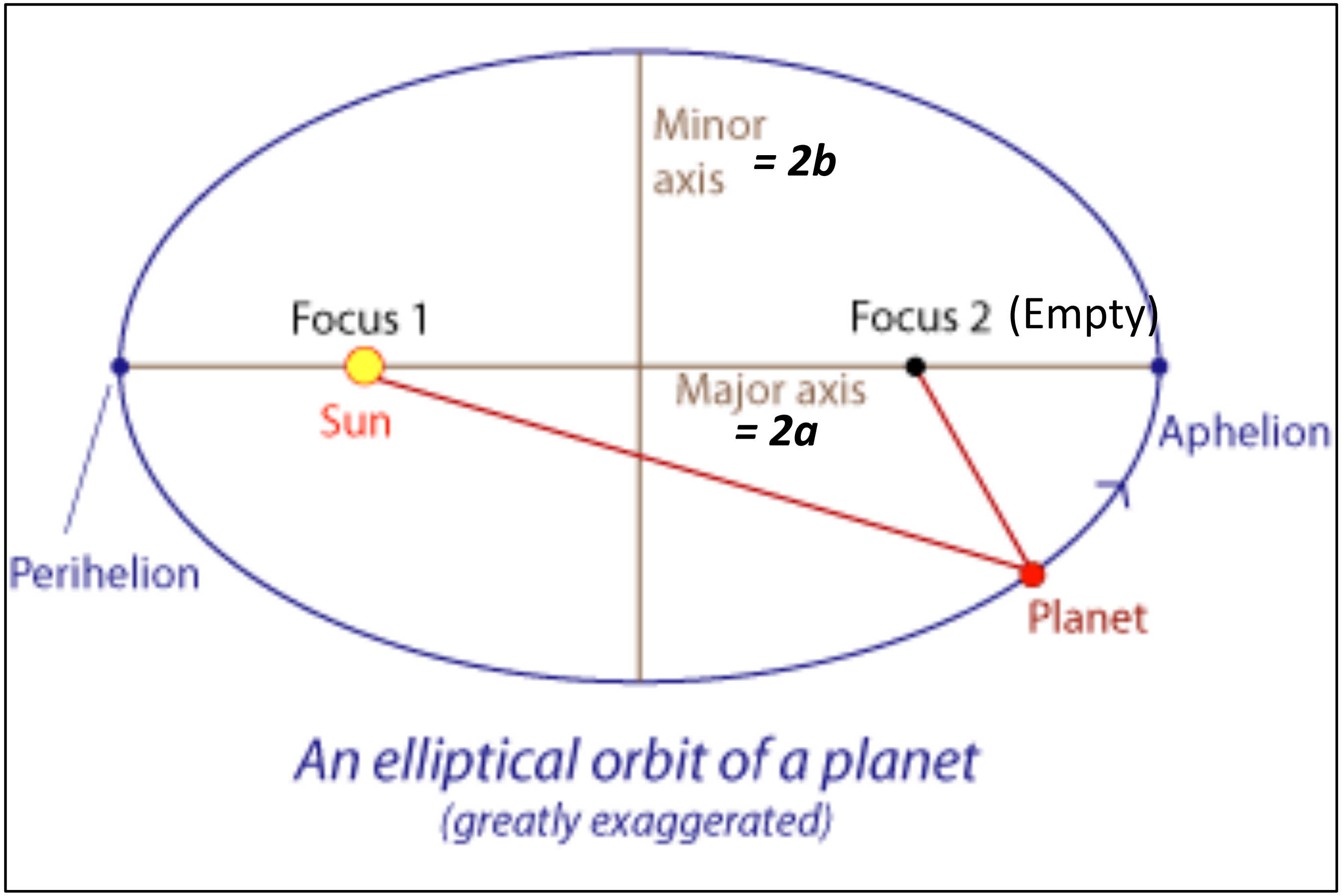
PH1a: ORBITS AND KEPLER



Source: https://oneminuteastronomer.com/8626/keplers-laws/

PS: we usually speak about semi-major axis, ***a***, and semi-minor axis, ***b***.

All orbital problems can be solved with these formulas

1. Relevant Formulas

Planetary motion:
1) Kepler's first law from the focus (barycenter):

$$r = \frac{a(1-e^2)}{1+e\cos\theta}, \tag{1}$$

where a is the *semi-major axis*, e is the ellipticity of the orbit and θ is measured from the focus with the origin $\theta = 0$ when the object is closer to the focus. Notice that r has a minimum and a maximum value called perigee and apogee, respectively.

$$r_p = r_{\min} = a(1-e), \tag{2}$$ ← !

$$r_a = r_{\max} = a(1+e). \tag{3}$$ ← !

2) Kepler's second law ("equal areas in equal times")

$$\frac{dA}{dt} = \frac{L}{2M} = \text{constant} \Rightarrow A = \frac{L}{2M}T. \tag{4}$$

This result is not linked to the particular $1/r^2$ dependence of Newton's gravitational force but it is a *general* property of any radial force (a force directed from a fixed origin).

PS: orange arrows mark the most relevant ones

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All orbital problems can be solved with these formulas

An *important* relationship for an orbiting body derived from the conservation of angular momentum is:

(5) $$r\, v\, \sin\alpha = r_a v_a = r_p v_p,$$

where α is the angle between the position vector and the velocity vector. For the apogee and perigee, $\alpha = \pi/2$ (and only for these two positions), so that $v_a r_a = v_p v_p$. This relationship will *often* be used in the problems.

3) Kepler's third law for elliptical orbits:

(6) $$T^2 = \frac{4\pi^2 a^3}{GM},$$

where M is the reduced mass of the system. Although, in practice, we will consider the mass of the most massive body (for instance, the Sun).

This result *only* holds for a central force with an inverse of square distance behavior. A fact that Newton used to derive his universal gravitational force.

All orbital problems can be solved with these formulas

4) Energy:

(7) $$E = K + U = \text{constant}$$

(8) $$K = \frac{1}{2}mv^2, \qquad U = -\frac{GMm}{r}$$

Moreover, for an *elliptical* orbit:

(9) $$E_{\text{elliptical}} = -\frac{GMm}{2a}$$

This result will be used in the resolution of problems.

The total energy, since it is conserved, will be calculated at the perigee and apogee, together with $r_a v_a = r_p v_p$, often resulting in a second order polynomial for r_a, r_p.

The total energy is ***E* = 0** for a **parabolic** orbit. And for an **hyperbolic** orbit, ***E* > 0**, that is, unconstrained.

First exercise: prove this identity

$$E_{elliptical} = - GMm/2a$$

On one hand, we have the expression for the energy (kinetic + potential) of an object in an orbit:

$$E = \frac{1}{2} m v^2 - \frac{GMm}{r} ;$$

Let's write it for the apogee and perigee:

$$E_a = \frac{1}{2} m v_a^2 - \frac{GMm}{r_a} ; \quad E_p = \frac{1}{2} m v_p^2 - \frac{GMm}{r_p} ;$$

These two energies must be the same since the total energy is conserved:

$$E_a = E_p \Rightarrow \frac{1}{2} m v_a^2 - \frac{GMm}{r_a} = \frac{1}{2} m v_p^2 - \frac{GMm}{r_p} .$$

On the other hand, conservation of angular momentum means:

$$r_a v_a = r_p v_p$$

And by definition:

$$r_a = a(1+e)$$
$$r_p = a(1-e) .$$

We have to mix them together to get E with only "a", the semi-major axis.

One way is: $v_a = v_p \cdot \frac{r_p}{r_a} = v_p \cdot \frac{a(1-e)}{a(1+e)} = v_p \frac{(1-e)}{(1+e)}$

$$\frac{1}{2} m v_a^2 = \frac{1}{2} m v_p^2 \frac{(1-e)^2}{(1+e)^2}$$ $\Rightarrow$ Substituting it together with $r_a = a(1+e)$

and $r_p = a(1-e)$, we get:

$$\frac{1}{2} m v_p^2 \frac{(1-e)^2}{(1+e)^2} - \frac{GMm}{a(1+e)} = \frac{1}{2} m v_p^2 - \frac{GMm}{a(1-e)}$$

$$\Rightarrow \frac{1}{2} \cancel{m} v_p^2 \left[\frac{(1-e)^2}{(1+e)^2} - 1 \right] = \frac{GM\cancel{m}}{a} \left(\frac{1}{1+e} - \frac{1}{1-e} \right) = - \frac{GM\cancel{m}}{a} \cdot \frac{2e}{(1-e^2)}$$

$(1-e)^2 - (1+e)^2 = -4e$

$$\Rightarrow \frac{1}{2} v_p^2 \frac{(-4e)}{(1+e)^2} = - \frac{GM}{a} \cdot \frac{2e}{(1-e^2)} \Rightarrow \boxed{v_p^2 = \frac{GM}{a} \left(\frac{1+e}{1-e} \right)}$$

(NOTICE: $v_a^2 = v_p^2 \frac{(1-e)^2}{(1+e)^2} = \frac{GM}{a} \frac{(1-e)}{(1+e)}$ symmetric "±")

So that, finally:

$$E = \frac{1}{2} m v_p^2 - \frac{GMm}{r_p} = \frac{1}{2} \frac{GMm}{a} \left(\frac{1+e}{1-e} \right) - \frac{GMm}{a(1-e)} = \frac{GMm}{a} \left[\frac{1}{2} \frac{(1+e)}{(1-e)} - \frac{1}{1-e} \right]$$

$$= \frac{GMm}{a} \frac{1+e-2}{2(1-e)} = \frac{GMm}{a} \frac{e-1}{2(1-e)} = \boxed{- \frac{GMm}{2a}}$$!!

Journey to Mars

**46. The Earth has an orbit of radius 1.50×10^8 km around the Sun; Mars has an orbit of radius 2.28×10^8 km. In order to send a spacecraft from the Earth to Mars, it is convenient to launch the spacecraft into an elliptical orbit whose perihelion coincides with the orbit of the Earth and whose aphelion coincides with the orbit of Mars (Figure 9.46); this orbit requires the least amount of energy for a trip to Mars.

(a) To achieve such an orbit, with what speed (relative to the Earth) must the spacecraft be launched? Ignore the pull of the gravity of the Earth and Mars on the spacecraft.

(b) With what speed (relative to Mars) does the spacecraft approach Mars at the aphelion point? Assume that Mars actually is at the aphelion point when the spacecraft arrives.

(c) How long does the trip from Earth to Mars take?

(d) Where must Mars be (in relation to the Earth) at the instant the spacecraft is launched? Where will the Earth be when the spacecraft arrives at its destination? Draw a diagram showing the relative positions of Earth and Mars at these two times.

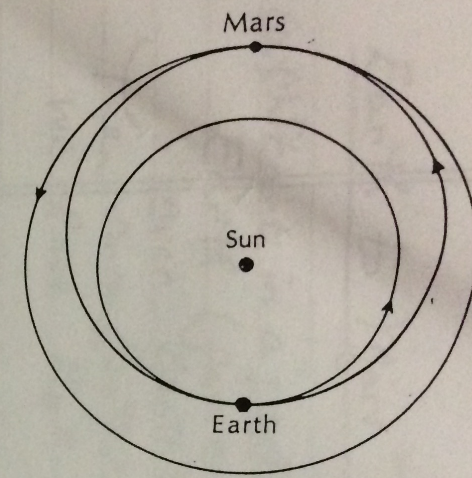

Fig. 9.46 Orbit for a space probe on trip to Mars.

Journey to Mars

a) This problem is solved using conservation of energy and angular momentum. Notice that the apogee is at the orbit of Mars and the perigee is at the orbit of Earth.

$$r_a v_a = r_p v_p,$$

$$\frac{1}{2}v_a^2 - \frac{GM}{r_a} = \frac{1}{2}v_p^2 - \frac{GM}{r_p},$$

$$\frac{1}{2}\left(v_p^2 - v_a^2\right) = GM\left(\frac{1}{r_p} - \frac{1}{r_a}\right),$$

$$\frac{v_a^2}{2}\left[\left(\frac{r_a}{r_p}\right)^2 - 1\right] = GM\left(\frac{1}{r_p} - \frac{1}{r_a}\right),$$

$$v_a^2 = 2GM\frac{r_p}{r_a}\frac{1}{r_a + r_p} \quad \Rightarrow \quad v_a = \sqrt{\frac{2GMr_p}{r_a(r_a + r_p)}}.$$

re I have used $v_p^2 - v_a^2 = (v_p + v_a)(v_p - v_a)$.

Journey to Mars

Now, the Earth is orbiting with a speed given by (circular orbit approximation) $v_E = \sqrt{GM/r_a}$. Therefore the relative velocity is:

(34) $$\Delta v_E = v_a - v_E = \sqrt{\frac{2GMr_p}{r_a(r_a + r_p)}} - \sqrt{\frac{GM}{r_a}} = \sqrt{\frac{GM}{r_a}}\left(\sqrt{\frac{2r_p}{r_a + r_p}} - 1\right).$$

Notice that $r_p > (r_a + r_p)/2$ (the semi-major axis), so the previous expression tells us $\Delta v_E > 0$.

Substituting the values for $r_a = r_E$ and $r_p = r_M$, we have $\Delta v_E = v_E * 0.098 = 2.93$ km/s, where $v_E = 29.8$km/s ($v_E = \sqrt{\frac{GM}{r_E}}$, or also you can divide $2\pi r_E$ by a year. The speed for the probe, 2.93 km/s, is about 8.6 times the speed of sound, still a respectable speed to provide to the probe.

Journey to Mars

b) Using conservation of angular momentum, as before, $v_p = v_a r_a / r_p$, or

$$v_p = \sqrt{\frac{2GMr_a}{r_p(r_a + r_p)}} \tag{35}$$

And

$$\Delta v_M = v_p - v_M = \sqrt{\frac{2GMr_a}{r_p(r_a + r_p)}} - \sqrt{\frac{GM}{r_p}} = \sqrt{\frac{GM}{r_p}} \left(\sqrt{\frac{2r_a}{r_a + r_p}} - 1 \right). \tag{36}$$

Now Δv_M is negative, since $r_a < (r_p + r_a)/2$.

The velocity of Mars can be obtained as $v_M = \sqrt{\frac{GM}{r_M}}$, or by dividing $2\pi r_M$ by a Martian year. The Martian year can be found by direct use of Kepler's third law comparing Mars and Earth. The result is $v_M / v_E = \sqrt{r_E / r_M}$.

Its value is: $\Delta v_M = v_M * (-0.109) = -2.64$ km/s, where $v_M = 24.2$ km/s.

The fact that Δv_M is negative means that the probe will arrive with less velocity than Mars, so it will be necessary to speed it up. Such speed is not negligible, about 7.76 times the speed of sound.

Journey to Mars

c) It takes half a period. Remember that for elliptical orbits, Kepler's third law can be applied:

(37) $$T^2 = \frac{4\pi^2 a^3}{GM}.$$

a is $(r_E + r_M)/2$. Taking ratios with respect to Earth (1 year):

(38) $$\frac{T_{\text{spacecraft}}}{T_{\text{Earth}}} = \left(\frac{r_E + r_M}{2r_E}\right)^{3/2} = 1.4143 \sim \sqrt{2}.$$

The last fact is just a curiosity. Anyway, it is half the orbital period, or $T_{\text{voyage}} = 0.7071$ years.

Journey to Mars

d) We can know the angular speed of Mars: $\omega_M = \sqrt{GM/r_M^3}$ or from its speed found before and its orbital radius. On the other hand we have the semi-period of the orbital probe found before. Therefore, Mars will have swept an angle:

$$\alpha_M = \omega_M \frac{T_{\text{spaceccraft}}}{2} = \frac{T_{\text{Earth}}}{2}\sqrt{\frac{GM}{r_M^3}\left(\frac{r_E + r_M}{2r_E}\right)^3} = \frac{\pi}{2\sqrt{2}}\sqrt{\left(\frac{r_E + r_M}{r_M}\right)^3} \sim 0.754\pi = 135.85^\circ. \tag{39}$$

Where I have used $T_{\text{Earth}} = 2\pi\sqrt{r_E^3/GM}$. The best opportunity is then when Mars is ahead of Earth by $180^\circ - 135.85^\circ = 44.14^\circ$.

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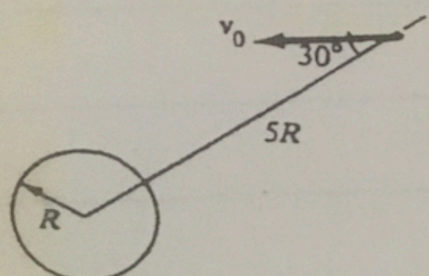
Determining an orbit

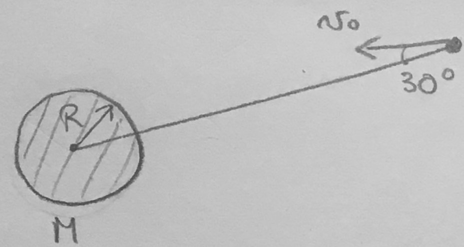
From a distance of $5R$ from the center of a planet of mass M and radius R, a satellite of mass m is launched with a speed $v_0 = (0.2GM/R)^{1/2}$ in the direction shown in the sketch. In terms of G, m, M, and R, determine the following quantities for the orbit: (a) the energy, (b) the angular momentum, and (c) the eccentricity.



30. Using conservation of energy and angular momentum, calculate the speed of the satellite in Problem 29 at both its perigee and apogee in terms of G, M, and R.

Both exercises: 29 and 30. Source: The Mechanical Universe. Chapter 17.

Ex. 29 & 30:



$v_0 = \sqrt{\frac{GM}{5R}}$

a) $\boxed{E = K + U = \frac{1}{2} m v_0^2 - \frac{GMm}{5R} = \frac{1}{2} m \left(\frac{GM}{5R}\right) - \frac{GMm}{5R} = -\frac{GMm}{10R}}$

b) $\boxed{L = m v_0 (5R) \cdot \sin(150^\circ) = \frac{5}{2} m R \sqrt{\frac{GM}{5R}} = \frac{m}{2}\sqrt{5GMR}}$

c) e: Let's get to know the distance to the apogee and perigee

Eq. 1: $\vec{L} = ct \Rightarrow \boxed{r_a v_a} = (5R) v_0 \cdot \sin 150^\circ = \frac{5}{2} v_0 R = \frac{1}{2}\sqrt{5GMR}$ ($v_0 = \sqrt{\frac{GM}{5R}}$)

Eq. 2: $\frac{1}{2} m v_a^2 - \frac{GMm}{r_a} = -\frac{GMm}{10R} \Rightarrow v_a^2 - \frac{2GM}{r_a} = -\frac{GM}{5R} \Rightarrow$

$\frac{1}{4 r_a^2}(5GMR) - \frac{2GM}{r_a} = -\frac{GM}{5R} \Rightarrow \frac{5}{4}\frac{R}{r_a^2} - \frac{2}{r_a} + \frac{1}{5R} = 0$

$(\times r_a^2) \Rightarrow \frac{r_a^2}{5R} - 2r_a + \frac{5}{4}R = 0 \Rightarrow \boxed{r_a^2 - 10 R r_a + \frac{25}{4} R^2 = 0}$

$$r_a = \frac{10R \pm \sqrt{100R^2 - 25R^2}}{2} = 5R \pm \frac{5}{2}\sqrt{3}R = \boxed{5R\left(1 \pm \frac{\sqrt{3}}{2}\right)}$$

Recall:

$$\left.\begin{array}{l} r_a = r_0(1+e) \\ r_p = r_0(1-e) \end{array}\right\} \rightarrow \left\{\begin{array}{l} r_a + r_p = 2r_0 \\ r_a - r_p = 2r_0 e \end{array}\right\} \boxed{e = \frac{r_a - r_p}{r_a + r_p}}$$

$$\boxed{e = 2\cdot\left(\sqrt{3}/2\right)/2 = \sqrt{3}/2}$$

d) $$r_a v_a = \frac{1}{2}\sqrt{5GMR} \Rightarrow v_a = \frac{\sqrt{5GMR}}{2\cdot 5R(1+\sqrt{3}/2)} = \sqrt{\frac{GM}{5R}}\cdot\frac{1}{2+\sqrt{3}}$$

$$= \boxed{(2-\sqrt{3})\sqrt{\frac{GM}{5R}}}$$

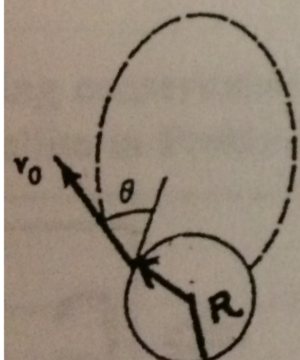
$$r_p v_p = \frac{1}{2}\sqrt{5GMR} \Rightarrow v_p = \frac{\sqrt{5GMR}}{2\cdot 5R(1-\sqrt{3}/2)} = \sqrt{\frac{GM}{5R}}\,\frac{1}{2-\sqrt{3}} = \boxed{(2+\sqrt{3})\sqrt{\frac{GM}{5R}}}$$

NB: $v_p > v_a$ (makes sense, faster when closer)

NB2: UNITS are OK $\left[\sqrt{\frac{GM}{R}}\right] = L/T$ = speed.

True ‘parabolic’ shot (*)

A rocket is fired from Cape Canaveral with an initial speed v_0 at an angle θ from the horizon as shown. Neglecting air resistance and the earth's rotation, calculate the maximum distance from the center of the earth that the rocket reaches in term of the mass and radius of the earth, M and R, v_0, θ, and G.



(*) It’s actually … elliptical!

True ‘parabolic’ shot

2. Rocket from cape Canaveral

The maximum distance is the apogee. As mentioned in the summary of main formulas, we will use conservation of angular momentum and conservation of energy to find the solution.

Let's now use conservation of energy:

$$\frac{1}{2}mv_0^2 - \frac{GMm}{R} = \frac{1}{2}mv_a^2 - \frac{GMm}{r_a} \quad \Rightarrow \quad \frac{1}{2}m(v_0^2 - v_a^2) = GMm\left(\frac{1}{R} - \frac{1}{r_a}\right)$$

Substituting $v_a = v_0 R\cos\theta/r_a$, and multiplying all by r_a^2, we get a second order polynomial:

$$r_a^2\left(\frac{2GM}{R} - v_0^2\right) - 2GMr_a + v_0^2R^2\cos^2\theta = 0,$$

$$r_a^2 - br_a + c = 0, \quad b = \frac{2GM}{\frac{2GM}{R} - v_0^2}, \quad c = \frac{v_0^2R^2\cos^2\theta}{\frac{2GM}{R} - v_0^2}$$

Notice that $\sqrt{2GM/R}$ is the escape velocity from the planet (Earth). v_0 must be less than the escape velocity or it would escape to infinity. That's why I have written it as $\frac{2GM}{R} - v_0^2$ because it is a *positive* quantity. The solution is:

$$r_a = \frac{b}{2}\left(1 \pm \sqrt{1 - \frac{4c}{b^2}}\right)$$

True 'parabolic' shot

Notice that $\sqrt{2GM/R}$ is the escape velocity from the planet (Earth). v_0 must be less than the escape velocity or it would escape to infinity. That's why I have written it as $\frac{2GM}{R} - v_0^2$ because it is a *positive* quantity. The solution is:

$$r_a = \frac{b}{2}\left(1 \pm \sqrt{1 - \frac{4c}{b^2}}\right)$$

Clearly we need $4c/b^2 < 1$ (for having a real solution). Or:

$$2v_0 \cos\theta \sqrt{1 - \frac{v_0^2}{v_E^2}} < 1,$$

where $v_E = \sqrt{2GM/R}$ is the escape velocity and I have done some algebra (check it!).

Also notice that for an ellipse we need to have an apogee and a perigee, so that there is another condition (prove it!):

$$\frac{4c}{b^2} > 0 \Rightarrow v_0 < v_E,$$

This condition is what we expected before. Otherwise, the rocket would go to infinity after all. In summary the farthest point is:

$$r_a = \frac{GM}{v_E^2 - v_0^2}\left(1 + \sqrt{1 - \frac{4c}{b^2}}\right)$$

True ‘parabolic’ shot

Now, using $v_E^2 = 2GM/R$, one can simplify a bit the previous expression and finally get:

$$r_a = \frac{R}{2}\left(\frac{1}{1-\frac{v_0^2}{v_E^2}}\right)\left(1+\frac{1}{v_E^2}\sqrt{v_E^4+4v_0^4\cos^2\theta-4v_0^2v_E^2\cos^2\theta}\right)$$

Units are okay as can easily be checked. If $\theta = \pi/2$, a vertical launch, $r_a = R/(1 - v_0^2/v_E^2)$ and for a tangential launch: $r_a = R$. Obviously, these extreme cases could have been solved directly from the very first equations above. I suggest you verify that one obtains indeed these solutions.

ROCKET FROM CAPE CANAVERAL FOR LOW SPEEDS

If $v_0 << v_E$, we are in a regime where the rocket will not go too high, so that we should recover an expression that resembles the one of the maximum height in a parabolic shot.

If $v_0 << v_E$, then $\frac{v_0}{v_E} << 1$. Notice that:

$$\frac{1}{1-\frac{V_0^2}{V_E^2}} \sim 1 + \frac{V_0^2}{V_E^2} \quad \left(1/(1-x) \sim 1 + x + O(x^2),\ |x| < 1\right).$$ And:

$$\frac{1}{v_E^2}\sqrt{v_E^4 + 4V_0^4\cos^2\theta - 4v_0^2 v_E^2 \cos^2\theta} = \sqrt{1 + \frac{4V_0^4}{V_E^4}\cos^2\theta - 4\frac{v_0^2}{V_E^2}\cos^2\theta}$$

$\frac{V_0^4}{V_E^4} << \frac{V_0^2}{V_E^2}$, since $\frac{V_0^2}{V_E^2} << 1$, so we can do:

$$\sqrt{1 + \frac{4V_0^4}{V_E^4}\cos^2\theta - 4\frac{V_0^2}{V_E^2}\cos^2\theta} \sim \sqrt{1 - \frac{4V_0^2}{V_E^2}\cos^2\theta} \sim 1 - \frac{2V_0^2}{V_E^2}\cos^2\theta,$$ where

we have used $\sqrt{1+x} \sim 1 + \frac{x}{2} + O(x^2)$, $|x| << 1$. Therefore, we have:

$$r_a \sim \frac{R}{2}\left(1+\frac{V_0^2}{V_E^2}\right)\left(1+1-2\frac{V_0^2}{V_E^2}\cos^2\theta\right) = R\left(1+\frac{V_0^2}{V_E^2}\right)\left(1-\frac{V_0^2}{V_E^2}\cos^2\theta\right)$$

$$\sim R\left(1+\frac{V_0^2}{V_E^2}-\frac{V_0^2}{V_E^2}\cos^2\theta\right),$$ disregarding terms of order V_0^4/V_E^4 and higher.

$$r_a \sim R\left(1+\frac{V_0^2}{V_E^2}\sin^2\theta\right)$$

The height with respect the surface is: $h_{max} = r_a - R = \frac{R V_0^2 \sin^2\theta}{V_E^2}$.

Finally, since $V_E^2 = 2GM/R$, we get:

$$h_{max} = \frac{R^2 V_0^2 \sin^2\theta}{2GM};$$ Now $g = \frac{GM}{R^2}$ $(=9{,}81\,m/s^2, \dots)$

$$\Rightarrow \boxed{h_{max} = \frac{V_0^2 \sin^2\theta}{2g}}$$

This is the result from Kinematics!

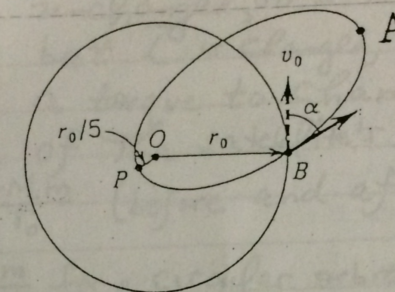
$$V_f^2 = V_0^2 + 2ad \Rightarrow 0 = V_0^2\sin^2\theta - 2g\,h_{max}$$

$$\Rightarrow \boxed{h_{max} = \frac{V_0^2 \sin^2\theta}{2g}}.$$

Sudden change in an orbit: collisions, explosions, …

ORBIT PROBLEM

A satellite of mass m is traveling at speed v_0 in a circular orbit of radius r_0 under the gravitational force of a large uniform planet of mass $M >> m$ centered at O, as shown in the figure.



(2 points) (a) Taking the potential energy to be zero at $r = \infty$, find the *total* mechanical energy E of the satellite in terms of m and v_0. Also find E in terms of G, M, m and r_0.

At a certain point B in the orbit (see the figure), the direction of motion of the satellite is suddenly changed *without any change in the magnitude of the velocity.* As a result the satellite goes into an elliptical orbit. Its distance of closest approach to O (at point P) is now $r_0/5$.

(2 points) (b) Find the speed of the satellite v_P at point P, expressed as a multiple of v_0.

(2 points) (c) Through what angle α (see the figure) was the angle of the satellite turned at the point B.

(1 point) (d) Find the product of the speed and the distance $r_A v_A$ at the point in the elliptical orbit farthest from point O, in terms of $r_0 v_0$.

(2 points) (e) Find the values of the ratios r_A/r_0 and v_A/v_0.

(1 point) (f) Now suppose that just after the satellite has returned to point B, an internal explosion on the planet instantly vaporizes part of its mass. The vaporized mass resulting from the explosion does not hit the satellite, and it moves rapidly out to large distances ($>> r_0$) in a negligible time. The satellite subsequently follows a *parabolic orbit.*

Find the remaining mass of the planet M_{final} in terms of its initial mass M.

Sudden change in an orbit: collisions, explosions, …

3. Orbit problem with explosion

a) This question could be solved immediately because for an elliptical orbit (circular in this case), the total energy is given by $E = -GMm/2r_0$.

However, we need to give the solution in terms m and v_0 as well. Let's recall the whole process.

First, from Newtonian dynamics, the equilibrium equation gives us:

$$GMm/r_0^2 = mv_0^2/r_0, \qquad \text{or} \qquad mv_0^2 = GMm/r_0.$$

Therefore,

$$E = -GMm/2r_0 = -mv_0^2/2 = -K$$

As a side note, notice that in the case of a circular orbit $E = K + U = -K$, or $K = -U/2$, which is an expression of the virial theorem for a potential that is proportional to r^{-1}.

Sudden change in an orbit: collisions, explosions, …

b) We use conservation of energy. The reason to prefer conservation of energy to conservation of angular momentum is because the magnitude of the velocity does not change and we do not need to know the angle (this will be considered in the next question). Using conservation of the total energy, we can write:

$$\begin{aligned} \frac{1}{2}mv_0^2 - \frac{GMm}{r_0} &= \frac{1}{2}mv_P^2 - \frac{GMm}{r_P} \\ -\frac{GMm}{2r_0} &= \frac{1}{2}mv_P^2 - \frac{5GMm}{r_0} \\ \frac{9GMm}{2r_0} &= \frac{1}{2}mv_P^2 \\ v_P &= 3\sqrt{\frac{GM}{r_0}} = 3v_0. \end{aligned}$$

c) Here we can use conservation of angular momentum:

$$v_0 r_0 \cos\alpha = (r_0/5)(3v_0),$$

where I have obviated m and I have used the fact that α is the complementary angle to the one between the position vector and the velocity. Therefore, $\cos\alpha = 3/5$, or $\alpha = 51.13°$.

Sudden change in an orbit: collisions, explosions, …

d) Again conservation of angular momentum gives us the answer: $r_a v_a = \frac{3}{5} v_0 r_0 (= v_p r_p)$.

e) The easiest way is noticing that A is the apogee. We have

$$r_A = a(1+e), \qquad \text{and} \qquad r_P = a(1-e),$$

so that

$$r_P = r_0/5 = r_0(1-e),$$

because $a = r_0$ due to the fact that the magnitude of the velocity did not change in the explosion. Therefore, the total energy before and after the explosion is the same.

Since the total energy is $-GMm/(2a)$ for an elliptical orbit and it was $-GMm/(2r_0)$ in the circular orbit, one must have $a = r_0$.

In summary, $e = 4/5$ and $r_A/r_0 = 1 + e = 9/5$.

For v_A/v_0 we can use conservation of angular momentum again: $v_A r_A = v_P r_P$ or $v_A(1+e) = v_P(1-e)$. We know $v_P = 3v_0$ so that $v_A/v_0 = 3(1-e)/(1+e) = 1/3$.

Sudden change in an orbit: collisions, explosions, …

f) For the last question, the kinetic energy of the mass m does not change, only the potential energy due to the disappearance of part of the mass of the central body. For a *parabolic* orbit, the total energy is 0, so we can write:

$$0 = \frac{1}{2}mv_0^2 - \frac{G\tilde{M}m}{r_0} = \frac{GMm}{2r_0} - \frac{G\tilde{M}m}{r_0} \quad \Rightarrow \quad \tilde{M} = \frac{M}{2}. \tag{28}$$

Additional material beyond the course:
Escape velocity from a planet and Black Holes?

Black Holes: escape velocity from a planet greater than c?

$$v_{escape} = 11.2\ km/s$$

$$\frac{1}{2}mv^2 = \frac{GMm}{r}$$

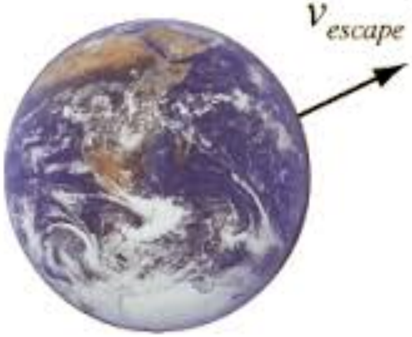
$$v_{escape} = \sqrt{\frac{2GM}{r}}$$

Can we think of an object where light cannot escape?

1. Assume the mass of the sun, find the radius of the object?
2. Assume the radius of the sun, find the mass of the object?
3. Assume density of lead, find the radius and mass of the object?

Use SI: $c \sim 3.00\text{x}10^8$ m/s, $G \sim 6.67\ \text{x}10^{-11}$ Nm^2/kg^2, $M_{sun} \sim 2.00\text{x}10^{30}$ kg, $r_{Sun} \sim 4.38\times10^9$ m, ρ_{LEAD}=11,340 $kg/m^{3.}$

Black Holes: escape velocity from a planet greater than c?



$$v_{escape} = 11.2\ km/s$$

$$\frac{1}{2}mv^2 = \frac{GMm}{r}$$

$$v_{escape} = \sqrt{\frac{2GM}{r}}$$

Can we think of an object where light cannot escape?

1. Assume the mass of the sun, find the radius of the object?
   - r = 2.96 km! All the sun should be shrunk to that radius.

2. Assume the radius of the sun, find the mass, "M"?
   - M = 2.96x10^{36} kg (~1,000,000 Sun mass)! All this mass should be put into the size of the sun.
3. Assume density of lead, find the radius and mass of the object? (Use $M = 4/3\pi r^3$)
   - r = 27 R_{SUN}!, M = 40 million M_{SUN}. That's the option first thought in … See next slide.

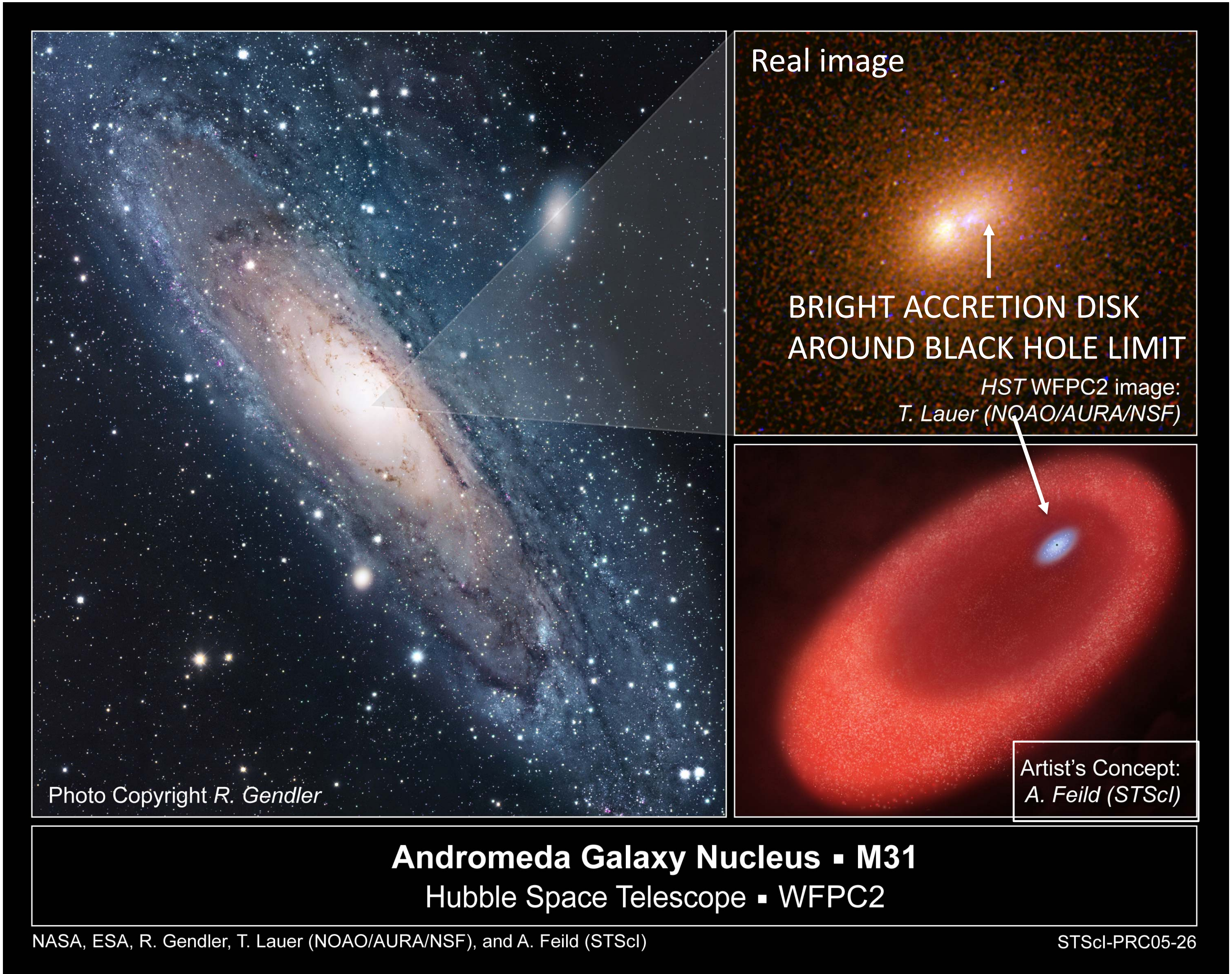
How old is the idea of a Black Hole?

English John Michell and French Pierre-Simon Laplace in 1784 and 1796

They assumed large objects where the density was not higher than the one found in the known elements, ~ 10 g/cm^3. Not small objects with incredible densities. In reality, we have only detected BH that correspond to ultra high densities –singularity, and 'small' sizes.

Keeping mass and modifying the radius (what really happens in a Black Hole), we would get:

- Mass of 1 Earth: Black Hole radius is 9 mm
- At Andromeda, M31, super massive Black Hole: 4.7×10^{11} m. About 26 light-minutes in radius. Much less 'dense'. Less dangerous than an Earth-like Black Hole. That is, much weaker 'spaghetti' effect.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