PH1a: rotational dynamics II

Additional set of exercises

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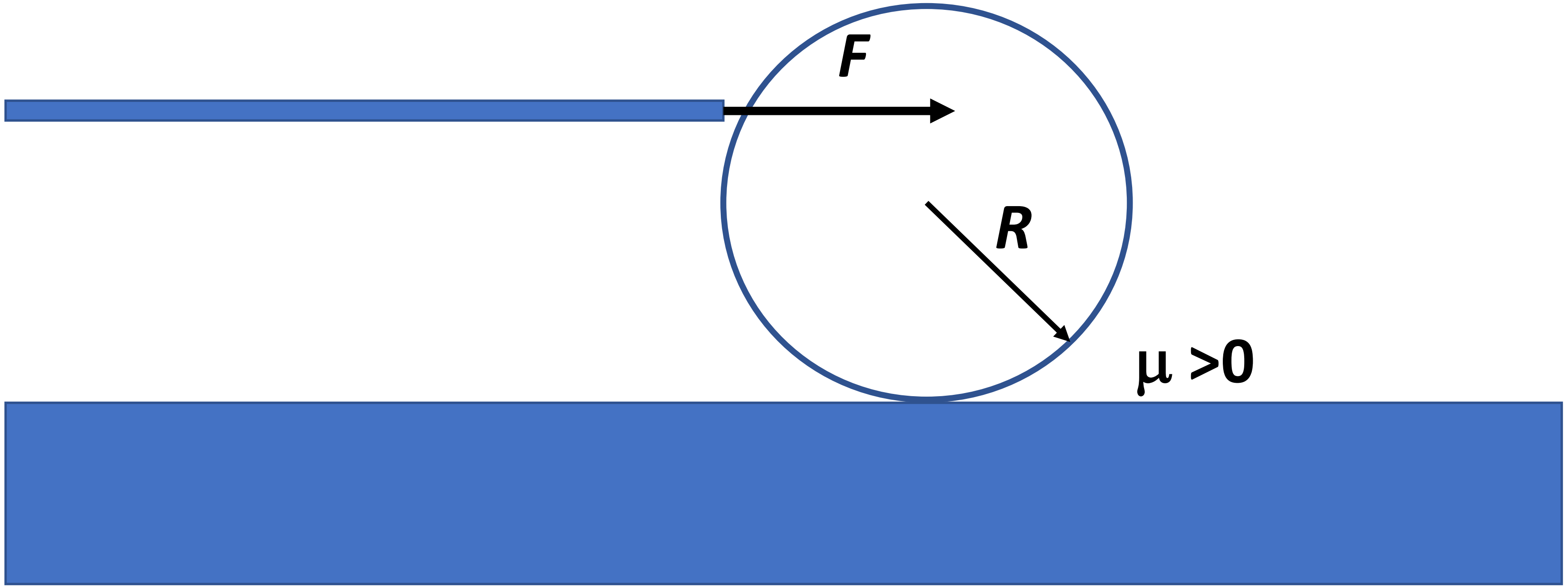
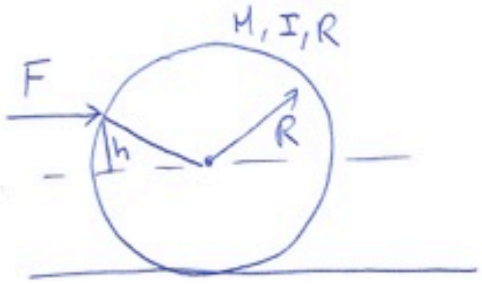
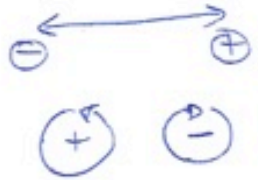
inspired from Prof. Harvey
Newman’s exercises, section 10.

Sweet spot of a billiard ball

What’s the height we have to hit the ball at? To make it roll without slipping immediately after the kick.







We want no friction, so:

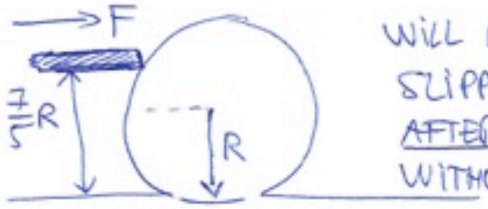
$$\left.\begin{aligned} F &= M\cdot a \\ -Fh &= I\alpha \\ a &= -\alpha R \end{aligned}\right\} \quad \begin{aligned} F &= Ma \\ Fh &= Ia/R \end{aligned}$$

$(\alpha < 0 \circlearrowright, \Rightarrow a \oplus)$

$$\Rightarrow (F \neq 0) \Rightarrow \boxed{h = \frac{I}{MR}}$$

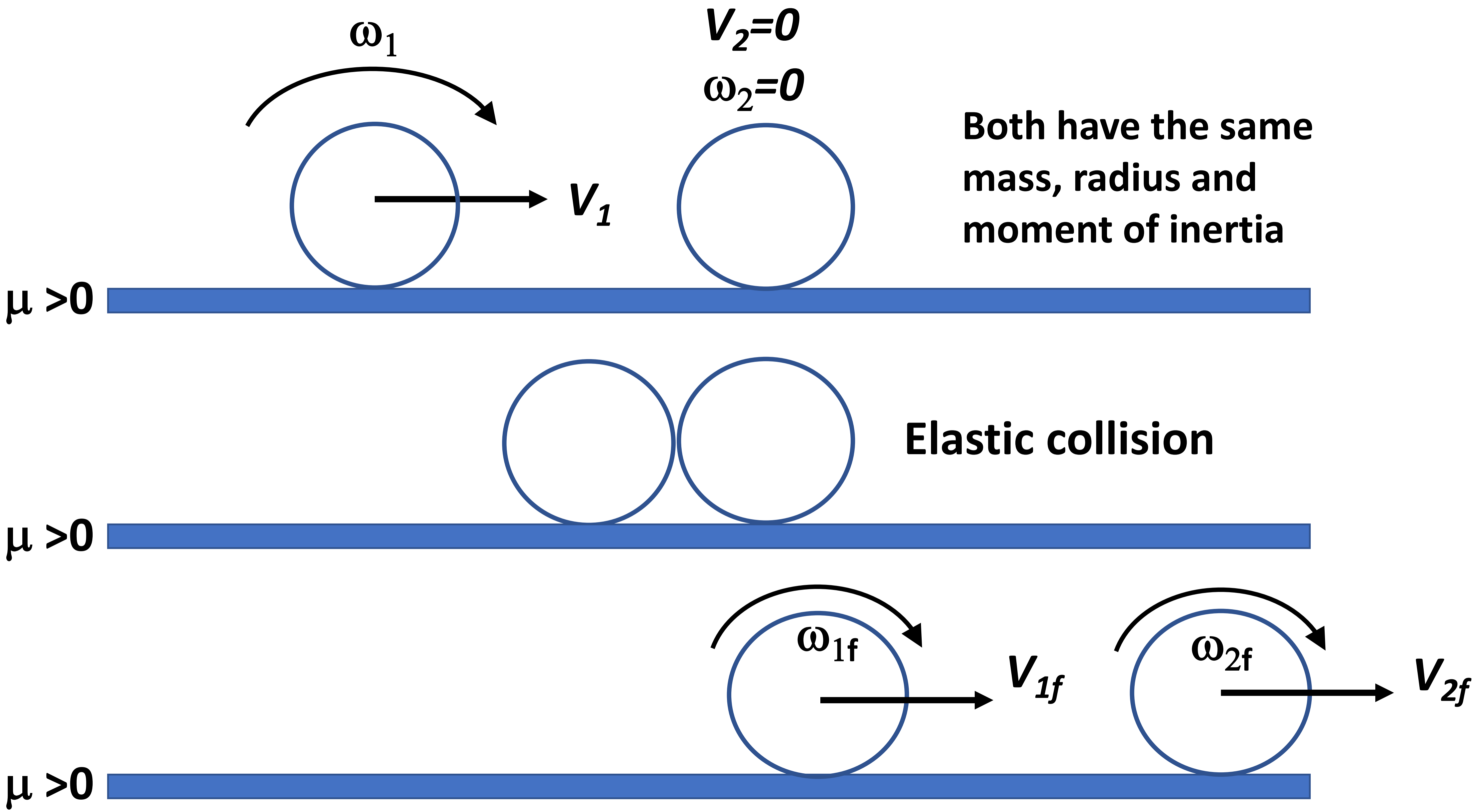
For a sphere (billiard ball):

$$h = \frac{2/5\, MR^2}{MR} = 2/5\, R \Rightarrow$$



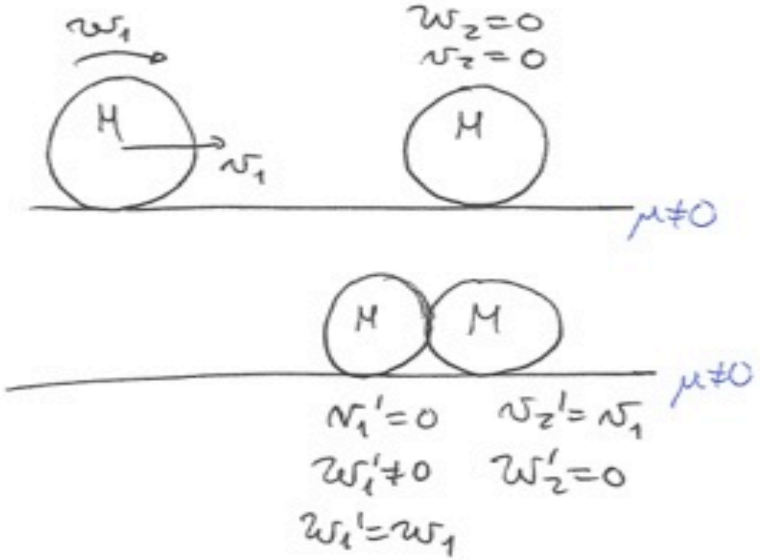
WILL ROLL WITHOUT SLIPPING RIGHT AFTER THE STRIKE, WITHOUT BEING SLOWED DOWN.

Collision between two identical billiard balls



Both have the same mass, radius and moment of inertia

Elastic collision



This is an interesting problem.

Since the collision is elastic, we have that linear momentum and kinetic energy are conserved.

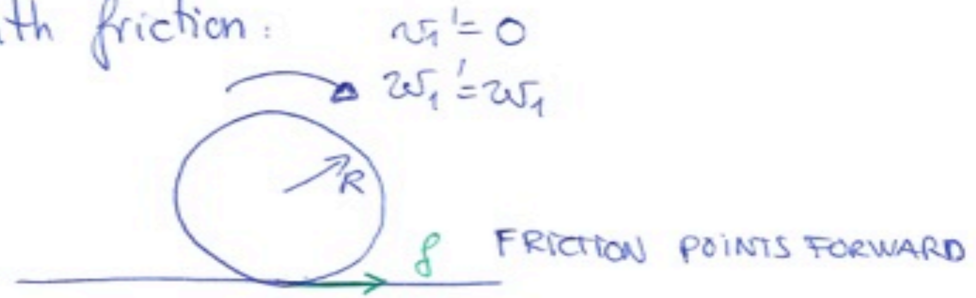
We already gave the general solution for a 1-D elastic collision. In this case, there's so much symmetry that it is clear that the speeds of the CM after the collision are $v_1' = 0$ (stops!) and $v_2' = v_1$ (just gets the speed of the incoming ball 1. Recall the two billiard balls are identical.

On the other hand, during the collision, the internal forces do not create any torque, since they are directed along the line that joins both CM. That implies that angular momentum is conserved during the collision. Therefore, the angular speed of each ball does <u>not</u> change.

Immediately after the collision, however, friction will change the speeds, both linear and angular. It is important to notice, though, that without friction, the first ball would keep spinning without moving, and the second ball would move forward without rolling.

Now, with friction:

1st Ball:



Notice that we can't apply the condition of rolling without slipping. The ball slips. Therefore, we have:

Newton: $f = M a_{CM}$

Torque: $f R = I \alpha$;

a_{CM} will increase the speed from $\emptyset$ to some value, while α will decrease the angular speed. Once "$v = \omega R$", the ball will continue to move, rolling without slipping.

$$\boxed{v_{1f}} = v_{10} + a_1 \Delta t = 0 + \frac{f}{M}\Delta t = \boxed{\frac{f}{M}\Delta t}$$

$$\boxed{\omega_{1f}} = \omega_{10} + \alpha_1 \Delta t = \boxed{-|\omega_1| + \frac{fR}{I}\Delta t}$$

Notice how I have made it clear that the ball begins with some clockwise angular speed and, as time goes by, it becomes less negative, $\frac{fR}{I}\Delta t > 0$. It slows down.

$v_{1f} = -\omega_{1f} R$ (↻ → Moves to the right, which is ⊕)

Initially: $v_1 = |\omega_1| R$

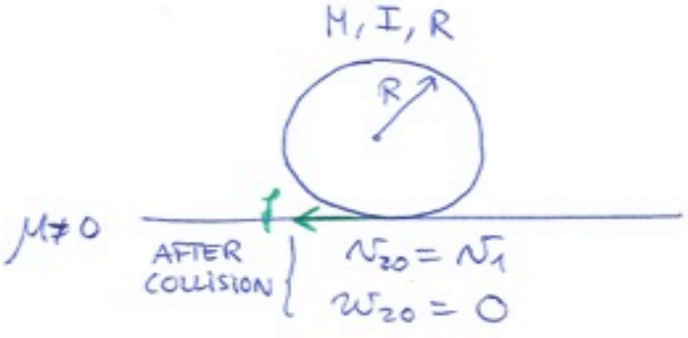
$$\rightarrow \quad \frac{f}{M}\Delta t = |\omega_1| R - \frac{fR^2}{I}\Delta t \Rightarrow \boxed{f\Delta t} = \frac{|\omega_1| R}{\frac{1}{M} + \frac{R^2}{I}} \stackrel{\checkmark}{=} \boxed{\frac{M v_1}{1 + \frac{MR^2}{I}}}$$

UNITS Make sense ($f\Delta t$ = Force $\cdot \Delta t = \Delta p$ = change in linear momentum).

Finally, the linear speed and angular speed after the time it takes to roll again without slipping are:

$$\boxed{v_{1f} = \frac{f}{M}\Delta t = \frac{v_1}{1 + \frac{MR^2}{I}}} \quad \text{and} \quad \boxed{\omega_{1f} = \frac{\omega_1}{1 + \frac{MR^2}{I}}}$$

For the second ball, we have:



The ball slides towards the right side. Therefore, friction goes to the left. The equations and method are similar as in the previous case. Beware of the signs.

$$v_{2f} = v_1 - \frac{f}{M}\Delta t$$

$$w_{2f} = \alpha \Delta t = -\frac{fR}{I}\Delta t$$

$$\alpha = -f\frac{R}{I}$$

$v_{2f} = -w_2 R \Rightarrow v_1 - \frac{f}{M}\Delta t = f\frac{R^2}{I}\Delta t$

NO SLIPPING

$\Rightarrow f\Delta t = \dfrac{M v_1}{\left(1 + \frac{MR^2}{I}\right)}$. Notice that $f\Delta t$ is the same as before, but the friction force cancels each other when considered overall.

This means that the total $\vec{P}$ is conserved even after the collision. Substituting the expression:

$$\boxed{v_{2f} = v_1 - \frac{M v_1}{\left(1 + \frac{MR^2}{I}\right)M} = \frac{MR^2}{I} \cdot \frac{v_1}{1 + \frac{MR^2}{I}} = \frac{v_1}{1 + I/MR^2}}$$

and

$$\boxed{w_{2f} = \frac{w_1}{1 + \frac{I}{MR^2}}}$$

($w_{2f} = -v_{2f}/R$, $(v_1/R = -w_1)$)

Notice how:

$$M v_1 + M \cdot 0 = M \frac{v_1}{1+\frac{MR^2}{I}} + M \frac{v_1}{1+\frac{I}{MR^2}} = M v_1 \frac{1+\frac{MR^2}{I}+1+\frac{I}{MR^2}}{\left(1+\frac{MR^2}{I}\right)\left(1+\frac{I}{MR^2}\right)} = M v_1 \frac{2+\frac{MR^2}{I}+\frac{I}{MR^2}}{2+\frac{MR^2}{I}+\frac{I}{MR^2}}$$

$$= M v_1 \,!$$

So $\vec{P}_I = \vec{P}_F$, even if the balls were <u>sliding</u> and friction was acting.

On the other hand, $L_I = L_F$ is also satisfied.

$$L_I = I \cdot \omega_1 \quad \text{and} \quad L_F = I\omega_{1f} + I\omega_{2f} = I(\omega_{1f}+\omega_{2f})$$

$$= I\omega_1 \left(\frac{1}{1+\frac{MR^2}{I}} + \frac{1}{1+\frac{I}{MR^2}} \right) = I\omega_1;$$

The reason is that friction acted accelerating/deaccelerating the 1st ball and deaccelerating/accelerating the 2nd ball in a way that all changes in $\vec{P}$ and $\vec{L}$ are conserved.

This is just a coincidence given the great symmetry of the problem. See the general solution at the end of these notes.

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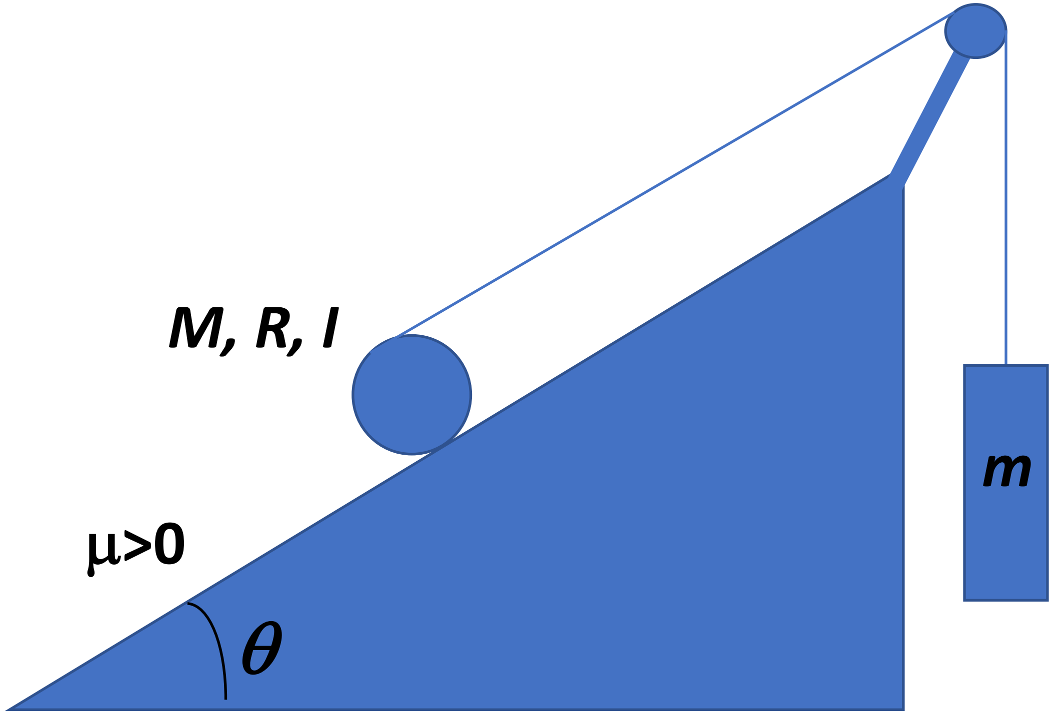
Incline with a disk and a pulley



Consider a fixed incline with friction. There's a yo-yo with its string going through a pulley, and the other extreme is attached to a body of mass *m.*

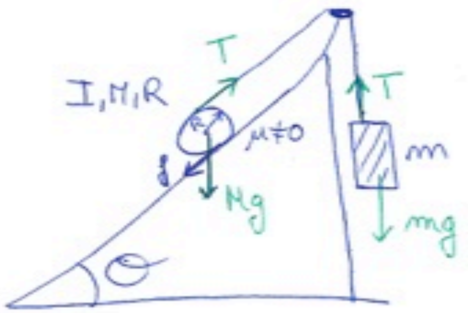
Neglect the mass of the string, the mass of the pulley, and its moment of inertia, and assume the mass *m* is falling. Find in terms of the variables of the problem:

1) The expression for the acceleration of the mass *m.*

2) The tension along the string, and

3) The force of friction.

Assume that the yo-yo rolls without slipping.



As the problem says, let's assume that the system is such that the block of mass m moves down. Then friction on the incline points as depicted (otherwise, one needs to change the sign of "f" and re-do the problem)

The equations of motion are: ⊖ ⊕ ... ⊖ ⊕ and ⊕ ⊖

$$T - Mg\sin\theta - f = Ma$$
$$-T + mg = ma$$
$$-(T+f)R = I\alpha$$
$$a = -\alpha R \quad (a \oplus \text{ when } \alpha \oplus \text{, which is } \ominus)$$

T, f, a, α : 4 unknowns & 4 linear equations.

$$\left.\begin{aligned} T - Mg\sin\theta - f &= Ma \quad (1) \\ -T + mg &= ma \quad (2) \\ T + f &= Ia/R^2 \quad (3) \end{aligned}\right\}$$

$\overset{(2)}{} T = m(g-a) \Rightarrow \overset{(3)}{} mg - ma + f = \frac{Ia}{R^2} \Rightarrow$

$mg + f = \left(m + \frac{I}{R^2}\right)a$; Now (1) with (2):

$mg - ma - Mg\sin\theta - f = Ma \Rightarrow$

$$(m - M\sin\theta)g - f = (M+m)a.$$

Summing up this equation with $mg + f = \left(m + \frac{I}{R^2}\right)a$, we get:

$$(2m - M\sin\theta)g = (2m + M + I/R^2)a \Rightarrow$$

$$\boxed{a = \frac{(2m - M\sin\theta)}{(2m + M + I/R^2)}\, g}$$

(NOTICE THAT THE DENOMINATOR HAS ALL TERMS ADDING UP: GOOD!)

The tension is thus:

$$T = m(g-a) = mg\left[1 - \frac{(2m - M\sin\theta)}{(2m + M + I/R^2)}\right] = mg\left[\frac{M(1+\sin\theta) + I/R^2}{2m + M + I/R^2}\right]$$

CHECKS: 1) Units are ok. 2) If no rotation $\rightarrow I = 0 \Rightarrow T = mg\,\frac{M(1+\sin\theta)}{(2m+M)}$, ... the factor $2m+M$ looks odd. We'll check it soon...

The force of friction is:

$$f = \left(m + \frac{I}{R^2}\right)a - mg = \left[\left(m + \frac{I}{R^2}\right)\left(\frac{2m - M\sin\theta}{2m + M + I/R^2}\right) - m\right]g =$$

$$= \frac{\left(2m^2 - Mm\sin\theta + 2mI/R^2 - M\sin\theta\, I/R^2 - 2m^2 - 2mM - mI/R^2\right)}{2m + M + I/R^2}\, g$$

$$\boxed{f = \frac{(m - M\sin\theta)I/R^2 - Mm(1+\sin\theta)}{2m + M + I/R^2}}$$

These expressions are difficult to check. The fact that rolling without slipping adds a constraint between the angular motion of the rolling object and its center of mass acceleration, allows us to determine the force of friction without use of the Normal force. That makes the expressions different to the ones we are more accustomed to with $f = \mu N$.

In order to be consistent, $f \geq 0$, since we already set its sign in the equations. Therefore:

$$f \geq 0 \iff m - M\sin\theta \geq \left(\frac{MR^2}{I}\right) m(1+\sin\theta), \text{ and}$$

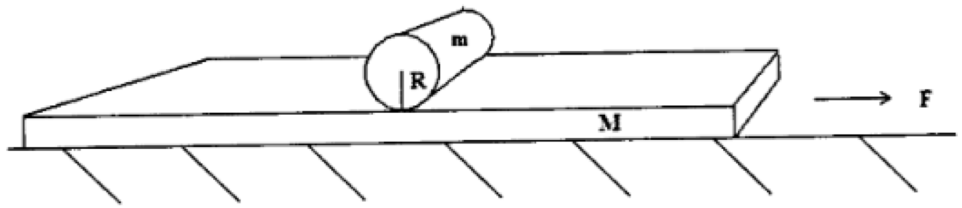
$$f \leq \mu_s N = \mu_s M g \cos\theta$$

Otherwise the system would slip and the solution would be different.

FP23

Rolling on moving surfaces

Roll Out the Barrel



As depicted in the diagram above, a large flat board, sitting on the ground, of mass M is pulled with a force F, while a cylinder of mass m, radius R and moment of inertia $I = (1/2)mR^2$ rolls without slipping on the board. Assume there is friction between the board and the cylinder to ensure rolling without slipping, but for simplicity, assume there is no friction between the board and the ground. In addition, assume that the board is sufficiently long that the cylinder remains on the board for the problem. Finally, for calculations in this problem, define a to be the acceleration of the board with respect to the ground, a_1 to be the acceleration of the centre of mass of the cylinder *with respect to the accelerating board*, and a_2 to be the acceleration of the cylinder *with respect to the ground.*

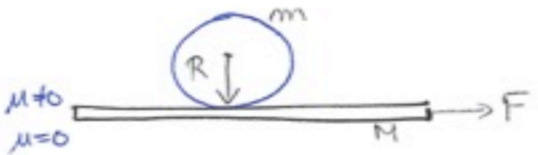
(2 points) (a) Draw free body diagrams for the cylinder and the board.

(3 points) (b) Write down Newton's equations (listing all relevant forces) for the linear motion of the board and cylinder. In addition, write down an expression describing the rotational motion of the cylinder. Finally, find a constraint associated with rolling without slipping. For each of the equations, list which frame of reference it is in.

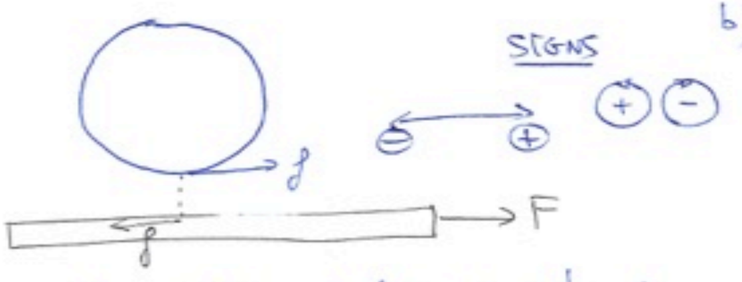
(1 point) (c) Does the cylinder roll clockwise or counter-clockwise?

(2 points) (d) Solve the equations you found for a, in terms of F, M, and m.

(2 points) (e) Solve the equations for a_1 and a_2, again in terms of F, M, and m.



a)



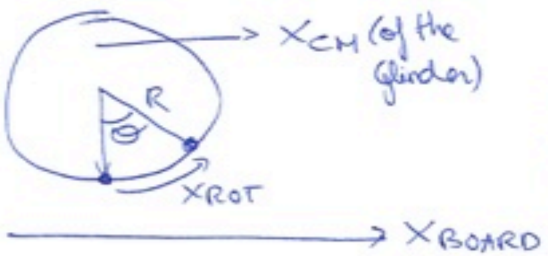
PS: weights and Normal forces do not play any relevant role in this problem, and I don't plot it, but it's fine if you add them to the plot.

b) $F - f = Ma$ (board)

$f = ma_2$ (cylinder with respect to ground)

$fR = I\alpha$ (torques with signs)

Rolling without slipping: In this problem, the surface in contact with the cylinder is moving (the board). Therefore, we need to take that into account. One way of approaching this question is by thinking about how the displacements of each element sum up:



$x_{BOARD} = x_{CM} + x_{ROT}$, so that there will be no slipping.

[Signs are fine: ⊕ to the right and $x_{ROT} = \theta \cdot R > 0$, $\theta > 0$ CCW]

Differentiating twice, we get:

$$a_{BOARD} = a_{CM} + a_{ROT} \quad \text{or} \quad \boxed{a = a_2 + \alpha R}$$

PS: notice that for non slipping in the standard case of the board being at rest, $x_{CM} + x_{ROT} = 0$. That is, the contact point moves forward due to the CM motion, but backwards due to the rotation. That is, if the board is not moving, and the object rotates without slipping $v_{CM} = -\omega R$. Beware that a different sign convention ends up with the typical $v_{CM} = \omega R$.

Another way of dealing with a moving surface is to just use our last relationship, without further derivation, but remember to use the same sign convention:

$$a_{SURFACE} = a_{CM,\ \text{ROLLING OBJECT WITHOUT SLIPPING}} + \alpha R$$

The equations of motion are:

$$F - f = Ma, \quad f = ma_2, \quad fR = I\alpha, \text{ and } \quad a = a_2 + \alpha R.$$

c) The cylinder will roll counter clock-wise.

d) $f = ma_2 \Rightarrow ma_2 R = I\alpha \Rightarrow \alpha = \left(\frac{mR}{I}\right) a_2 \Rightarrow a = a_2 + \left(\frac{mR^2}{I}\right) a_2.$

Summing up the first 2 equations (we get rid of the friction), we get:

$$F = Ma + ma_2 = \left\{M\left[1+\left(\frac{mR^2}{I}\right)\right] + m\right\} a_2$$

$$\Rightarrow \quad a_2 = \frac{F}{m + M\left[1+\left(\frac{mR^2}{I}\right)\right]} \; ; \quad \text{so that} \quad \boxed{a = \frac{\left[1+\left(\frac{mR^2}{I}\right)\right] F}{m + M\left[1+\left(\frac{mR^2}{I}\right)\right]}}$$

(PS: NOTICE the ⊕ sign in the denominator → GOOD!)

For the case of a cylinder: $I = \frac{1}{2} mR^2 \Rightarrow \frac{mR^2}{I} = 2$.

$$\Rightarrow \boxed{a = \frac{3F}{m+3M}}$$

e) $$\boxed{a_2 = \frac{F}{m+3M}}$$

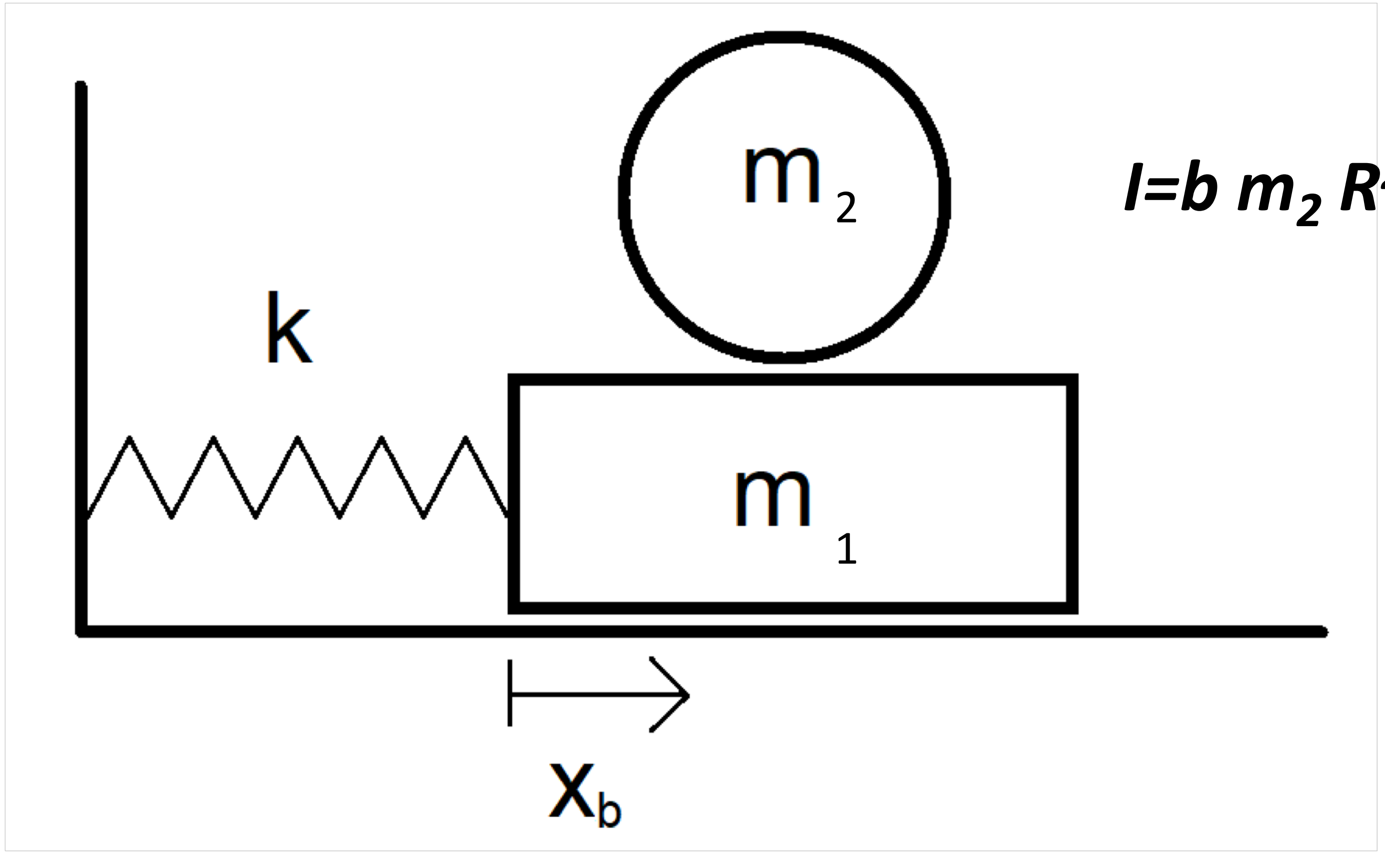
a_1 is the acceleration of the CM of the cylinder with respect to the board. We subtract a to the CM motion:

$$a_1 = a_2 - a = \frac{-2F}{(m+3M)}.$$

Notice that $a_1 = -\alpha R$, the rolling part.

Rolling on moving surfaces: oscillating surface

Find the period of the oscillations of the object on top of the oscillating block (assume the surface of the oscillating block is long enough, so that the rolling object does not fall off)



$I = b\, m_2 R^2, \quad 0 < b < 1$

Solution with rotational dynamics and energy

Notice that we cannot impose ***R*α$_s$ =*a*$_s$** because the contact point is also in motion. See equation (3) later on.

1. Using $I\alpha_s = Rf$ and $ma_s = f$ we get $I\alpha_s = Rma_s$ or
$$\frac{2}{5}R\alpha_s = a_s. \quad \text{(For a sphere)} \tag{1}$$

2. Integrating and remembering that the system starts from rest (ω_s and v_s should be zero initially) we see that
$$\frac{2}{5}R\omega_s = v_s. \tag{2}$$

3. The velocity of the point of contact is v_b so
$$v_b = R\omega_s + v_s \tag{3}$$
since the sphere doesn't slip.

4. The energy is
$$E = \frac{1}{2}mv_b^2 + \frac{1}{2}kx_b^2 + \frac{1}{2}I\omega_s^2 + \frac{1}{2}mv_s^2 \tag{4}$$
$$= \frac{1}{2}\left(\frac{9}{7}\right)mv_b^2 + \frac{1}{2}kx_b^2 \tag{5}$$
after using our expressions above and doing a little algebra.

5. The frequency of oscillation is
$$\omega_0^2 = \frac{7}{9}\frac{k}{m}. \tag{6}$$

Solution with rotational dynamics only: step by step solution

1. The system as a whole: the friction forces cancel each other. Let's call x_1 the initial displacement of the block from equilibrium. Let's call a_1 the acceleration of the (center of mass of the) block, and a_2 the acceleration of the (center of mass of the) rotating object, one can write:

$$m_1 a_1 + m_2 a_2 = -k x_1.$$

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2. The force on the rotating object: f is the force of friction that points to the left (think about the same problem without rotation)

$$-f = m_2 a_2.$$

3. Torques:

$$-fR = I\alpha$$

4. Rolling condition: the displacement at the contact point must satisfy (x_2 is the displacement of the center of the mass of the rotating object, and θ is the angle rotated from the equilibrium position. We use the sign conventions defined above.)

$$x_1 = x_2 + R\theta.$$

Differentiating twice we get:

$$a_1 = a_2 + R\alpha.$$

5. In order to find the solution, combine previous equations by noticing that Hooke's law applies to x_1, so that a_1 is the preferred acceleration to solve first:

$$-fR^2 = I\alpha R \Rightarrow m_2 a_2 R^2 = I(a_1 - a_2)$$

$$a_2 = \frac{Ia_1}{m_2R^2 + I} = \frac{b}{1+b}a_1 \Rightarrow m_1 a_1 + \frac{m_2 b}{1+b}a_1 = -kx_1$$

$$a_1 = -\omega_1^2 x_1, \quad \text{with} \quad \omega_1^2 = \frac{1+b}{(m_1+m_2)b + m_1}k = \frac{(1+b)m_1}{(m_1+m_2)b+m_1}\tilde{\omega}^2,$$

where $\tilde{\omega} = \sqrt{k/m_1}$ is the angular frequency of oscillations if there was no object on top of the block. Notice that a_2 is also a simple harmonic motion with the *same* frequency ω_1, because is directly proportional to a_1. Similarly, the rotation of the second object is a simple harmonic motion with the same frequency. This is a consequence of the system being *linear*.

Using the particular case $m_1 = m_2$ and $I = \frac{2}{5}mR^2$, or $b = \frac{2}{5}$, we have:

$$\omega_1 = \sqrt{\frac{1+2/5}{1+2*2/5}}\tilde{\omega} = \sqrt{\frac{7}{9}}\tilde{\omega}.$$

It's interesting to notice that the system behaves as a single body attached to a spring with constant k and with an effective mass given by:

$$m_{\text{eff}} = m_1 + \frac{b}{1+b}m_2.$$

Let's finally remark that we have assumed that the static friction will always be as large as necessary to keep the bodies together without slipping. One could use the expression of the maximum acceleration of an harmonic motion to set a constraint on the maximum initial amplitude x_1 given some μ_{static} or some other questions in a new version of this problem.

From the previous expressions, one can easily integrate a_1 to find $x_1(t)$ and then $x_2(t)$ and $\theta(t)$ imposing the initial conditions of the problem. Let's get them for the case $m_1 = m_2 = m$.

From $a_1 = -\omega_1^2 x_1$, we get

$$x_1 = x_b \cos \omega_1 t,$$

where we have imposed $x_1(t=0) = x_b, v_1(t=0) = 0$. For a_2 we obtained $a_2 = b a_1/(1+b)$, so that

$$a_2 = -\frac{b}{1+b}\omega_1^2 x_b \cos \omega_1 t,$$
$$v_2 = -\frac{b}{1+b}\omega_1 x_b \sin \omega_1 t,$$
$$x_2 = \frac{x_b}{1+b}\left(1 + b\cos \omega_1 t\right),$$

where we have imposed $x_2(t=0) = x_b, v_2(t=0) = 0$. PS: notice that $x_{2,\mathrm{min}} = x_b(1-b)/(1+b)$ and $x_{2,\mathrm{max}} = x_b$.

Now, the time solution for θ (recall $x_1 = x_2 + R\theta$) is

$$\theta = \frac{x_1 - x_2}{R} = -\frac{x_b}{(1+b)x_b}\left(1 - \cos\omega_1 t\right).$$

Notice that $\theta \leq 0$! That is the rotation is contrary to the motion of x_2 (this can be seen pulling a sheet of paper with a pencil on top of it). Also, $\theta_{\max} = 0$, and $\theta_{\min} = -\frac{2x_b}{R(1+b)}$. The latter implies that the circular object does N turns per cycle while rolling on top of the block, with $N = E[\frac{x_b}{\pi R(1+b)}]$, where $E(x)$ is the integer part of x.

The center of mass satisfies (recall we are considering $m_1 = m_2$):

$$x_{\mathrm{CM}} = \frac{x_1 + x_2}{2} = \frac{x_b}{2(1+b)}\left[1 + (1+2b)\cos\omega_1 t\right].$$

We have $x_{\mathrm{CM}}(t=0) = x_b$ and $v_{\mathrm{CM}} = 0$, as expected. The acceleration of the center of mass is

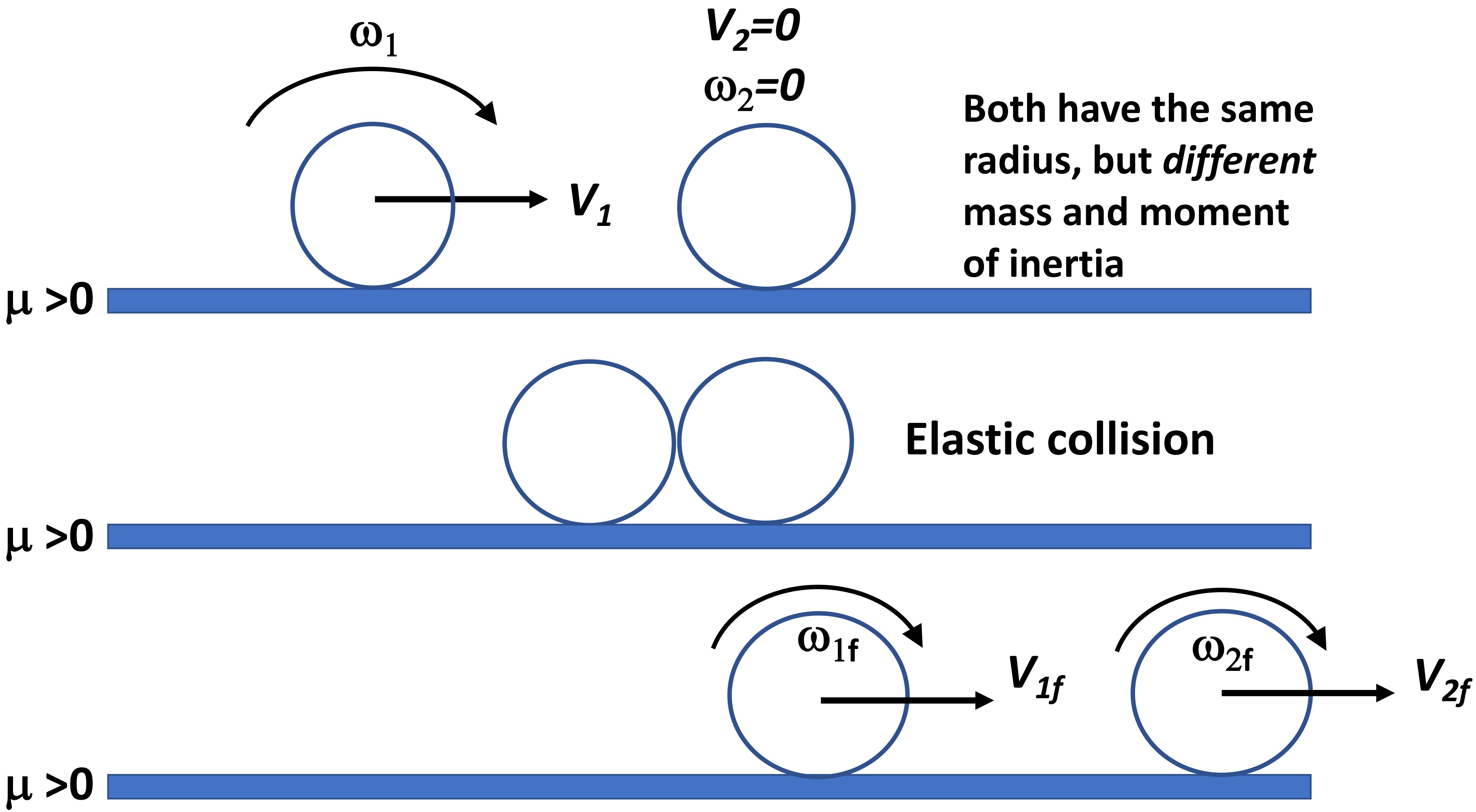
$$a_{\rm CM} = -\frac{x_b(1+2b)\omega_1^2}{2(1+b)}\cos\omega_1 t,$$

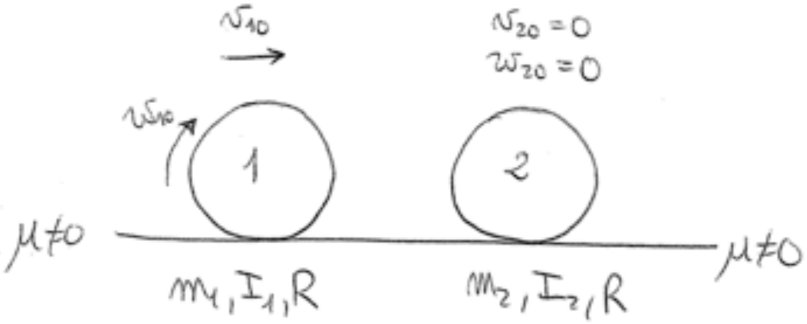
$$\text{(using Eq. 16)}\quad \cos\omega_1 t = \frac{2(1+b)x_{\rm CM} - x_b}{x_b(1+2b)},$$

$$a_{\rm CM} = \frac{x_b\omega_1^2}{2(1+b)} - \omega_1^2 x_{\rm CM}.$$

This shows that the center of mass follows a simple harmonic motion with frequency ω_1, and a shifted equilibrium position. That is, $a_{\rm CM}(t=0) = -\frac{x_b(1+2b)\omega_1^2}{2(1+b)} \neq -\omega_1^2 x_b$, in fact, $\forall b$.

General solution for two balls of equal radius colliding





Two balls of masses m_1, m_2 with moment of inertia I_1, I_2 and same radius, R, collide elastically as shown in the picture. The first ball rolls without slipping before the collision. Some time after the collision both balls roll without slipping again. Derive the change of angular momentum and linear momentum between the time before the collision and after the collision when both balls roll without slipping again.

1) Let's first derive the final speeds after the collision: (general formula for elastic collisions. See my PDF notes)

$$\boxed{v_1'} = \frac{(m_1-m_2)}{(m_1+m_2)} v_{10} + \frac{2m_2}{(m_1+m_2)} \cdot 0 = \boxed{\frac{(m_1-m_2)}{(m_1+m_2)} v_{10}}$$

If neither m_1 nor m_2 are 0, which we can assume, $|v_1'|<v_{10}$

$$\boxed{v_2'} = \frac{(m_1-m_2)}{(m_1+m_2)} v_{20} + \frac{2m_1}{(m_1+m_2)} v_{10} = \boxed{\frac{2m_1 v_{10}}{(m_1+m_2)}}$$

Notice that during the collision, the angular speed of each ball does not change. The torques of the contact forces are zero since they are directed towards the center of each ball. So, even if it is an elastic collision, only the translational speeds can change. The equation to solve during the collision is the same as if it were for two blocks.

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Right after the collision, $\omega_1' = \omega_{10}$ and $\omega_2' = 0$, the same as before the collision, because there's no torque due to the elastic forces during the collision.

After the collision, friction slows down the angular rotation of ball #1 and increases the angular rotation of the #2. The balls slide until they reach the condition of rolling without slipping. Let's find $v_{1f}, v_{2f}, \omega_{1f}$ & ω_{2f}.

Let's call f_1 and f_2 the force of friction acting on each ball, respectively. Then, since it is a constant force, we have:

$$\left.\begin{aligned} v_{1f} &= v_1' + \frac{f_1 \Delta t_1}{m_1} \\ \omega_{1f} &= -|\omega_{10}| + \frac{f_1 R \Delta t_1}{I_1} \end{aligned}\right\} \quad \text{NON-SLIPPING @ THE END}: \; v_{1f} = -\omega_{1f} R$$

$$\Rightarrow v_1' + f_1 \frac{\Delta t_1}{m_1} = |\omega_{10}|R - \frac{f_1 R^2}{I_1}\Delta t_1 \Rightarrow f_1 \Delta t_1 \left(\frac{1}{m_1} + \frac{R^2}{I_1} \right) = -v_1' + |\omega_{10}|R.$$

Let's use this notation: $I_1 = m_1 R^2 b_1$ and $I_2 = m_2 R^2 b_2$; clearly $0 \leq b_1 \leq 1, 0 \leq b_2 \leq 1$.

$$\Rightarrow \quad f_1 \Delta t_1 = \frac{m_1 b_1}{(1+b_1)} \left(-v_1' + |\omega_{10}| R\right) = \frac{m_1 b_1}{1+b_1} \left(v_{10} - v_1'\right).$$

Therefore:

$$\boxed{v_{1f} = v_1' + \frac{f_1 \Delta t_1}{m_1} = v_1' + \frac{b_1}{(1+b_1)} \left(v_{10} - v_1'\right) = \frac{v_1' + b_1 v_{10}}{1+b_1}}$$

Now, for the second ball, we have:

$$\left.\begin{aligned} v_{2f} &= v_{2'} - \frac{f_2}{m_2} \Delta t_2 \\ \omega_{2f} &= 0 + \frac{f_2 R \Delta t_2}{I_2} \end{aligned}\right\} v_2 = \omega_{2f} R \Rightarrow \quad v_2' - \frac{f_2 \Delta t_2}{m_2} = \frac{f_2 R^2 \Delta t_2}{I_2} \Rightarrow$$

$$f_2 \Delta t_2 = \frac{v_2'}{\frac{1}{m_2} + \frac{1}{m_2 b_2}} = \frac{m_2 b_2 v_2'}{1+b_2}, \text{ and}$$

$$\boxed{v_{2f} = v_2' - \frac{b_2 v_2'}{1+b_2} = \frac{v_2'}{1+b_2}} \; ;$$

Let's now write the angular momentum once the balls roll without slipping and compare it with the beginning.

$$L_f = I_1 \omega_{1f} + I_2 \omega_{2f} = \frac{I_1}{R} v_{1f} + \frac{I_2}{R} v_{2f} = (m_1 b_1 v_{1f} + m_2 b_2 v_{2f})R.$$

And $L_0 = (m_1 b_1 v_{10})R$. Therefore, if $L_f \stackrel{?}{=} L_0$, we must have:

$$m_1 b_1 v_{1f} + m_2 b_2 v_{2f} \stackrel{?}{=} m_1 b_1 v_{10}.$$

Substituting previous results:

$$\frac{m_1 b_1 (v_1' + b_1 v_{10})}{1+b_1} + \frac{m_2 b_2 v_2'}{(1+b_2)} \stackrel{?}{=} m_1 b_1 v_{10} \Rightarrow$$

$$\frac{m_1 b_1 v_1'}{1+b_1} + \frac{m_2 b_2 v_2'}{1+b_2} \stackrel{?}{=} m_1 b_1 v_{10}\left(1 - \frac{b_1}{1+b_1}\right) = \frac{m_1 b_1 v_{10}}{1+b_1}$$

Now, we got that for an elastic collision $v_1' = \frac{(m_1 - m_2)}{(m_1+m_2)} v_{10}$ and $v_2' = \frac{2 m_1 v_{10}}{(m_1+m_2)}$,

so: $$\frac{m_1 b_1}{(1+b_1)}\left(\frac{m_1 - m_2}{m_1+m_2}\right) v_{10} + \frac{m_2 b_2}{1+b_2} \frac{2 m_1 v_{10}}{(m_1+m_2)} \stackrel{?}{=} \frac{m_1 b_1}{(1+b_1)} v_{10}$$

$$(v_{10} \neq 0) \Rightarrow \frac{m_1 b_1}{1+b_1}(m_1 - m_2) + \frac{m_2 b_2}{1+b_2}(2 m_1) \stackrel{?}{=} \frac{m_1 b_1}{1+b_1}(m_1 + m_2) \Rightarrow$$

$$2m_1m_2\left(\frac{b_2}{1+b_2}\right) \stackrel{?}{=} 2m_1m_2\left(\frac{b_1}{1+b_1}\right) \Rightarrow \frac{b_2}{1+b_2} = \frac{b_1}{1+b_1}$$

$$\Rightarrow b_1 \stackrel{?}{=} b_2 \quad \text{or} \quad \frac{I_1}{m_1} = \frac{I_2}{m_2}.$$

Therefore, $L_f = L_0$, if and only if $\frac{I_1}{m_1} = \frac{I_2}{m_2}$, which does not hold in general. For two identical balls, it does.

PS: checking $\vec{P}_F = \vec{P}_I$ (I = Before collision, F = after collision, once they roll without slipping – so friction acted upon), gives a similar equation:

$$m_1v_{1f} + m_2v_{2f} \stackrel{?}{=} m_1v_{10} = \dots \Rightarrow \frac{b_1}{m_1} = \frac{b_2}{m_2}$$ and that means that $\vec{P}_F = \vec{P}_0$ if and only if $\frac{I_1}{m_1^2} = \frac{I_2}{m_2^2}$ $\left(I_1 = m_1b_1R^2 \;\&\; I_2 = m_2b_2R^2\right)$.

For $\vec{P}_F = \vec{P}_0$ & $L_F = L_I$ iff ("if and only if") $m_1 = m_2$ & $I_1 = I_2 \Rightarrow$ Iff IDENTICAL BALLS

Very particular case.