Ph1a - Flipped Section

Problem Set 9 - Solutions

November 4, 2019

1. Strike it good!

Let V be the speed of the ball after the collision. Let v be the speed of the CM of the stick after the collision. Let ω be the angular speed of the stick after the collision. Conservation of momentum, angular momentum (around the initial center of the stick), and energy give:

$$MV_0 = MV + mv \ , \tag{1}$$

$$MV_0 d = MVd + \eta m l^2 \omega \ , \tag{2}$$

$$MV_0^2 = MV^2 + mv^2 + \eta m l^2 \omega^2 \ . \tag{3}$$

We must solve these three equations for V, v, and ω and determine the numerical factor η in the expression for the angular momentum of the stick about its center. In order to determine η, consider

$$I = \int dm r^2 = \int_{-l/2}^{l/2} \rho dr \ r^2 = \int_{-l/2}^{l/2} \frac{m}{l} r^2 dr = \frac{mr^3}{3l}\Big|_{-l/2}^{l/2} = \frac{ml^2}{12}. \tag{4}$$

Therefore, $\eta = 1/12$.

The first two equations, quickly give $vd = \eta l^2 \omega$. Solving for V in the first equation and plugging it into the third, and then eliminating ω through $vd = \eta l^2 \omega$ gives

$$v = \frac{2V_0}{1 + m/M + d^2/(\eta l^2)} \implies \omega = \frac{2V_0 d/(\eta l^2)}{1 + m/M + d^2/(\eta l^2)} \ . \tag{5}$$

Knowing v, the first equation above gives V as,

$$V = V_0 \frac{1 - \frac{m}{M} + \frac{d^2}{\eta l^2}}{1 + \frac{m}{M} + \frac{d^2}{\eta l^2}} \ . \tag{6}$$

Another solution is of course $V = V_0, v = 0$, and $\omega = 0$. Nowhere in the equations we have written does it say the ball must hit the stick. It is important to verify the physical meaning of all solutions.

2. Spheres and Cavities

The mass M is distributed in the larger sphere uniformly wherever it is present, which can help us find the density ρ of the material filling the larger sphere,

$$M \ = \ \rho \frac{4\pi}{3} \left[(2R)^3 - R^3 \right] \ , \tag{7}$$

which gives us,

$$\rho = \frac{3M}{28\pi R^3} \ . \tag{8}$$

Now for an axis passing through the center of both the sphere and the cavity, one can use the fact that moment of inertia algebraically adds up for objects. The integral $\int r^2 dm$ will be very hard to solve. One can think of the "sphere with cavity" as a solid sphere of radius $2R$ completely filled minus the mass filled in the cavity of radius R. Were the sphere completely filled, its mass would be,

$$M_{solid} = \rho \frac{4\pi}{3} (2R)^3 \ = \frac{8M}{7}. \tag{9}$$

and the mass that would have filled the cavity is,

$$M_{cavity} = \rho\frac{4\pi}{3}(R)^3 \; = \frac{M}{7}. \tag{10}$$

Now the moment of inertia about an axis through the centers(call it the z-axis)is simply,

$$I_z = I_{solid} - I_{cavity} \; , \tag{11}$$

$$I_z = \frac{2M_{solid}(2R)^2}{5} - \frac{2M_{cavity}R^2}{5} \; , \tag{12}$$

which gives us,

$$I_z \; = \; \frac{62MR^2}{35} \; . \tag{13}$$

Now to get the moment of inertia about an axis perpendicular to this and passing through the center of the large sphere, we use a similar argument, but now the moment of inertia of the cavity about this new axis is found using the Parallel Axis Theorem,

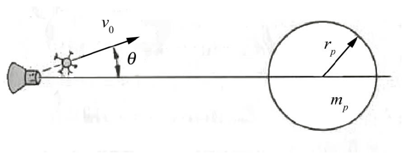
$$I'_{cavity} = \frac{2MR^2}{35} + \frac{MR^2}{7} = \frac{MR^2}{5} \; , \tag{14}$$

and therefore the moment of inertia of the sphere with cavity about the required axis is,

$$I = I_{solid} - I'_{cavity} \; , \tag{15}$$

$$I = \frac{2 \times 8M(2R)^2}{5 \times 7} - \frac{MR^2}{5} \; = \; \frac{57MR^2}{35} \; . \tag{16}$$

3. Spaceships



The gravitational force by the planet on the spaceship $\vec{F}^G_m$ always points towards the center of the planet. The torque about the center of the planet (point labeled O) due to the gravitational force is given by the expression

$$\vec{\tau}_O = \vec{r}_{O,m} \times \vec{F}^G_m.$$

The vector $\vec{r}_{O,m}$ points from the center of the planet to the spaceship so $\vec{r}_{O,m}$ and $\vec{F}^G_m$ are anti-parallel hence the torque $\vec{\tau}_O$ is zero. Therefore the angular momentum of the spaceship about the center of the planet is constant. The initial angular momentum of the spaceship about the center of the planet is

$$\vec{L}_{O,i} = \vec{r}_{O,i} \times m_p\vec{v}_i = 5r_p\hat{r} \times (-m_i v_0 \cos\theta\hat{r} + m_i v_0 \sin\theta\hat{\theta}) = 5r_p m_i v_0 \sin\theta\hat{k}.$$

The angular momentum about the center of the planet when the spaceship just grazes the planet is

$$\vec{L}_{O,f} = \vec{r}_{O,f} \times m_p\vec{v}_f = r_p m_i v_f\hat{k}.$$

Since angular momentum is conserved,

$$5r_p m_i v_0 \sin\theta = r_p m_i v_f,$$

and $v_f = 5v_0 \sin\theta$.

There is no non-conservative work done on the package throughout the process so mechanical energy is conserved. Choosing infinity as the zero point for potential energy, the energy equation becomes

$$\frac{1}{2}m_i v_0^2 - \frac{G m_i m_p}{5 r_p} = \frac{1}{2}m_i v_f^2 - \frac{G m_i m_p}{r_p}.$$

Substituting for the final speed v_f gives

$$\frac{1}{2}m_i v_0^2 - \frac{G m_i m_p}{5 r_p} = \frac{1}{2}m_i (5 v_0 \sin\theta)^2 - \frac{G m_i m_p}{r_p}.$$

We can now solve for θ and get

$$\theta = \sin^{-1}\left(\frac{1}{5}\sqrt{\frac{G m_p}{5 r_p v_0^2} + 1}\right).$$