PH1a: oscillations

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x.$$

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = constant = \frac{1}{2}kA^2$$

If the forces or torques of your system end up being written with a similar equation as the ones above, the system will undergo *Simple Harmonic Oscillations*, SHO.

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Simple Harmonic Oscillation

SHO are completely characterized by 2 initial conditions. Many times we use an expression such as:

$$x(t) = A\sin(\omega t) \quad \text{or} \quad x(t) = A\cos(\omega t).$$

However, in some problems we can not use those expressions. In general, we need to impose an initial position, x_0, and an initial speed, v_0. For instance, if we write $x(t) = A\sin(\omega t)$, it is because we are assuming that initially, the spring was at its equilibrium position ($x(0)=0$) and it was moving with its maximum speed $\dot{x}(0) = +A\omega$. If the problem is telling us other initial conditions we need to use them. Let's look at the general formula and some examples.

Given x_0 and $\dot{x}_0$, the initial position and initial speed at $t=0$, the equation for the SHO in terms of time, looks like:

$$x(t) = x_0 \cos \omega t + \frac{\dot{x}_0}{\omega} \sin \omega t \quad (1)$$

$\omega = \sqrt{\frac{k}{m}}$, where $\underline{k}$ is the Hooke's constant and $\underline{m}$ is the mass of the moving object attached to the spring. It has units of rad/s. Also notice that $\dot{x}_0$ may be ⊕ or ⊖, depending on your sign choice. Another remark is that the expression above may be written in a more compact form:

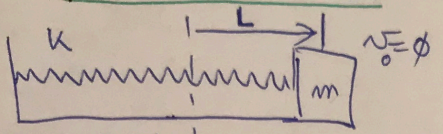
$$x(t) = A \cos(\omega t + \varphi_0), \text{ where}$$

$$A = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega^2}} \quad \text{and} \quad \varphi_0 = \operatorname{atan}\left(\frac{-\dot{x}_0}{\omega x_0}\right).$$

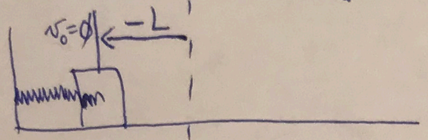
This expression helps understand the Amplitude that corresponds to Eq. (1) above, but is generally less helpful than 1, except to know the amplitude.

Let's look at different cases:

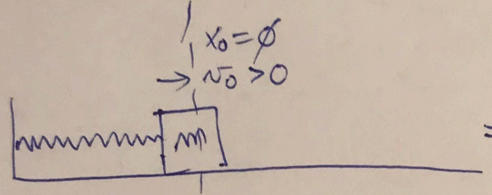
EQUILIBRIUM POSITION

1) 

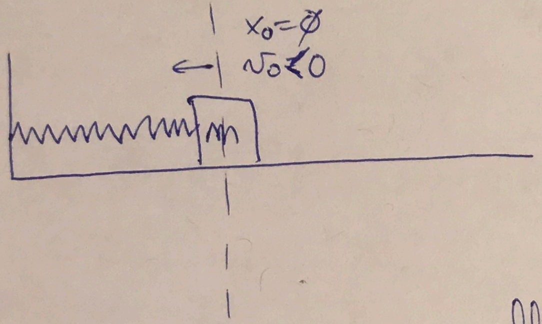
$\Rightarrow \begin{cases} x_0 = +L \\ \dot{x}_0 = 0 \end{cases} \Rightarrow x(t) = L\cos\omega t.$

2) 

$\Rightarrow \begin{cases} x_0 = -L \\ \dot{x}_0 = 0 \end{cases} \Rightarrow x(t) = -L\cos\omega t.$

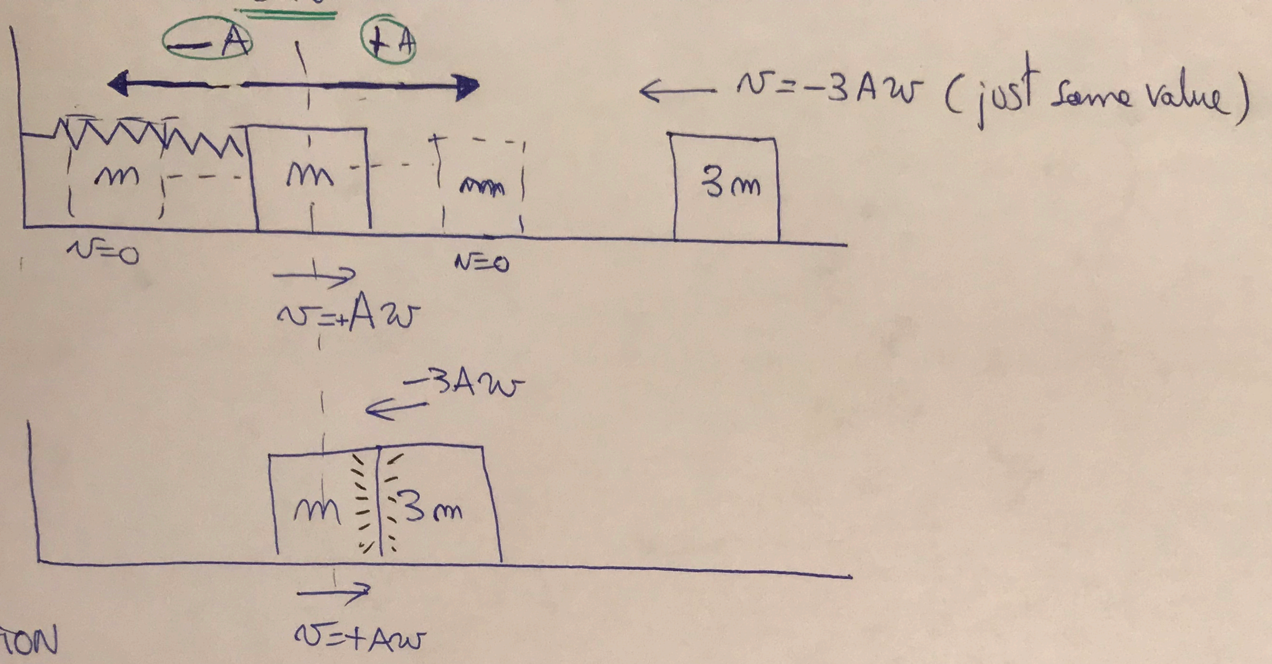
3) 

$\Rightarrow \begin{cases} x_0 = 0 \\ \dot{x}_0 = v_0 \end{cases} \Rightarrow x(t) = \frac{v_0}{\omega}\sin\omega t.$

4) 

$\Rightarrow \begin{cases} x_0 = 0 \\ \dot{x}_0 = -v_0 \end{cases} \Rightarrow x(t) = -\frac{v_0}{\omega}\sin\omega t.$

5) A block of mass m is oscillating with SHO when another block of mass 3m collides with it. The moment of the collision is when the block of mass m is at its equilibrium position and moving with speed $+A\omega$, whereas the 3m body is moving to the left with speed $v = -3A\omega$.
Find the new equation of motion if the collision is inelastic or elastic.



SOLUTION

In both cases the value of x_0 is $\emptyset$ (we neglect the real size of the blocks). The value of $\dot{x}_0$ will depend on the type of collision.

Ⓐ INELASTIC COLLISION:

$$\vec{P}_I = \vec{P}_F$$

$$m\,(A\omega) + 3m\,(-3A\omega) = 4m\,v_f$$

$$v_f = -\frac{8\,m\,A\omega}{4m} = -2A\omega.$$

Therefore (Recall Eq. 1)

$$x(t) = \emptyset \cdot \cos\tilde{\omega}t - \frac{2A\omega}{\omega}\sin\tilde{\omega}t = -2A\sin\tilde{\omega}t$$

Notice that $\tilde{\omega} = \sqrt{\frac{k}{4m}}$, since both blocks will oscillate together.

(B) <u>ELASTIC COLLISION</u>: $\vec{P}_I = \vec{P}_F$ & $K_I = K_F$

We may solve those equations, or use the results we found for 1-dimensional collisions. Namely:

$$v_{1f} = \frac{(m_1 - m_2)v_{1I} + 2m_2 v_{2I}}{m_1 + m_2}$$

(we don't need to calculate v_{2f} since it is not oscillating after the collision)

$$\boxed{v_{1f}} = \frac{(m - 3m)A\omega + 6m(-3A\omega)}{4m} = \frac{-20mA\omega}{4m} = \boxed{-5A\omega}$$

thus, $\dot{x}_0 = -5A\omega$, $x_0 = \emptyset \Rightarrow$

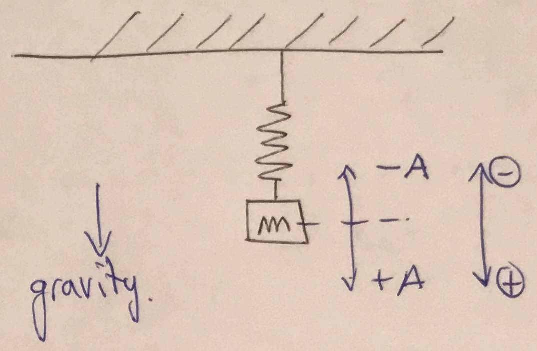
$$\boxed{x(t) = -\frac{5A\omega}{\omega}\sin\omega t} = \boxed{-5A\sin\omega t}$$

with $\omega = \sqrt{\frac{k}{m}}$, since only the block with mass <u>m</u> oscillates after the collision.

PS: (CHECK) $v_{2f} = \frac{(m_2 - m_1)v_{2I} + 2m_1 v_{1I}}{m_1 + m_2} = \frac{2m\cdot(-3A\omega) + 2m(A\omega)}{4m} = \boxed{-A\omega}$ & $P_F = +m(-5A\omega) + 3m(-A\omega) = -8mA\omega$

$P_I = m(A\omega) + 3m(-3A\omega) = -8m$

VERTICAL SPRINGS



The case of a vertical spring is different to the one horizontal, because gravity pulls the spring.

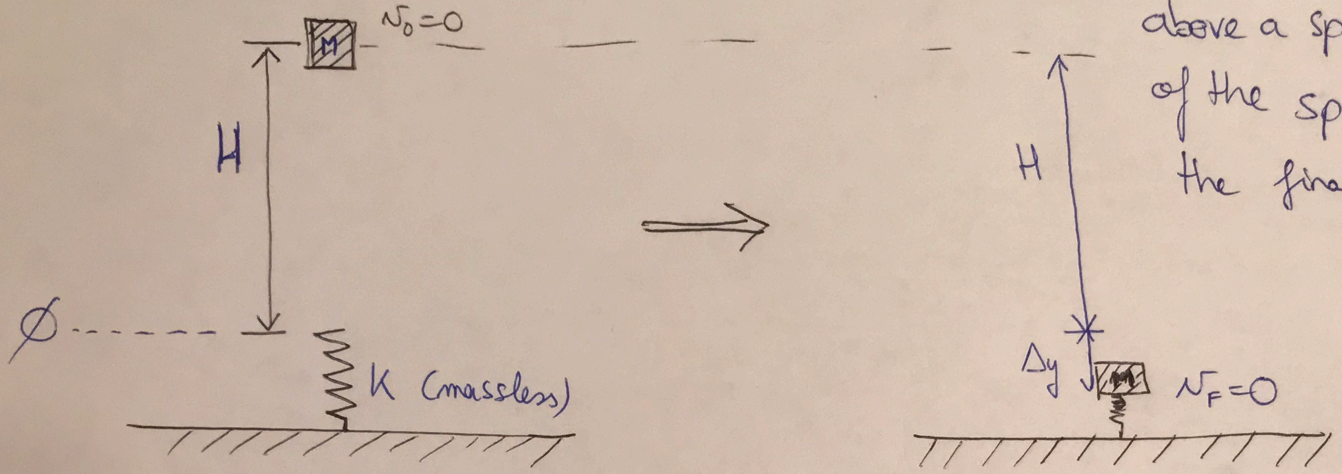
However, gravity is a constant force all the time, (which is not the case for the spring force). The effect of such additional constant force, mg, is simple: $mg = K\Delta y$ That is, gravity pulls the string an additional length. The general solution can be written in similar terms as the horizontal case:

$$y(t) = \underbrace{\frac{mg}{K}}_{\text{New}} + \underbrace{y_0 \cos \omega t + \frac{\dot{y}_0}{\omega} \sin \omega t}_{\text{Same as horizontal case}}, \text{ with } \omega = \sqrt{\frac{K}{m}}.$$

I have chosen ↑⊖ ↓⊕

Energy and springs

Let's solve one problem as an example: A mass M is released from a height H above a spring. Consider the mass of the spring negligible. Find the final stretch of the spring.



Initial mechanical energy: $E_m = K + U = \emptyset + mgH + \frac{1}{2}k \cdot \emptyset^2 = mgH$

(ORIGIN

Final mechanical energy: $E_m = K + U = \emptyset - mg\Delta y + \frac{1}{2}k\Delta y^2$

$$\Rightarrow \quad mgH = -mg\Delta y + \frac{1}{2}k\Delta y^2 \Rightarrow mg(H + \Delta y) = \frac{1}{2}k\Delta y^2$$

$$\boxed{\frac{1}{2}k\Delta y^2 - mg\Delta y - mgH = 0} \qquad \boxed{\Delta y = \frac{mg \pm \sqrt{m^2g^2 + 2mgHK}}{K}}$$

There are 2 solutions. The one we are interested in is the one with the sign (+), maximum compression.

$$\Delta y = \frac{mg}{K} \pm \sqrt{\frac{m^2g^2}{K^2} + \frac{2mgH}{K}}$$

You can check units are fine

[Units of K: $[K] = [N/m] = \frac{ML}{T^2 \cdot L} = M/T^2$]

Two interesting things to notice are:

1) $\Delta y = \frac{mg}{K} \pm \sqrt{\cdots}$; there is an average position:

$$\boxed{\frac{mg}{K}}$$

and this corresponds to the equilibrium position where $F_g = F_{spring}$ ($mg = K\Delta y_{EQ}$).

2) The amplitude can be found by subtracting the two maximum stretches of the spring and dividing by 2:

$$\boxed{A = \left[\left(\frac{mg}{k} + \sqrt{\cdots}\right) - \left(\frac{mg}{k} - \sqrt{\cdots}\right)\right]/2 = \sqrt{\cdots} = \sqrt{\frac{m^2g^2}{k^2} + \frac{2mgH}{k}}}$$

In other words, in order to find the EQUILIBRIUM position of a SHO and its AMPLITUDE, we can use energy conservation, find the 2 solutions that correspond to the maximum stretch of the spring and derive them.

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FINAL COMMENT: When solving $\frac{1}{2}k\Delta y^2 - mg\Delta y - mgH = 0$, we could 'use physics'. We know that the SHO will be about the new equilibrium position: mg/k, so it should be symmetric about it. Define:

$$\Delta y = z + mg/k \quad \text{and let's find "} z \text{"}$$

$$\frac{1}{2}k\left(z + \frac{mg}{k}\right)^2 - mg\left(z + \frac{mg}{k}\right) - mgH = \frac{1}{2}k\left(z^2 + \frac{2mgz}{k} + \frac{m^2g^2}{k^2}\right) -$$

$$mgz - \frac{m^2g^2}{k} - mgH = \frac{1}{2}kz^2 + mgz + \frac{1}{2}\frac{m^2g^2}{k} - mgz - \frac{m^2g^2}{k} - mgH$$

$\frac{1}{2}Kz^2 - \frac{1}{2}\frac{m^2g^2}{K} - mgH$. Therefore,

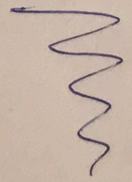
$\frac{1}{2}K\Delta y^2 - mg\Delta y - mgH = 0$ is equivalent to solve:

$$\frac{1}{2}Kz^2 - \frac{1}{2}\frac{m^2g^2}{K} - mgH = 0 \Rightarrow \frac{1}{2}Kz^2 = \frac{1}{2}\frac{m^2g^2}{K} + mgH$$

$$\Rightarrow z = \pm\sqrt{\frac{m^2g^2}{K^2} + \frac{2mgH}{K}};$$

So that z measures the amplitude: $\sqrt{\frac{m^2g^2}{K^2} + \frac{2mgH}{K}}$.

$$\Delta y = \frac{mg}{K} \pm \sqrt{\frac{m^2g^2}{K^2} + \frac{2mgH}{K}}$$, as we found before.



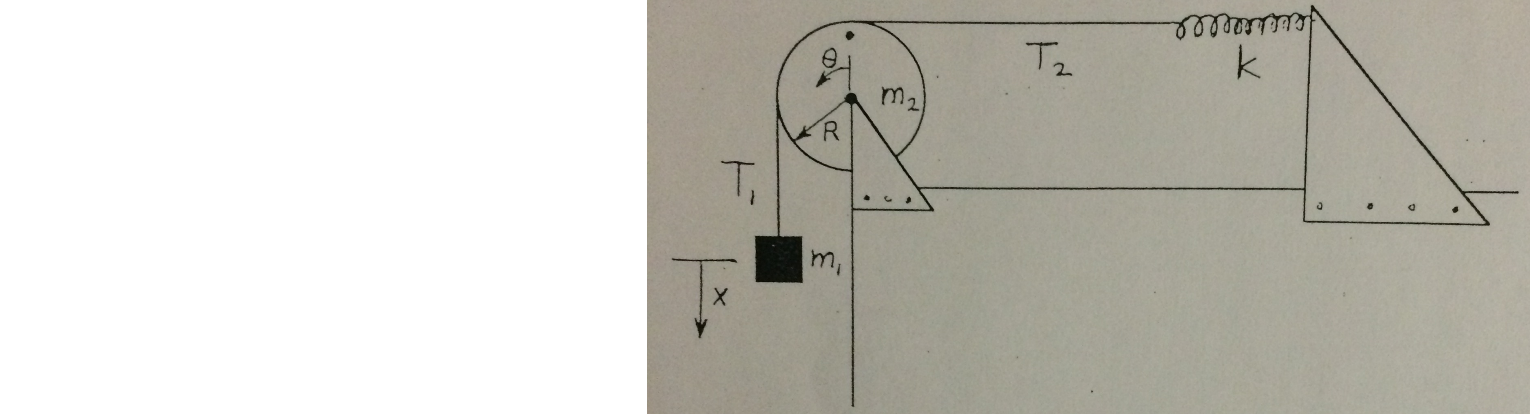
PS: The next 3 pages are not necessary this year

Rotational dynamics, oscillations and friction

Physics 1a

Quiz 4

A spring with spring constant k is fastened to a string, which runs over a pulley and is tied to a weight of mass m_1, as shown in the figure. There is friction between the string and the pulley so the string does not slip and the tensions in the string below and above the pulley, T_1 and T_2, are generally not the same. The bearings of the pulley have negligible friction. Consider the pulley to be a uniform solid disk of radius R and mass m_2.



The weight is held at a point such that the spring is unstretched. At time $t = 0$ the weight is released. The position of the weight x and the angle θ of the disk undergo simple harmonic motion.

a) (2 points) What is the potential energy of the system as a function of x?

b) (2 points) What is the kinetic energy (linear and rotational) of the system in terms of the speed v of the weight?

c) (1 point) What is the maximum speed v of the weight?

d) (2 points) Write the Newton's second Law for the vertical motion of the weight (x) and for the rotational motion of the pulley (θ) in terms of g, m_1, m_2, T_1, T_2, k, and R.

e) (2 points) Combine these equations to find the period of the oscillation in terms of g, m_1, m_2, k, and R (note that the system oscillates around an equilibrium point, not the starting point).

f*) (1 point) If there is some friction in the pulley bearings so that the torque on the pulley caused by friction is a constant times the angular speed, $\tau = -C\omega$, what is the period?

Rotational dynamics, oscillations and friction

1. QP4: The spring and the weight

a) We define $x > 0$ downwards, as in the picture. Notice that when $x > 0$ there is a loss of gravitational potential energy. The solution is:

$$U = \frac{1}{2}kx^2 - mgx. \tag{1}$$

b) The kinetic energy is the sum of the two moving objects (massless spring):

$$K = \frac{1}{2}m_1v^2 + \frac{1}{2}I\omega^2, \quad I = \frac{1}{2}m_2R^2, \quad \omega = v/R, \tag{2}$$

$$K = \frac{1}{4}(2m_1 + m_2)v^2. \tag{3}$$

Rotational dynamics, oscillations and friction

c) It is said that the resulting motion is a SHO. The maximum speed happens at the equilibrium position. The equilibrium position happens when the weight equals the stress force in the spring, which is not $x = 0$:

(4) $$kx_0 = m_1 g, \quad x_0 = m_1 g/k.$$

Using conservation of energy (initially at $x = 0$, the total energy is 0):

(5) $$0 = K + U(x_0) = \frac{1}{4}(2m_1 + m_2)\,v_{\max}^2 + \frac{1}{2}kx_0^2 - mgx_0, \Rightarrow v_{\max} = m_1 g\sqrt{\frac{2}{k(2m_1 + m_2)}}.$$

Dimensions are correct (check it, using that k has dimensions of force/length).

Rotational dynamics, oscillations and friction

d) For the mass m_1, we have ($a > 0$, when $x > 0$):

(6) $$F = ma \Rightarrow m_1 a = m_1 g - T_1.$$

For the pulley:

(7) $$\tau = I\alpha = Ia/R, \quad \tau = (T_1 - T_2)R, \Rightarrow Ia = (T_1 - T_2)R^2.$$

1

Notice the signs of T_1 and T_2 that have been set in accordance with $a > 0$ if $x > 0$, and therefore $\alpha > 0$ (use the right hand rule).

Rotational dynamics, oscillations and friction

e) In order to find the period of the oscillation, we need to arrive to some equation of the type:

$$\frac{d^2x}{dt^2} = -\omega^2 x. \tag{8}$$

We need the previous equations, but still we need an expression for T_2.
T_2 is in fact the force felt by the spring, so:

$$T_1 = m_1(g-a), \quad T_2 = kx, \Rightarrow Ia = (m_1 g - m_1 a - kx)R^2, \tag{9}$$

$$a = \frac{m_1 g - kx}{m_1 + \frac{I}{R^2}} = \frac{m_1 g}{m_1 + \frac{I}{R^2}} - \frac{k}{m_1 + \frac{I}{R^2}} x. \tag{10}$$

One can realize that we have arrived at an equation of the type

$$a = -\omega^2 x + \text{cnt}. \tag{11}$$

Rotational dynamics, oscillations and friction

The constant term does not affect the period of the oscillations. The constant term appears simply because the initial position $x = 0$ is not the equilibrium position. You can find in any text book (including ours) that the solution is the same as a SHO. The ω is

$$\omega = \sqrt{\frac{k}{m_1 + \frac{I}{R^2}}} = \sqrt{\frac{2k}{2m_1 + m_2}}, \tag{12}$$

from where the period can be found as $T = 2\pi/\omega$.

NB: Notice how the relationship $v_{\max} = A\omega$ is satisfied. We have that the maximum amplitude is $A = x_0 = m_1 g/k$ and we had obtained $v_{\max}$ before. If you check it, you will see it works. In other words, ω could have been found so easily. The only reason to have deduced the equations is (besides being asked) that we can prove it is a SHO motion.

Rotational dynamics, oscillations and friction

For question (**f**), we need to introduce ***damped*** oscillations. That is, oscillations with ***friction***. We will also introduce **forced oscillations**, where an external force is applied to the oscillatory system (long term or asymptotic solution)

Damped oscillations

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt}.$$

After rearranging terms and dividing througl

$$\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \omega_0^2 x = 0,$$

where $\beta = \gamma/m$ and, as before, $\omega_0^2 = k/m$.

$$x = Ce^{-bt}\cos(\omega_1 t + \theta_0).$$

$$\omega_1 = \sqrt{\omega_0^2 - \frac{\beta^2}{4}} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}.$$

Rotational dynamics, oscillations and friction

f) If there is friction, we have to add it to the equation of the torque ($\omega = v/R$).

$$(T_1 - T_2)R - Cv/R = Ia/R. \tag{13}$$

Rotational dynamics, oscillations and friction

We can now write the following equation (substituting T_1, T_2)

$$(m_1R^2 + I)a + Cv + kxR^2 = m_1gR^2. \tag{14}$$

This expression resembles the differential equation of a damped oscillator, but for a constant term on its right hand side, instead of being equal to 0. Re-arranging terms to make it clearer, we get:

$$a + \beta v + \omega_0^2 x = \text{constant} = \frac{m_1 g}{m_1 + \frac{I}{R^2}}, \tag{15}$$

where $\beta = C/(m_1R^2 + I)$ and $\omega_0^2 = k/(m_1 + I/R^2)$. Notice that the units are correct (C has units of ML^2/T).

Rotational dynamics, oscillations and friction

The general solution to Eq. 15 is found in 2 steps. The first one consists in solving the differential equation, making it equal to 0. This is called the *homogeneous* solution, sometimes also called the *transitory* solution. The second step consists in finding a *particular* solution to the differential equation. This is called the *inhomogeneous* solution, sometimes also called the *steady state* solution. The *general* solution is the sum of both solutions because the differential equation is *linear*. Let's find each solution for our problem.

Rotational dynamics, oscillations and friction

First, the homogenous solution is the one corresponding to an *underdamped* oscillatory motion, because the problem mentions that there is a period (otherwise, it might also be critically damped or overdamped). The expression is:

(16) $$x(t) = x_0 e^{-\beta t/2} \cos\left(\omega_1 t + \phi_0\right), \quad \omega_1 = \sqrt{\omega_0^2 - \frac{\beta^2}{4}}$$

Where the period is $T = 2\pi/\omega_1$. Notice that it is ω_1, and *not* ω_0 the variable that defines the period. The final expression for ω_1 can be written in terms of the parameters of the exercise recalling that $I = 1/2 m_2 R^2$, as well as the expressions for β and ω_0 found before. It is not a simple expression. It is enough to make clear we have solved it.

Rotational dynamics, oscillations and friction

Finally, the inhomogeneous solution is found by setting $x = x_1$ with x_1 some unknown constant. The reason is that the inhomogeneous solution to a *linear* differential equation can be found by 'guessing' (aka *Ansatz* from the German word). The *Ansatz* is always a function of the same type as the inhomogeneous term. In this case, it is a constant, but it could be a sinusoidal function or a polynomial of degree 4, say. In our case, we try $x = x_1$ =constant, but for a sinusoidal function we would try a combination of sin/cos functions with arbitrary coefficients. For a polynomial of 4^{th} degree, we would try $x = a+bx+cx^2+dx^3+ex^4$. That is, a generic polynomial of 4^{th} degree. We substitute our *Ansatz* in the equations and we solve for the unknown coefficients. In our current case, it is simpler since $x = x_1, dx/dt = 0$, and $d^2x/dt^2 = 0$. Thus, Eq. 15 becomes:

$$\omega_0^2 x_1 = \frac{m_1 g}{m_1 + \frac{I}{R^2}} \Rightarrow \qquad x_1 = m_1 g/k, \tag{17}$$

where we have used the expression for ω_0. That is, x_1 is the contraction of the spring due to the weight of m_1. The final solution is the sum of both, the homogeneous and inhomogeneous solutions. That is,

Rotational dynamics, oscillations and friction

The final solution is the sum of both, the homogeneous and inhomogeneous solutions. That is,

$$x(t) = m_1 g/k + x_0 e^{-\beta t/2} \cos(\omega_1 t + \phi_0) \,. \tag{18}$$

The 2 constants of integration (x_0, and ϕ_0) correspond to 2 initial conditions. We could set them by setting some initial conditions such as: for $t = 0$, the mass m_1 is at the equilibrium position and the speed is v_0, or another set of 2 conditions. The problem does not set any particular conditions, so we do not need to set them either, but we know what we would do in case we had some.