Ph1a - Flipped Section

Problem Set 8 - Solutions

October 31, 2019

1. Rockets

a. We shall start by considering a body with velocity $\vec{v}$ and external forces $\vec{F}$, gaining mass at a rate $\dot{m} = dm/dt$. Let us look at the process of gaining a small amount of mass dm. Let $\vec{v'}$ be the velocity of dm before it is captured by m, and let $\vec{f}$ represent the average value of the impulsive forces that dm exerts on m during the short interval dt, in which the capturing takes place. By Newton's third law, dm will experience a force $-\vec{f}$, exerted by m, over the same dt.

We can now examine the capture process from the point of view of dm and equate $-\vec{f}dt$, to the change in linear momentum of dm,

$$-\vec{f}dt = dm(\vec{v} + d\vec{v} - \vec{v'})\,. \tag{1}$$

Here, $\vec{v} + d\vec{v}$ is the velocity of m (and dm) after impact. Analogously, from the point of view of m, we write,

$$\vec{F}dt + \vec{f}dt = m(\vec{v} + d\vec{v}) - m\vec{v} = md\vec{v}\,. \tag{2}$$

The term $dm\, d\vec{v}$ in equation (1) is a higher order term and will disappear when we take limits. The impulse due to the contact force can be eliminated by combining equations (1) and (2),

$$\vec{F}dt - dm(\vec{v} - \vec{v'}) = md\vec{v}\,,$$

or, dividing through by dt,

$$m\frac{d\vec{v}}{dt} = \vec{F} - (\vec{v} - \vec{v'})\frac{dm}{dt} = \vec{F} + (\vec{v'} - \vec{v})\frac{dm}{dt}\,,$$

where $\vec{u} = \vec{v'} - \vec{v}$ is the velocity of dm relative to m. This equation is known as the variable-mass force law.

b. This is simply a matter of plugging in the correct variables in the variable-mass force law we just derived in part (a). We take the vertically upward direction to be conventionally positive, and hence both the gravitational force and the relative speed of the ejecta are in the "negative" direction. The force law looks like,

$$m\frac{dv}{dt} = -mg - u\frac{dm}{dt}\,.$$

Do not substitute $dm/dt = -k$ at this stage since m is a dynamical variable too which depends on time. The assumption $uk > m_0 g$ makes sure that the rocket starts accelerating right from $t = 0$. Now, solving this differential equation with the initial condition of $v(t = 0) = 0$ and $m(t = 0) = m_0$, we obtain,

$$v(t) \;=\; u\log\left(\frac{m_0}{m(t)}\right) - gt\,,$$

and $m(t)$ can be very easily obtained from $dm/dt = -k$ as,

$$m(t) = m_0 - kt\,.$$

2. Bouncing Balls

As the basketball hits the ground, both are moving downward at a speed $v = \sqrt{2gh}$ (since they fell a height h, $\frac{1}{2}mv^2 = mgh$). Since $m_2 >> m_1$, the basketball bounces back with the same speed v. Immediately after the basketball bounces back, in the frame of the basketball, tennis ball hits it with speed $2v$ so bounces back with speed $2v$; from the point of view of the outside observer, the tennis ball bounces back with speed $3v$, from a height d above the ground. It has energy $mgd + \frac{1}{2}m(3v)^2 = mgd + 9mgh$ So it rises to a height $H = d + 9h$.

3. Angular momentum when total linear momentum vanishes

Total angular momentum about some point with the origin at that point is

$$\sum_i \vec{L}_i = \sum_i (\vec{r}_i \times \vec{p}_i).$$

Now, consider a shift of the origin by $\vec{d}$, so that

$$\sum_i \vec{L'}_i = \sum_i (\vec{r'}_i \times \vec{p}_i)$$

$$= \sum_i (\vec{r'}_i - \vec{d}) \times \vec{p}_i) = \sum_i (\vec{r'}_i \times \vec{p}_i) - \sum_i (\vec{d} \times \vec{p}_i)$$

$$= \sum_i \vec{L'}_i - \vec{d} \times \sum_i \vec{p}_i = \sum_i \vec{L'}_i$$

since $\sum_i \vec{p}_i = 0$.

4. Different collisions on each side

a. First, the linear momentum of the system vanishes. Thus, by the result of the previous problem, angular momentum is the same regardless of the reference point chosen, and it makes sense to speak of the angular momentum of the system.

The simplest way to approach the problem is to take the reference point to be one end (say, the bottom) of the rod. Then, the mass m which is coming towards it contributes nothing to the angular momentum, and the masss m on the top contributes mvL, counterclockwise.

b. The point which does not rotate will not be undergoing any circular motion, hence its centripetal acceleration vanishes. The forces acting on it must cancel.

Now, after the collision, one end of the rod has mass $m + M$ and the other end has mass M, and both points rotate with angular frequency ω. If the point about which the rod rotates is a distance d away from the $m + M$ mass, then the $m + M$ mass will have a centripetal force (coming from the tension in the rod) $(m+M)\omega^2 d$ pulling on it; similarly, the M mass will have $M\omega^2(L-d)$. These exert equal and opposite forces on the rod, and they must cancel so that the rod doesn't itself move. Hence we get

$$(m + M)\omega^2 d = M\omega^2(L - d),$$

and

$$d = ML/(m + 2M).$$

c. First, we find v_f, the velocity of the mass m at the top after it collides elastically with the mass M, by restricting to the top and considering the collision of the two bodies (this is a good approximation for $M \gg m$, otherwise what's happening at the bottom would complicate things).

Momentum conservation gives

$$mv = MV + mv_f,$$

where we let V be the velocity of the mass M after the collision. Energy conservation gives

$$\frac{1}{2}mv^2 = \frac{1}{2}MV^2 + \frac{1}{2}mv_f^2.$$

Together these equations give

$$v_f = \frac{m-M}{m+M}v.$$

d. Initial angular momentum is mvL counterclockwise, which should equal the final angular momentum, which gets three contributions: the $m+M$ side gives $(m+M)d^2\omega$, the M side gives $M(L-d)^2\omega$, and the free mass m moving straight gives $-mv_f(L-d)$. Hence,

$$mvL = (m+M)d^2\omega - mv_f(L-d) = \frac{(m+M)M^2L^2}{(m+2M)^2}\omega - m\frac{m-M}{m+M}v(L - \frac{ML}{m+2M})$$

and we can solve for ω.