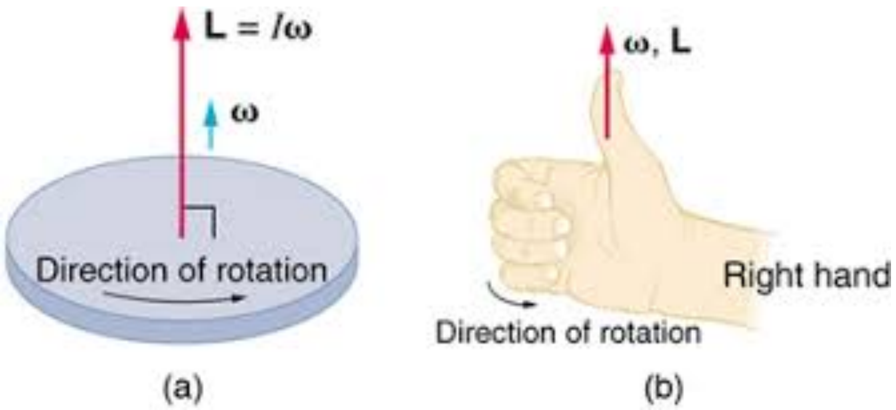
PH1a: angular momentum

$$\vec{L} = \vec{r} \times \vec{p}$$



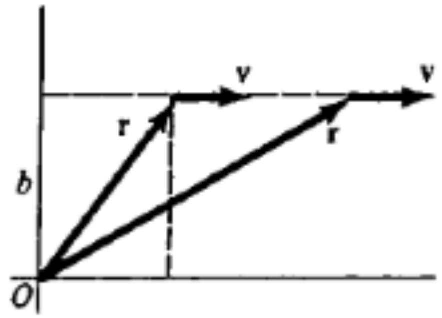
First example: angular momentum also applies to linear motion

Example 1

A particle of mass m moves with constant velocity $\mathbf{v}$ along a straight line which is a distance b from the origin of a coordinate system.

(a) Find the angular momentum of the particle at any instant.

(b) Show explicitly that the angular momentum is conserved.



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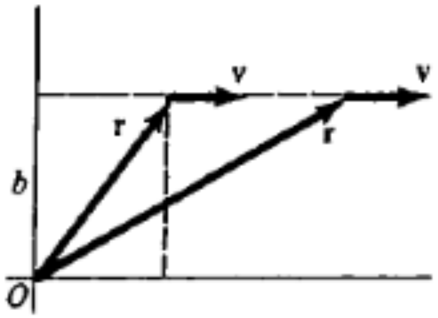


L = *b m v*,

Always use the perpendicular distance from the axis of rotation to ***v*:** *b* in this case.

b, m and *v* are all constants, so is *L*.

Leonhard Euler introduced the concept of moment of inertia

If each element of a body be multiplied into the square of its distance from the axis 0A and all these products be collected into one sum and if this is put = *Mkk*, which I call the moment of inertia of the body with respect to the axis 0A, then the moment of force required to produce acceleration α will be *Mkk* · α.

- Leonhard Euler, in *Theoria Motus Corporum Solidorum seu Rigidorum* (1765)

$$I = \int_{\text{body}} r^2 \, dm.$$

Netwon's Principia had been published 78 years earlier.

Moments of inertia: use tables

Table 14.1 Moments of Inertia for Uniform Bodies

Body	Axis	I
Rod (length L)	Perpendicular axis through center	$\frac{1}{12}ML^2$
Thin ring (radius R)	Perpendicular axis through center	MR^2
Circular cylinder	Axis of cylinder	$\frac{1}{2}MR^2$
Thin disk	Transverse axis through center	$\frac{1}{4}MR^2$
Solid sphere	Any axis through center	$\frac{2}{5}MR^2$
Thin spherical shell	Any axis through center	$\frac{2}{3}MR^2$
Rectangular plate (length a, height b)	Axis through center perpendicular to the plate	$\frac{1}{12}M(a^2 + b^2)$

Source: The Mechanical Universe

Several formulas are similar between linear and angular motion

Quantity	Translation	Rotation
Displacement	x	θ
Speed	$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$
Inertia	m	$I = \sum mr^2$
Force	$\mathbf{F}$	$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$
Momentum	$\mathbf{p} = m\mathbf{v}$	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$
Impulse	$\int \mathbf{F}\, dt$	$\int \boldsymbol{\tau}\, dt$

General Laws

Law	Translation	Rotation
Newton's second law	$\mathbf{F} = \frac{d\mathbf{p}}{dt}$	$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$
Work	$W = \int F\, dx$	$W = \int \tau\, d\theta$
Power	$P = \frac{dW}{dt} = Fv$	$P = \frac{dW}{dt} = \tau\omega$ ← **Useful for HW QP33**
Impulse	$\int \mathbf{F}\, dt = \Delta\mathbf{p}$	$\int \boldsymbol{\tau}\, dt = \Delta\mathbf{L}$

Source: The Mechanical Universe

Parallel axis theorem

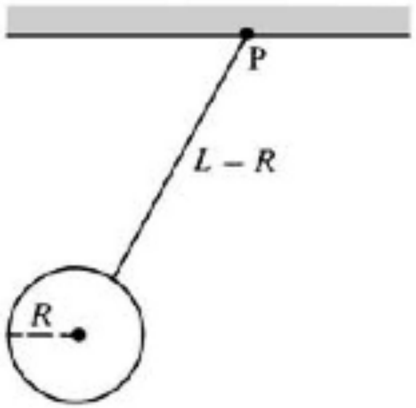
$$I_P = I_C + Mr^2.$$

Always use $\boldsymbol{I_C}$ around the Center of Mass, and *r* is the distance between the Center of Mass and the new rotation.

Parallel axis theorem

Example 9

Find the moment of inertia of a disk of radius R suspended by a string of length $L - R$, about an axis which passes through the point of suspension P and is perpendicular to the disk.



$$I_P = I_C + M_{DISK} L^2$$

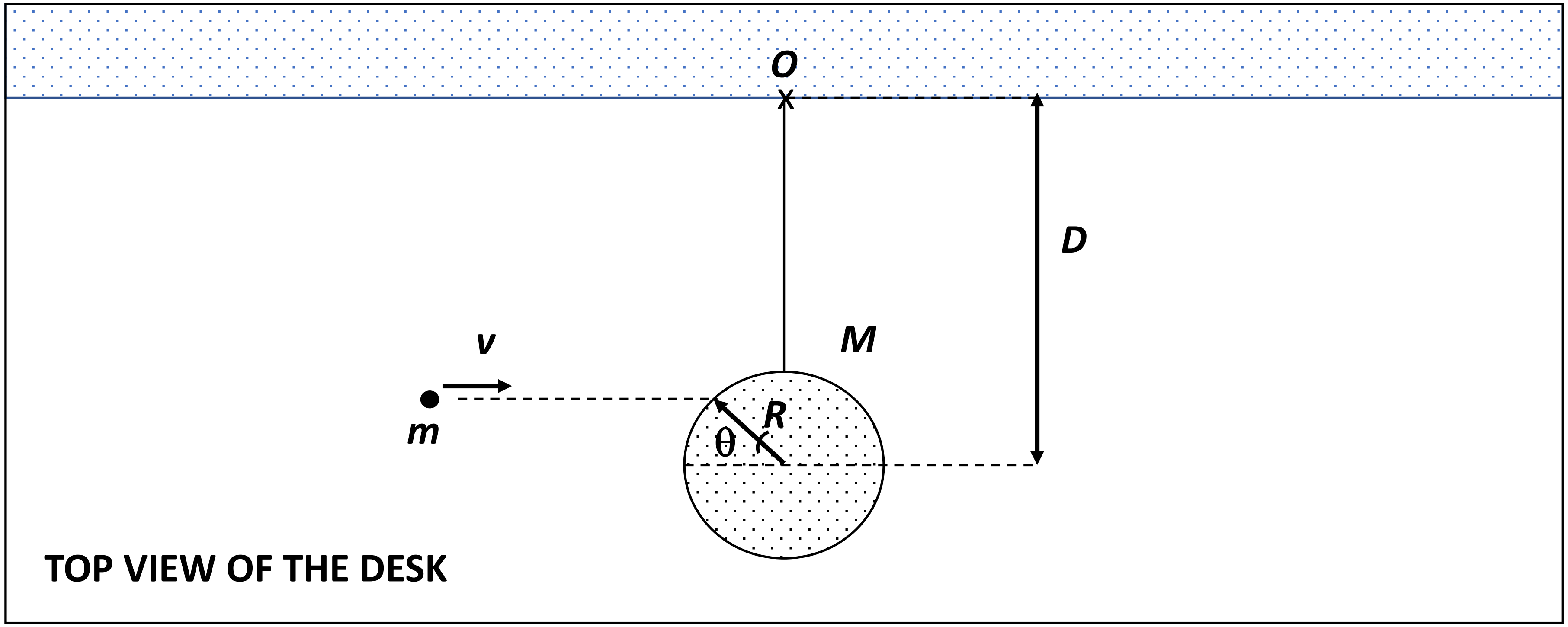
Notice that the Center of Mass of the Disk is at its center and the distance between ***P*** and the center of the disk is *L*.

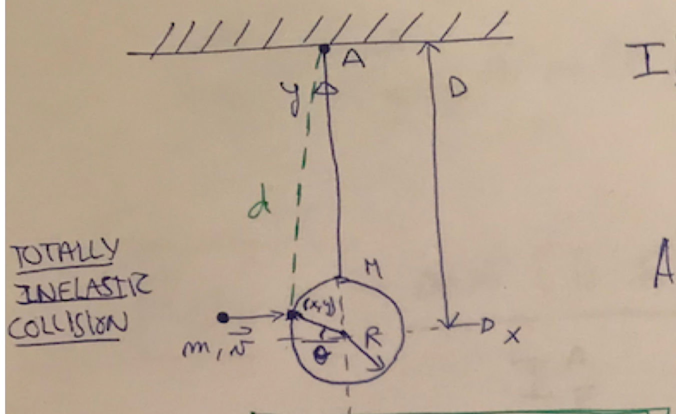
From the tables (or a direct calculation): $I_C = \frac{1}{2} M_{DISK} R^2$, therefore:

$$I_P = I_C + M_{DISK} L^2 = M_{DISK} \left(\frac{1}{2} R^2 + L^2 \right)$$

An interesting example of a collision and angular momentum

An object is composed of a rigid rod and a thin disk, see figure. Initially, the object lies at rest on a frictionless desk. One of the extremes of the rod is pinned down to a point ***O*** about which the rod is free to rotate without friction. A bullet moves along the surface of the desk with a velocity perpendicular to the rod. The projectile hits the disk on its side at an angle θ with respect to the center of the disk, and remains stuck to its rim. Consider the rod to be rigid and *massless*. The disk has mass ***M*** and radius ***R***. The bullet has mass ***m*** and speed ***v***, and no size, that is consider it *point-like*. The distance between ***O*** and the center of the disk is ***D***. Express the solution to the following questions in terms of the previous variables only. For which θ the angular speed after the collision is the greatest?





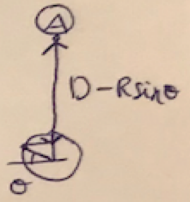
$I_{DISK}^{CM} = \frac{1}{2}MR^2$; $I_{DISK}^{A} = \frac{1}{2}MR^2 + MD^2$;

↑ Parallel axis theorem

$\vec{L}$: ⊖ , ⊕

Q1: What θ makes the system turn with the highest ω?

Angular momentum is conserved during the collision:



Initial: $L_0^A = -m v d \sin\alpha = -mv(D - R\sin\theta)$

$d\sin\alpha$: always the distance perpendicular to $\vec{v}$

Sense of rotation ↑ → Counter clockwise.

Final: $L_F^A = L_0^A = -mv(D - R\sin\theta)$.

Now $L_F^A = I_F \cdot \omega_F$, where I_F is the total moment of inertia after the collision.

It is composed of the disk suspended from A and the particle, which gets attached to the disk after the totally inelastic collision.

$$I_F = I_{DISK}^A + I_m^A = I_{DISK}^A + m \cdot (\text{distance A to m})^2.$$

$\vec{d} = (-R\cos\theta, D - R\sin\theta)$; $d^2 = R^2\cos^2\theta + (D - R\sin\theta)^2 = R^2 + D^2 - 2RD\sin\theta \Rightarrow$

(Vector from A to m)

$$I_F^A = I_{DISK}^A + m(R^2+D^2-2RD\sin\theta) = \boxed{\underbrace{\tfrac{1}{2}MR^2+MD^2+m(R^2+D^2)}_{\text{independent of } \theta}-2mRD\sin\theta}$$

$$\boxed{w_F} = -\frac{mv(D-R\sin\theta)}{I_F^A} = \boxed{\frac{-mv(D-R\sin\theta)}{\left(\tfrac{1}{2}MR^2+MD^2+m(R^2+D^2)-2mRD\sin\theta\right)}}$$

This expression is complicated. There is no obvious value of θ that makes it greatest. For instance, if $\theta \to \pi/2$ (closer to A), the numerator decreases (less "angular kick") but the total moment of inertia also decreases. So, the division might be greater, less than, etc... Similarly if $\theta \to 0$ or $\theta \to -\pi/2$ (as far as possible).

We must compute $\frac{dw_F}{d\theta} = 0$ (and check it is a maximum).

$$\frac{dw_F}{d\theta} = -mv\left[\frac{(-R\cos\theta)\left(\tfrac{1}{2}MR^2+MD^2+m(R^2+D^2)-2mRD\sin\theta\right)-(D-R\sin\theta)(-2mRD\cos\theta)}{(\ldots\ldots\ldots\ldots)^2}\right] =$$

$$\frac{d\omega_F}{d\theta} = \frac{-mv}{(\ldots - \ldots)^2}\Big[-R\cos\theta\left(\tfrac{1}{2}MR^2 + MD^2 + m(R^2+D^2)\right) + 2mR^2D\sin\theta\cos\theta +$$

$$+ 2mRD^2\cos\theta - 2mR^2D\sin\theta\cos\theta\Big] = \frac{-mvR}{(\ldots - \ldots)^2}\cos\theta\Big[2mD^2 - \tfrac{1}{2}MR^2 -$$

$$-MD^2 - mR^2 - mD^2\Big] = \frac{-mvR}{(\ldots - \ldots)^2}\left(mD^2 - \tfrac{1}{2}MR^2 - MD^2 - mR^2\right)\cos\theta.$$

Let's call $\Delta \overset{\text{(Definition)}}{\equiv} mD^2 - \frac{1}{2}MR^2 - MD^2 - mR^2 = m(D^2 - R^2) - M\left(\frac{R^2}{2} + D^2\right)$;

$$\boxed{\frac{d\omega_F}{d\theta} = -\frac{mvR}{(\ldots - \ldots)^2}\cdot\Delta\cdot\cos\theta}$$

$\frac{d\omega_F}{d\theta}$ depends on θ as $\cos\theta$ in the numerator and there's still a $\sin\theta$ in the denominator.

$\frac{d^2\omega_F}{d\theta^2} = -m \cdots$ complicated! There's a way out by looking at the increase/decrease of ω_F before and after the critical points:

1) $\frac{d\omega_F}{d\theta} = 0 \iff \begin{cases} \Delta = 0, \text{ which is only for some particular values of } m, M, R \text{ and } D. \\ \theta = \pm \pi/2 \end{cases}$

The sign of $\frac{d\omega_F}{d\theta}$ depends on the signs of Δ and $\cos\theta$ (there is a global minus sign and the denominator is square) $\Rightarrow$ For $\theta = +\frac{\pi}{2}$

$$\frac{\pi}{2}: \begin{cases} \text{sign}\left(\frac{d\omega_F}{d\theta}\right)_{\frac{\pi}{2}^-} = -\text{sign}(\Delta)\underbrace{(+)}_{\cos\frac{\pi}{2}^- > 0} = -\text{sign}(\Delta). \\ \text{sign}\left(\frac{d\omega_F}{d\theta}\right)_{\frac{\pi}{2}^+} = -\text{sign}(\Delta)(-) = \text{sign}(\Delta) \end{cases}$$

If $\text{sign}(\Delta)_{>0} \Rightarrow \theta = \frac{\pi}{2}$ is a Minimum $\frac{d\omega_F}{d\theta}: (-\ 0\ +)$

If $\text{sign}(\Delta) < 0 \rightarrow \theta = \frac{\pi}{2}$ is a Maximum. $(+\ 0\ -)$

and for $\theta = -\frac{\pi}{2} \Rightarrow$

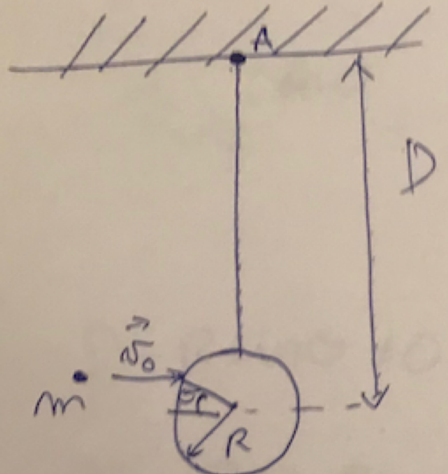
$$-\frac{\pi}{2}: \begin{cases} \text{sign}\left(\frac{d\omega_F}{d\theta}\right)_{-\frac{\pi}{2}^-} = -\text{sign}(\Delta)\cdot\underbrace{(-)}_{\cos(-\frac{\pi}{2}^-) < 0} = \text{sign}(\Delta) \\ \text{sign}\left(\frac{d\omega_F}{d\theta}\right)_{-\frac{\pi}{2}^+} = -\text{sign}(\Delta)\cdot(+) = -\text{sign}(\Delta) \end{cases}$$

If $\text{sign}(\Delta) > 0 \Rightarrow \theta = -\frac{\pi}{2}$ is a Maximum.

If $\text{sign}(\Delta) < 0 \Rightarrow \theta = -\frac{\pi}{2}$ is a Minimum

Notice that $\omega_F < 0$ so that <u>minimum</u>, means $\max|\omega_F|$! That's what we want.

$\Longrightarrow$



The maximum |angular speed| after the totally inelastic collision of m with the disk depends on the sign of $\Delta = m(D^2-R^2) - M(\frac{R^2}{2}+D^2)$:

If

- $mD^2 > mR^2 + M(R^2/2 + D^2)$, $\theta = \frac{\pi}{2}$ produces the maximum $|\omega|$
- $mD^2 < mR^2 + M(R^2/2 + D^2)$, $\theta = -\frac{\pi}{2}$ produces the maximum $|\omega|$

Notice that the case $m < M$ belongs to this case

What happens if $mD^2 = mR^2 + M(R^2/2 + D^2)$?

We know $\frac{d\omega_F}{d\theta} = 0, \forall\theta$; so that ω_F is a constant!? Let's see:

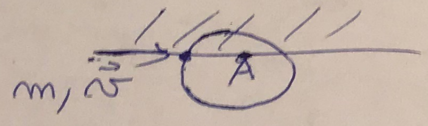
$$\omega_F^{(\Delta=0)} = \frac{-mv(D-R\sin\theta)}{(\frac{1}{2}MR^2 + MD^2 + m(R^2+D^2) - 2mRD\sin\theta)} = \frac{-mv(D-R\sin\theta)}{(2mD^2 - 2mRD\sin\theta)} = \longrightarrow$$

$\uparrow$ $mD^2 = mR^2 + M(R^2/2 + D^2)$

$$\omega_F^{(\Delta=0)} = -\frac{mv}{2mD}\frac{(D-R\sin\theta)}{(D-R\sin\theta)} = \boxed{-\frac{v}{2D}}$$

$D-R\sin\theta \neq 0$, because otherwise the collision would not rotate the system:

$D-R\sin\theta = 0 \Rightarrow$ particle hits at A, but disk inside wall:



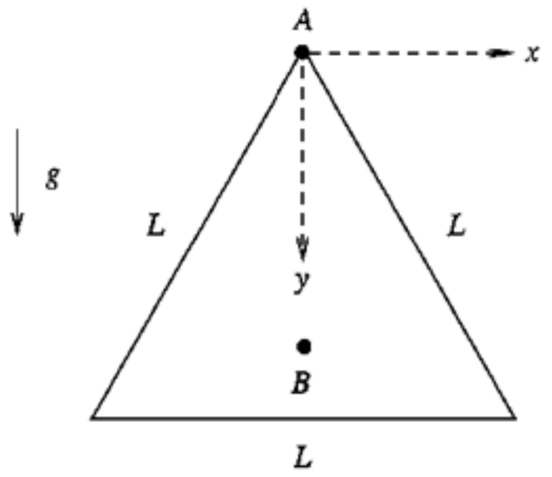
No sense.

If $mD^2 = mR^2 + M(R^2/2 + D^2)$, $\omega_F = -\frac{v}{2D}$ independent of any place where the particle hits the disk and ω_F is independent of the particle mass. Quite a curious result for this elementary setup but non-trivial equations.

Calculating the moment of inertia: a challenging example

FP5

A thin uniform plate (mass M), in the shape of an equilateral triangle (side L), is suspended from one vertex (at A in
the figure), forming a physical pendulum. The triangle swings about an axis perpendicular to the plate through point A. Take x-y coordinates as shown, so that $w(y) = \frac{2}{\sqrt{3}}\, y$ is the width of the triangle a vertical distance y from A.
Our goal is to calculate the period for small oscillations about A.



(3 points) (a) Find the coordinates of the center of mass (x_{cm} , y_{cm}). *Hint*: One method involves breaking the triangle into horizontal rectangular strips of mass dm and then integrating. There is also a symmetry argument.

(4 points) (b) Calculate the moment of inertia I_A about the axis through A. *Hint*: Apply the parallel axis theorem to each horizontal strip and then integrate.

(2 points) (c) What is the period for small oscillations about A? Leave your answer in terms of I_A if you were unable to solve part (b).

(2 points) (d) (**Extra Credit**) We now move the suspension to a second point B on the y axis such that, when the system is inverted, small oscillations have the same period as about A. Find the coordinates y_B of this point relative to the coordinate system centered on point A.

This problem is in the HW of this year.

For **a**) use the symmetry of the problem to justify that the CM is at the barycenter.

For **b**) In order to find the moment of inertia, you may think of the triangle as composed by many small (infinitesimal) bars, all aligned horizontally. The ones close to the vertex *A* will be much shorter than the ones close to the base.

Use the formula of the moment of inertia of a bar about its Center of Mass, keeping the mass of each bar as some small quantity Δm, then apply the parallel axis theorem to write down the total moment of inertia of that bar with respect to *A*. Then, write Δm in terms of the length of the bar and its width (infinitesimal, d*y*) using the fact that the triangle is homogeneous, so that the total mass is the surface density multiplied by the area of the triangle. Plug altogether and integrate,

The result should be $I_A = 5/12\, M L^2$.

For **c**) use I_A and the distance to the Center of Mass from *A* and apply the formula of the physical pendulum

For **d**) apply the Parallel axis Theorem for *B* and the new distance of the Center of Mass from *B*. There's a second order polynomial to solve for the position *B*.