PH1a: rotational dynamics

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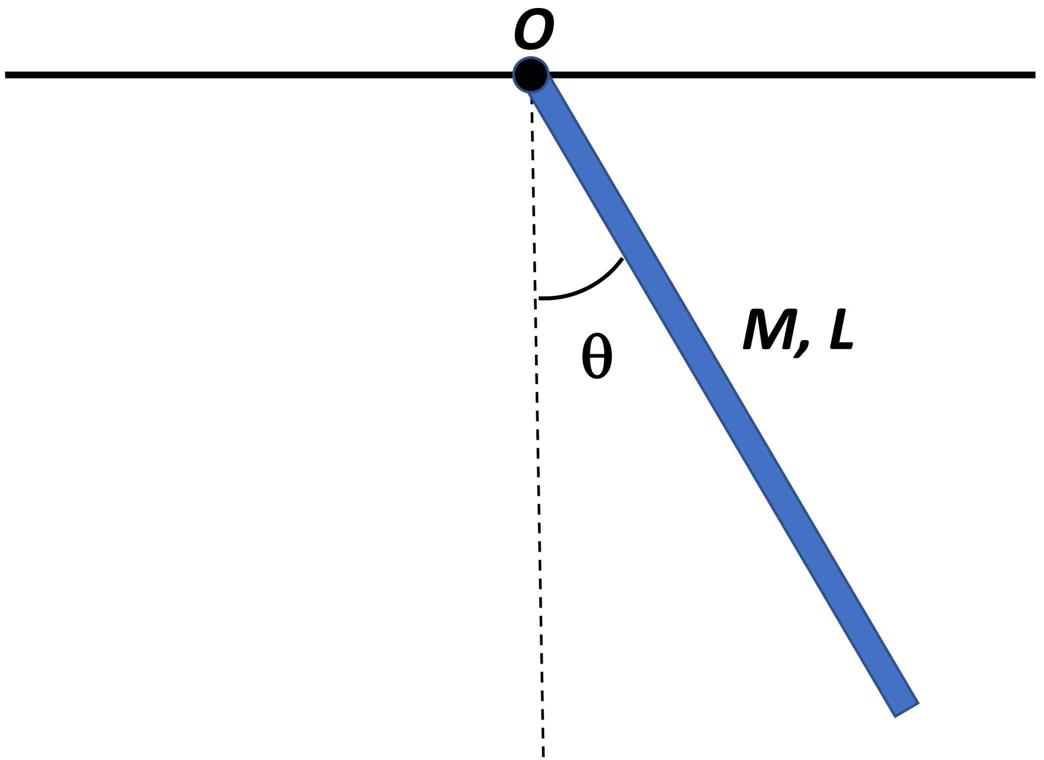
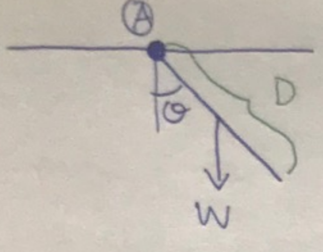
PH1a: rotational dynamics. First example

A homogeneous, thin bar is hanging from the ceiling and can freely rotate about the point ***O***.



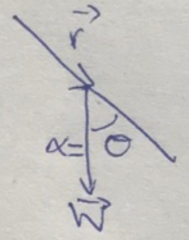
a) Find the angular acceleration of the bar as a function of θ

b) Find the period of the oscillations for small values of θ



The only force that creates a torque about Ⓐ is the weight of the bar. The weight of the bar is present along the bar. However, we can make use of the result that the sum of all contributions is equivalent as having <u>all</u> the mass of the bar at its Center of Mass. We are not giving a proof of this statement, though (I will add it at the end).

$$\tau = \frac{D}{2} \cdot W \cdot \sin\alpha$$



$$= D/2 \, Mg \sin\theta .$$

Which sign? ↻(+) ↺(−) (standard convention) ⇒

$$\boxed{\tau = -\frac{MgD}{2}\sin\theta}$$

I'll say something else about the sign in a moment.

Now, for small oscillations, $|\theta| << 1$, we have $\sin\theta \simeq \theta$ ⇒

$$\boxed{\tau \simeq -Mg\frac{D}{2}\theta}$$

On the other hand $\Sigma\tau = I\cdot\alpha$, where α is the second derivative of θ with respect to time:

$$\alpha = \frac{d^2\theta}{dt^2}.$$

Therefore:

$I_A = \frac{1}{3}MD^2$ (notice it is about Ⓐ, not about the C.M.)

$$\boxed{\frac{d^2\theta}{dt^2}} = -\frac{MgD}{2I}\theta \overset{\downarrow}{=} \boxed{-\frac{3}{2}\frac{g}{D}\theta}$$

CHECKS:

1) UNITS: Left: rad/s^2. Right: $\frac{m}{s^2 \cdot m}\cdot rad = rad/s^2$ OK.

2) The sign is ⊖. This means that when $\theta > 0$, $\frac{d^2\theta}{dt^2} < 0$, it makes it smaller → comes to the vertical (good). Once it crosses the vertical, $\theta < 0$ and $\frac{d^2\theta}{dt^2} = -\frac{3g}{2D}\theta > 0 \Rightarrow$ Makes it greater $\Rightarrow$ Closer to the vertical.

So we get oscillations: GOOD.

On the other hand, the equation for a spring in 1-D can be written as:

$$F = -Kx \rightarrow m\frac{d^2x}{dt^2} = -Kx \Rightarrow \frac{d^2x}{dt^2} = -\frac{K}{m}x.$$

From the results on the spring motion, we know that the period of the oscillations is given by:

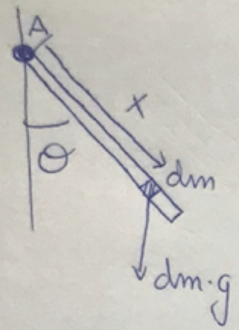
$$T = 2\pi\sqrt{\frac{m}{K}}$$

Comparing $\frac{d^2x}{dt^2} = -\frac{K}{m}x$ with $\frac{d^2\theta}{dt^2} = -\frac{3g}{2D}\theta$, we can identify θ as "x" and $\frac{3g}{2D}$ as "$\frac{K}{m}$"; therefore:

$$\boxed{T = 2\pi\sqrt{\frac{2D}{3g}}}$$

(units check out well)

Finally, let's prove that $\tau = -Mg\cdot\frac{D}{2}\sin\theta$ for the extended bar:



Every element of the bar creates a small torque given by:

$$d\tau = -x \cdot dm \cdot g \sin\theta$$

The total torque is, thus:

$$\tau = \int_0^D d\tau = -g\sin\theta \int_0^D x\,dm ;$$

The C.M. is by definition: $x_{CM} = \dfrac{\int_0^D x\,dm}{\int_0^D dm} = \dfrac{\int_0^D x\,dm}{M_{BAR}}$.

Therefore:

$$\Rightarrow \boxed{\tau = -g\sin\theta\, M_{BAR} \cdot x_{CM} = -x_{CM} \cdot W_{BAR} \sin\theta}$$

Notice that this proof does not make use of whether the bar is homogeneous or not (that is, whether the mass is distributed uniformly or not along the bar).

In summary, we have shown that the macroscopic object, the bar, can be treated as a point mass, with all its mass at the location of its CM (for torques)

Second example

14.12 ROLLING DOWN AN INCLINED PLANE

The rolling of a cylinder, sphere, or other symmetric object down a rough inclined plane (Fig. 14.14) is conveniently analyzed as an acceleration of the center of mass C along the plane and a simultaneous rotation of the object about C.

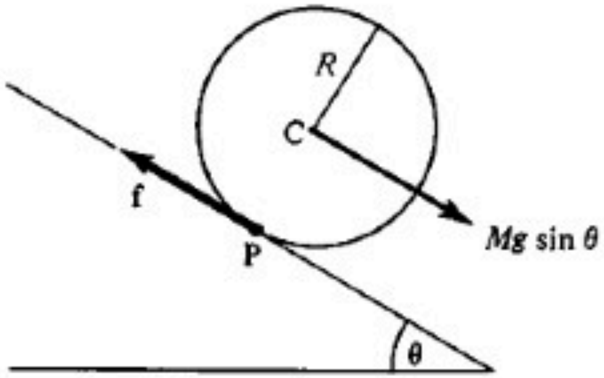


Figure 14.14 Rolling down an inclined plane.

What's the CM acceleration and the force of friction?

Second example

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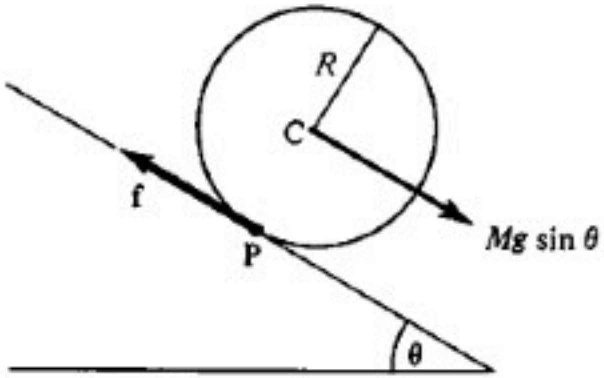

Figure 14.14 Rolling down an inclined plane.

$$Ma_C = Mg\sin\theta - f.$$

$$fR = I\alpha.$$

We impose $\alpha = a_C/R$ (no slipping):

$$f = \frac{Ia_C}{R^2}.$$

So that: $$Ma_C = Mg\sin\theta - \frac{Ia_C}{R^2}.$$

And: $$a_C = \frac{Mg\sin\theta}{M + I/R^2}.$$ Constant acceleration

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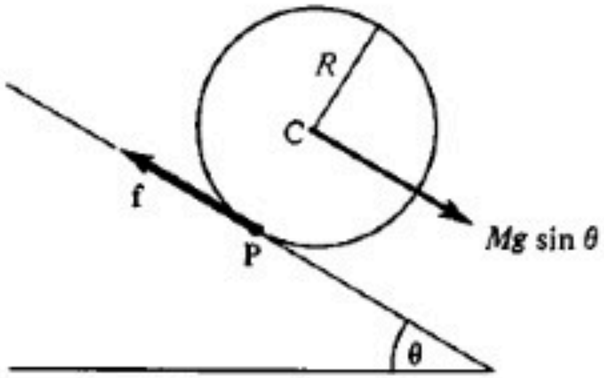

Figure 14.14 Rolling down an inclined plane.

What about the coefficient of friction?

$$f = \frac{Ia_C}{R^2}. \qquad a_C = \frac{Mg \sin\theta}{M + I/R^2}.$$

$$f \leq \mu N = \mu M g \cos\theta$$

Therefore, if $\mu < \mu_{min} = (I\, a_c / R^2)/N = \tan\theta/(1 + MR^2/I)$, the cylinder will roll and slip! And we can't set $\alpha = a_C/R$. Both α and a_C are `disconnected' if the object slips. We'll deal with all these situations when solving the problem of two billiard balls hitting each other later on.

Second example: with energy

14.12 ROLLING DOWN AN INCLINED PLANE

The rolling of a cylinder, sphere, or other symmetric object down a rough inclined plane (Fig. 14.14) is conveniently analyzed as an acceleration of the center of mass C along the plane and a simultaneous rotation of the object about C.

Let's assume no work from friction. And set $I=bMR^2$

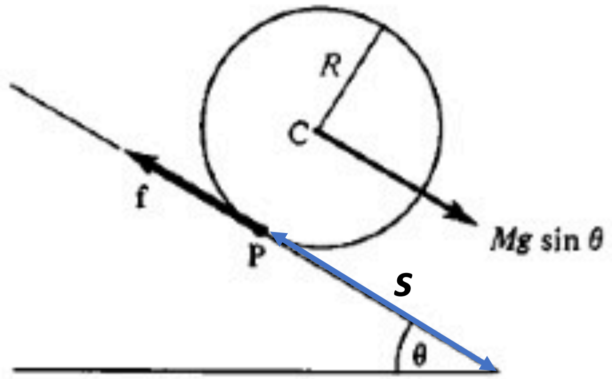

Figure 14.14 Rolling down an inclined plane.

Gravitational potential energy: $Mgh = Mgs \sin \theta .$

Kinetic energy: $\frac{1}{2} I \omega_C^2 + \frac{1}{2} M v_C^2 = \frac{1}{2} bMR^2 \omega_C^2 + \frac{1}{2} M v_C^2 .$

Conservation of energy: $Mgs \sin \theta = \frac{1}{2}(1 + b) M v_C^2 ,$

$$v_C = \sqrt[2]{\frac{2gs \sin \theta}{1+b}} = \sqrt[2]{2\, aC\, s} \longrightarrow a_C = \frac{g \sin \theta}{1 + b} ,$$

From kinematics with constant acceleration

The same result as before!

Second example

14.12 ROLLING DOWN AN INCLINED PLANE

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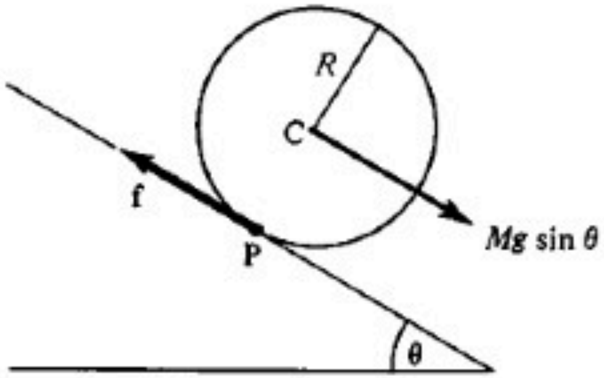


Figure 14.14 Rolling down an inclined plane.

Therefore, the friction does not do any work and energy is conserved in our simplified model of rolling without slipping.

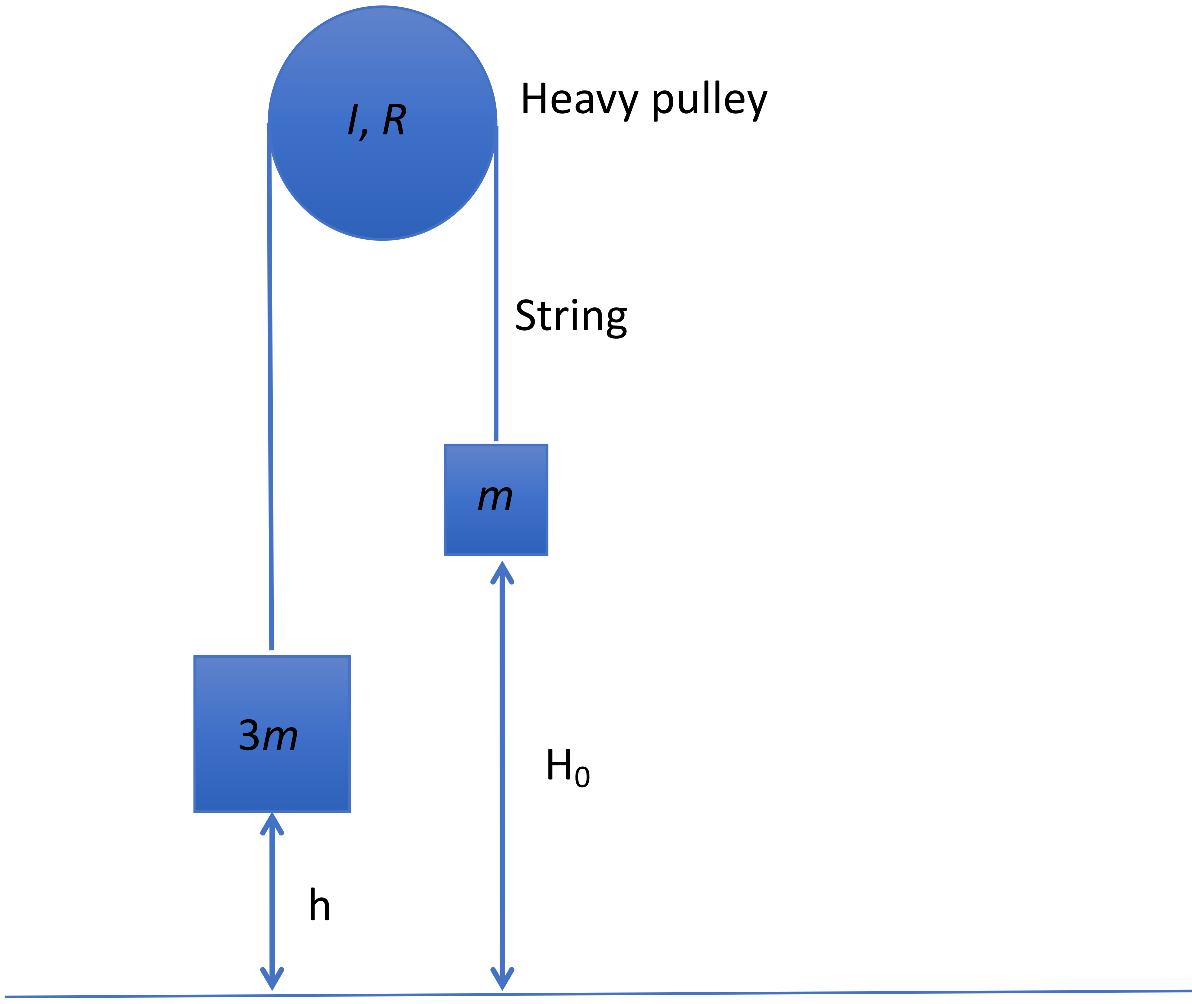
In fact, there are deformations of the body and contact forces do create a rolling friction and there's some energy loss.

Third example: A heavy pulley

A pulley, with the shape of a disk, has mass ***M***, moment of inertia ***I*** and radius ***R.*** Two blocks hang from each of its sides. The friction on the surface of the pulley allows the string to turn about the pulley without slipping. The string is massless. Initially, the system is at rest.

1) Find the speed of the blocks when the **3*m*** body reaches the ground.

2) Find the maximum height the mass ***m*** rises *after* the **3*m*** body reaches the ground (assume that somehow the string remains taut and in contact with the rotating pulley as it keeps moving upwards).



This problem can be solved with just energy conservation.

We have the gravitational energy and the kinetic energy of the masses and also the kinetic energy of the pulley. If the string does not slip, $v = \omega R$ at all times. Then,

$$mgH_0 + 3mgh = mg(H_0 + h) + \frac{1}{2}mv^2 + \frac{1}{2}(3m)v^2 + \frac{1}{2}I\omega^2 \quad \Rightarrow \tag{5}$$

$$2mgh = 2mv^2 + \frac{1}{2}I\frac{v^2}{R^2} \quad \Rightarrow \quad v = 2\sqrt{\frac{mgh}{4m + \frac{I}{R^2}}} = 2\sqrt{\frac{2mgh}{8m + M}}, \tag{6}$$

where I have used that for a disc $I = MR^2/2$.

Notice that the initial height of the block mass M is irrelevant: only changes of height will matter (this is because the change in height is small compared to the radius of the Earth).

Again, using energy conservation we can solve the second question.

Notice, however, that the block with $3M$ stays on the ground. Therefore, one only needs to consider the motion of mass M and the disc.

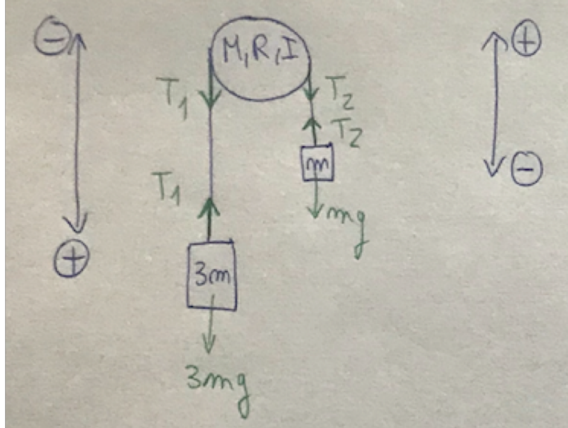
PS: recall the assumption that the string remains in contact with the pulley as it rotates after the block of mass 3*m* has reached the ground. In this situation, the rotational kinetic energy is also transferred to the block of mass *m* that keeps moving upwards.

Let's call h_1 the height that the mass M will be able to raise. At that point, the disc is not rotating (the string is not slipping and if the mass M is stopped, so will the disc be).

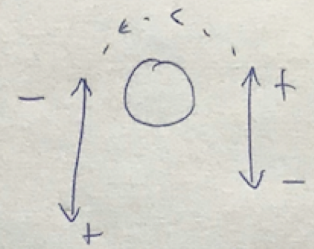
$$mgh_1 = \frac{v^2}{2}\left(m + I/R^2\right) \Rightarrow h_1 = 2h\left(\frac{2m+M}{8m+M}\right). \tag{7}$$

And the total height with respect where it was initially is $h_{\rm TOT} = h + h_1 = h(12m+3M)/(8m+M)$.

Let's show that we could solve the problem with dynamics as well.



Notice that I choose some sign convention on the right and I translate the sign convention to the left:



Although one may choose any convention on each side, doing so helps later when setting the torque's equation.

Newton's equations applied to each mass are:

$$\begin{cases} T_2 - mg = ma \\ -T_1 + 3mg = 3ma \end{cases}$$

Notice that I have not written Newton's equations for both masses at the same time, because we have a massive pulley.

[WRONG: $3mg - T_1 + T_1 - T_2 + T_2 - mg = (3m+m)a$
$\Rightarrow a = g/2$ WRONG]

Now, the rotational equation is:

$$\sum \tau = I \cdot \alpha \Rightarrow (T_1 - T_2)R = I \cdot \alpha .$$

$\circlearrowleft$ (+) $\circlearrowright$ (−)

The string rotates without slipping. Therefore, we can impose $a = \alpha \cdot R$.
<u>HOWEVER</u>, there may be a +/− sign! Let's look at it:

If $T_1 > T_2 \Rightarrow (T_1 - T_2) \cdot R > 0$ and so $\alpha > 0$. If $T_1 > T_2$, the system moves towards the left, lowering "3m" and rising "m", and we chose that motion as ⊕; so good:

$a = \alpha R$ (otherwise, impose $a = -\alpha R$)

$$\Rightarrow \begin{cases} T_2 - mg = ma \\ -T_1 + 3mg = 3ma \\ (T_1 - T_2)R = I\alpha = I\dfrac{a}{R} \Rightarrow T_1 - T_2 = \dfrac{I}{R^2}a \end{cases}$$

$$\Rightarrow \begin{array}{l} T_2 - T_1 + 2mg = 4ma \\ T_1 - T_2 = Ia/R^2 \\ \hline \emptyset + 2mg = \left(4m + \dfrac{I}{R^2}\right)a \end{array}$$

$$\Rightarrow \boxed{a = \left(\frac{2mg}{4m + I/R^2}\right)}$$

PS: NOTICE THAT "I" INCREASES THE DENOMINATOR → MORE EFFORT ⇒ SIGNS WERE CORRECT. OTHERWISE REVIEW IT.

Finally, since a = cnt, we can use a well-known result from kinematics:

$$v_f^2 = v_0^2 + 2a\,\Delta d \Rightarrow V_f^2 = \emptyset^2 + \frac{4mgh}{4m + I/R^2}$$

$$\Rightarrow \boxed{v_f = 2\sqrt{\frac{mgh}{4m + I/R^2}}}$$

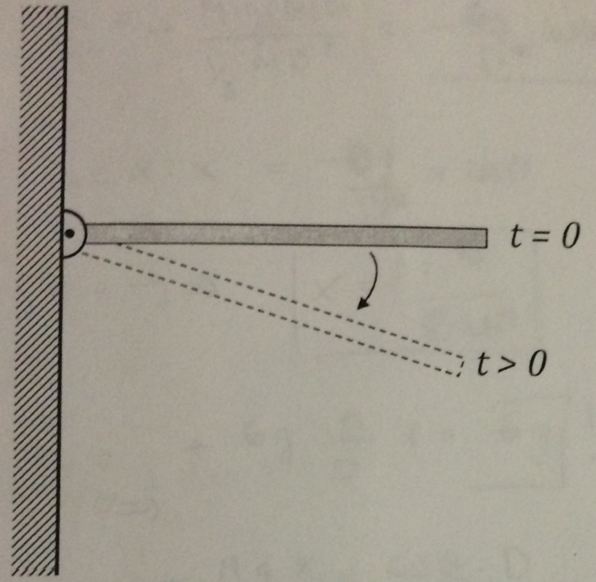
Same result as with energy.

Again the force of friction that is allowing the string to rotate without slipping is not creating any energy loss. Its work is zero. ~~If~~

PS: If the string slips, the force of friction (kinematic friction) will do some work and some mechanical energy will be lost.

An interesting problem: can rigid bodies exist?

1) A thin rod of length D and mass M is attached to a vertical wall via a frictionless pivot as shown. Initially the rod is horizontal.



At $t=0$ the rod is allowed to swing down under the influence of gravity. For this problem we are interested ONLY in the motion of the rod immediately after it is released from the horizontal.

a) Find the moment of inertia of the rod about the pivot point.

b) Find the angular acceleration of the rod immediately after it is released.

c) Find the initial vertical acceleration of the rod at a position $x<D$ measured along the rod.

d) At what position x along the rod is the initial vertical acceleration equal to $-g$, the ordinary acceleration due to gravity? (Measure x from the pivot point.)

e) Show that the initial vertical acceleration of the end of the rod ($x=D$) is larger (in magnitude) than g by a factor f which is independent of the length of the rod. Find f.

f) Does the result in e) violate Galileo's observation that all falling objects accelerate at the same rate? If not, in a sentence or two, explain why.

2. The rotating thin rod

a) It is solved using the parallel axis theorem that relates the moment of inertia about the center of the mass with another (parallel) axis at some distance:

$$I = \frac{1}{12}MD^2 + M\left(\frac{D}{2}\right)^2 = \frac{1}{3}MD^2. \tag{17}$$

b) Newton's second law for the rotation provides us the solution ($\tau = I\alpha$):

$$\alpha = \frac{\tau}{I} = -\frac{MgD}{2I} = -\frac{3g}{2D} \tag{18}$$

Notice that we have ignored a term $\cos\theta$, where θ is the angle rotated by the rod, because the problem says *immediately* after it is released. But it would be very simple to include $\cos\theta$. Finally, also noticethat units make sense.

c) It is simple to find it, because $a = \alpha x$, so

$$a = -\frac{3gx}{2D} = -\frac{3x}{2D}g. \tag{19}$$

Again units make sense.

d) Clearly that happens for $x = 2D/3$.

e) For $x = D$, $a = 3g/2 > g$. The 'surprising' fact is that it is higher than g.

f) There is no contradiction. Galileo principle applies to *free* fall otion. This is a rigid rod, where every piece is rigidly connected to the rest. Therefore, differential internal forces play a role in the motion.

NB: Even though it is not asked, one could use energy conservation to find the velocity of the center of mass in terms of θ, which would imply an $\omega(t)$ and knowing the rod is rigid, it would be possible to find the speed at any point on the rod.

An interesting problem: can rigid bodies exist?

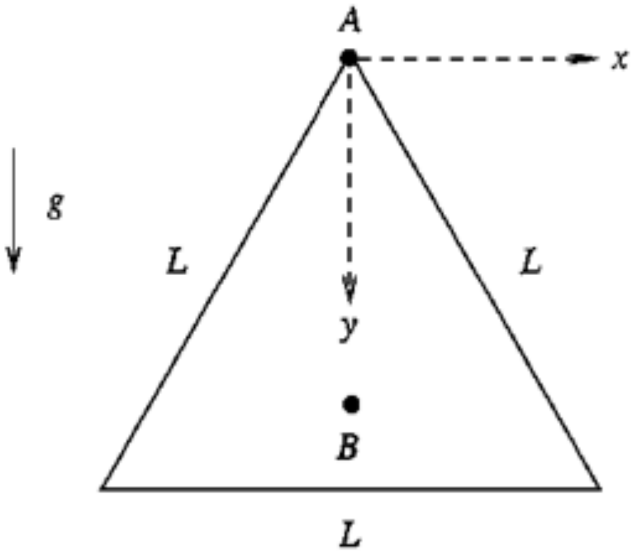
For some length (***D)***, the tip of the bar could end up moving with a linear speed ($v=\omega D$) greater than *c*, the speed of light. We know from observations that no object can travel faster than the speed of light. In order to be consistent with this fact, as the bar spins faster and some of its elements move with a speed close to *c*, the bar itself will bend (!). In other words, the moment of inertia *I* will no longer be 1/3 MD^2 , which is clearly independent of the speed, but it would be different at different times, so that no element of the bar moves faster than *c*. In summary, rigid bodies do not really exist. They are an *approximation*, which is valid as far as speeds are much lower than *c*.

An alternative way of viewing it is by considering the molecular forces that inside the bar. The interaction force between the elements of the bar can't be transmitted faster than the speed of light and, thus, a rigid body cannot exist: once the tip starts to move, the extreme of the bar in contact with the pivot will not know react to that motion until a time, at least, D/c. Similar with any other intermediate element. The bar will bend.

A (difficult) problem from HW that was in a final exam

FP5

A thin uniform plate (mass M), in the shape of an equilateral triangle (side L), is suspended from one vertex (at A in
the figure), forming a physical pendulum. The triangle swings about an axis perpendicular to the plate through point A. Take x-y coordinates as shown, so that $w(y) = \frac{2}{\sqrt{3}} y$ is the width of the triangle a vertical distance y from A.
Our goal is to calculate the period for small oscillations about A.



(3 points) (a) Find the coordinates of the center of mass (x_{cm} , y_{cm}). *Hint*: One method involves breaking the triangle into horizontal rectangular strips of mass dm and then integrating. There is also a symmetry argument.

(4 points) (b) Calculate the moment of inertia I_A about the axis through A. *Hint*: Apply the parallel axis theorem to each horizontal strip and then integrate.

(2 points) (c) What is the period for small oscillations about A? Leave your answer in terms of I_A if you were unable to solve part (b).

(2 points) (d) (**Extra Credit**) We now move the suspension to a second point B on the y axis such that, when the system is inverted, small oscillations have the same period as about A. Find the coordinates y_B of this point relative to the coordinate system centered on point A.

3. The triangular pendulum

This problem is more elaborated than usual. Especially in the queston about the calculation of the moment of inertia. The last question is indicated but the explicit final solution is left as an exercise.

a) Any vertex is equivalent. Therefore the center of the mass is located at the intersection of the three bisections of the vertices. With respect to the point A, taken as $(0, 0)$, the center of mass has coordinates: $\vec{r}_{\rm CM} = (0, -y_{\rm CM})$, where $y_{\rm CM} = L \sin 60^\circ - L \tan 30^\circ/2 = (\sqrt{3}/3)L$.

Think about it ...

b) The moment of inertia is composed of two parts. For each section of the triangle, we can consider a rod. That is an infinitessimally thin rod. Each of these rods is then rotating with respect A, so we will also use the parallel axis theorem (y is the vertical distance measured from A).

In summary:

$$dI_A = dI_C + dm\, y^2, \tag{22}$$

$$dI_C = \frac{1}{12} dm (2d)^2 = \frac{1}{3} dm\, d^2, \tag{23}$$

$$dI_A = \left(\frac{1}{3} d^2 + y^2\right) dm. \tag{24}$$

Now, dm can be found multiplying its mass density by the infinitessimal area:

$$dm = \sigma(2d)dy, \quad d = y \tan 30^\circ = \sqrt{3}y/3, \quad \sigma = M/\text{Area} = \frac{M}{\frac{1}{2}L\frac{\sqrt{3}}{2}L} = \frac{4\sqrt{3}}{3}\frac{M}{L^2} \Rightarrow dm = \frac{8M}{3L^2}y\,dy \tag{25}$$

Substituting back into dI_A, we get $dI_A = \frac{80M}{27L^2}y^3\,dy$. And, we can now integrate it:

$$I_A = \int_0^{\frac{\sqrt{3}}{2}L} dI_A = \frac{5}{12}ML^2. \tag{26}$$

c) To solve this part, we need to recall how Newton's second law is written for a pendulum:

$$\tau = I\alpha \Rightarrow \frac{5}{12}ML^2\frac{d^2\theta}{dt^2} = -\frac{MgL}{\sqrt{3}}\sin\theta = \tag{27}$$

And

$$\omega = 2\sqrt{\frac{\sqrt{3}g}{5L}} \tag{28}$$

Units are okay.

d) This part is easy if one recalls from the previous question or, from the theory of the physical pendulum, that

$$\omega_A^2 = \frac{MgD_A}{I_A}, \quad \omega_B^2 = \frac{MgD_B}{I_B}, \tag{29}$$

where D_A is the distance from A to the center of mass, and D_B is the distance from B to the center of mass. The problem asks about the position of B with respect to A, but knowing D_A from the first question and D_B, one can find the distance between A and B.

Another question is to know I_A and I_B. I_A has been derived in a previous question (b). I_B can be related to the moment of inertia about the center of mass as: $I_B = I_C + MD_B^2$. I_A can *also* be related, so that $I_A = I_C + MD_A^2$. This allows us to write $I_B = I_A + M(D_B^2 - D_A^2)$. As a consequence, we need to solve the following equation:

$$D_AI_A = D_BI_B \Rightarrow I_A + M(D_B^2 - D_A^2) = \left(\frac{I_A}{D_A}\right)D_B \Rightarrow D_B^2 - \left(\frac{I_A}{MD_A}\right)D_B + (I_A/M - D_A^2) = 0 \tag{30}$$

Units make sense (distance square). This is a second order polynomial where I_A and D_A have been derived before. The distance between A and B is $D_{AB,1} = D_A + D_B$ when B is close to the bottom, as in the figure. But there's also the possibility of being above the center of mass $D_{AB,2} = D_A - D_B$. I left as an exercise to find the two roots and check that only two positions of the possible four correspond to B lying inside the triangle and not being A.

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