Ph1a - Flipped Section

Problem Set 7 - Solutions

October 28, 2019

1. Mid-air explosions

a. $p = mv = 5000$ kg m/s, towards the right.

b. By conservation of momentum,

$$5000 = 20\ (-30) + 15\ (20) + 15\ (v_3)$$

Therefore, $v_3 = 353.33$ m/s.

c. Applying conservation of momentum in the vertical and horizontal directions,

$$0 = 25\ (10) + 25\ (v_{2y}),$$

$$5000 = 25\ (v_{2x}).$$

Therefore, $v_{2x} = 200$ m/s and $v_{2y} = -10$ m/s. The speed of the second fragment is 200.24 m/s at an angle of $-2.86°$ from the horizontal direction.

2. Spring railway gun

a. Let the velocity of the wagon be v_M and the velocity of the marble be v_m with the positive direction being along the velocity of the marble. Since the total momentum of the system is conserved,

$$0 = Mv_M + mv_m.$$

By conservation of energy,

$$\frac{1}{2}kx^2 = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2.$$

Substituting for the speed of the marble,

$$\frac{1}{2}kx^2 = \frac{1}{2}Mv_M^2 + \frac{1}{2}m\ (-\frac{M}{m}v_M)^2,$$

and

$$v_M = -\sqrt{\frac{k\ m}{M(m+M)}}x.$$

The velocity of the marble is

$$v_m = -\frac{M}{m}v_M = \sqrt{\frac{k\ M}{m(m+M)}}x.$$

b. The kinetic energies of the wagon and marble are $\frac{mkx^2}{2(m+M)}$ and $\frac{Mkx^2}{2(m+M)}$ respectively.

c. Since the marble bounces elastically, energy is conserved. Therefore, the new velocity of the marble is $v_b = -\sqrt{\frac{k\ M}{m(m+M)}}x$. The spring has maximum compression when the marble and train are moving at the same speed, v. At this point, conservation of energy gives

$$\frac{1}{2}Mv_M^2 + \frac{1}{2}mv_b^2 = \frac{1}{2}kd^2 + \frac{1}{2}(M+m)v^2,$$

where d is the maximum compression of the spring. We know that

$$\frac{1}{2}Mv_M^2 + \frac{1}{2}mv_b^2 = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2 = \frac{1}{2}kx^2,$$

and by momentum conservation

$$v = \frac{(Mv_M + mv_b)}{(m+M)} = \frac{-2mv_m}{m+M}.$$

Therefore,

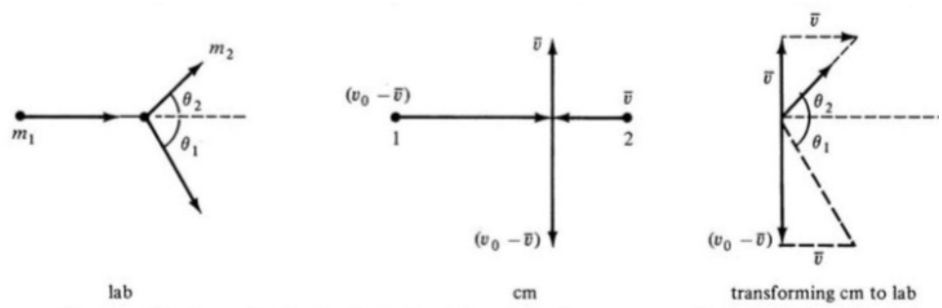
$$\frac{1}{2}kx^2 = \frac{1}{2}kd^2 + \frac{1}{2}(M+m)v^2,$$

and

$$d = x\sqrt{1 - 4\frac{mM}{(m+M)^2}}.$$

3. Maximizing transverse velocity

The answer is rather surprising, but is easily obtained by considering the center of mass frame.



The center of mass moves with a velocity $\vec{v}_{cm} = \vec{v}_0 m_1/(m_1+m_2)$ relative to the laboratory. In the center of mass frame, the incoming particles have velocities $\vec{v}_0 - \vec{v}_{cm}$ and $-\vec{v}_{cm}$, respectively. They emerge back-to-back with speeds unchanged. Because the target particle has speed v_{cm} in this frame no matter what angle it is scattered into, its transverse velocity reaches a maximum value v_{cm} when $\theta_{cm} = 90°$.

To transfer the velocity back to the laboratory frame one adds $\vec{v}_{cm}$ which, being along the initial direction, leaves the transverse velocity $v_{2y} = v_{cm}$ unchanged but makes the angle of maximal v_{2y} in the laboratory $\theta_2 = 45°$. Note that the corresponding θ_1 is

$$\theta_1 \;=\; \tan^{-1}\left((v_0 - v_{cm})/v_{cm}\right) \;=\; \tan^{-1}(m_2/m_1)\,. \tag{1}$$

which equals 45° only if the masses are equal. This is an example of the general statement that the particles emerge at right angles to one another only in the equal mass case.

4. Rockets

a. We shall start by considering a body with velocity $\vec{v}$ and external forces $\vec{F}$, gaining mass at a rate $\dot{m} = dm/dt$. Let us look at the process of gaining a small amount of mass dm. Let $\vec{v'}$ be the velocity of dm before it is captured by m, and let $\vec{f}$ represent the average value of the impulsive forces that dm exerts on m during the short interval dt, in which the capturing takes place. By Newton's third law, dm will experience a force $-\vec{f}$, exerted by m, over the same dt.

We can now examine the capture process from the point of view of dm and equate $-\vec{f}dt$, to the change in linear momentum of dm,

$$-\vec{f}dt = dm(\vec{v} + d\vec{v} - \vec{v'})\,. \tag{2}$$

Here, $\vec{v}+d\vec{v}$ is the velocity of m (and dm) after impact. Analogously, from the point of view of m, we write,

$$\vec{F}dt + \vec{f}dt = m(\vec{v}+d\vec{v}) - m\vec{v} = md\vec{v}\,. \tag{3}$$

The term $dm\ d\vec{v}$ in equation (2) is a higher order term and will disappear when we take limits. The impulse due to the contact force can be eliminated by combining equations (2) and (3),

$$\vec{F}dt - dm(\vec{v}-\vec{v'}) = md\vec{v}\,,$$

or, dividing through by dt,

$$m\frac{d\vec{v}}{dt} = \vec{F} - (\vec{v}-\vec{v'})\frac{dm}{dt} = \vec{F} + (\vec{v'}-\vec{v})\frac{dm}{dt}\ ,$$

where $\vec{u} = \vec{v'} - \vec{v}$ is the velocity of dm relative to m. This equation is known as the variable-mass force law.

b. This is simply a matter of plugging in the correct variables in the variable-mass force law we just derived in part (a). We take the vertically upward direction to be conventionally positive, and hence both the gravitational force and the relative speed of the ejecta are in the "negative" direction. The force law looks like,

$$m\frac{dv}{dt} = -mg - u\frac{dm}{dt}\ .$$

Do not substitute $dm/dt = -k$ at this stage since m is a dynamical variable too which depends on time. The assumption $uk > m_0 g$ makes sure that the rocket starts accelerating right from $t = 0$. Now, solving this differential equation with the initial condition of $v(t=0) = 0$ and $m(t=0) = m_0$, we obtain,

$$v(t)\ =\ u\log\left(\frac{m_0}{m(t)}\right) - gt\ ,$$

and $m(t)$ can be very easily obtained from $dm/dt = -k$ as,

$$m(t) = m_0 - kt\ .$$

5. Bouncing Balls

As the basketball hits the ground, both are moving downward at a speed $v = \sqrt{2gh}$ (since they fell a height h, $\frac{1}{2}mv^2 = mgh$). Since $m_2 >> m_1$, the basketball bounces back with the same speed v. Immediately after the basketball bounces back, in the frame of the basketball, tennis ball hits it with speed $2v$ so bounces back with speed $2v$; from the point of view of the outside observer, the tennis ball bounces back with speed $3v$, from a height d above the ground. It has energy $mgd + \frac{1}{2}m(3v)^2 = mgd + 9mgh$ So it rises to a height $H = d + 9h$.