Ph1a - Flipped Section

Problem Set 6 - Solutions

October 24, 2019

1. Shot Put

The purpose of this problem is to do a quick check on your understanding of work and energy. We have two integrals for the amount of work that Alice (W_A) and Bob (W_B) do respectively,

$$W_A = \int_0^l F_A dx = 20 \text{ J},$$

$$W_B = \int_0^l F_B(x)dx = \int_0^{0.5l} 60x dx + \int_{0.5l}^l 60(1-x)dx = 15 \text{ J}.$$

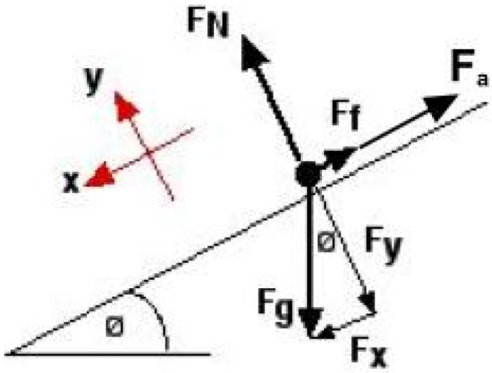
Now one can use the Work-Kinetic Energy Theorem to obtain,

$$v_A^2 = \frac{2W_A}{m},$$

$$v_B^2 = \frac{2W_B}{m},$$

since the initial speed in both cases is zero.

2. "Power" as you slide!



a. Since the box is sliding at a constant velocity, the net force on the box is zero, i.e.:

$$mg\sin\theta - \mu mg\cos\theta = 0 \longrightarrow \mu = \tan\theta.$$

Now we can find the work done by the frictional force as an integral over the distance it travels:

$$\text{Work} = \int_0^{\frac{h}{\sin\theta}} F\text{d}x = -\mu mg\cos\theta\left(\frac{h}{\sin\theta}\right) = -mgh.$$

b. The change in energy of the block is

$$E_{final} - E_{initial} = (0 + \frac{1}{2}mv^2) - (mgh + \frac{1}{2}mv^2) = -mgh.$$

c. No. Friction is a non-conservative force and does not have a corresponding potential energy function. Consider a scenario where the block slides down the slope and is then pushed back up. Do you recover the energy lost due to friction?

d. Since the frictional force is constant, we simply multiply by the distance travelled to find the work done. Now we can find the power by taking the time derivative:

$$\text{Power} \ = \frac{d}{dt}\text{Work} = \frac{d}{dt}(-\mu mg\cos\theta x) = -\mu mg\cos\theta v = -mgv\sin\theta.$$

3. Sledge Sliders

The Work-Kinetic Energy Theorem tells us,

$$W_{\text{net}} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \ .$$

Now, since there is are conservative forces involved, $W_{\text{cons}} = -\Delta U = 0$, and work is done by the (dissipative) friction force which is non-conservative, and hence a regular mechanical energy (kinetic + potential) conservation cannot be constructed here. Thus, the change in kinetic energy is provided by the work W_f done by the non-conservative friction force,

$$W_f = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \ .$$

Now, the friction force has a (constant) magnitude $f = \mu_k N = \mu_k mg$ and hence the work done by it is $W_f = -fx$ for a distance x moved by the sledge. Notice, the negative sign comes from the fact that the friction force and displacement vector are antiparallel and the dot product in the work picks up the negative sign. Now, say the sledge stops after travelling a distance d and hence has zero final speed, thus giving us,

$$0 - \frac{1}{2}mv_i^2 \ = \ -\mu_k mgd \implies d = \frac{mv_i^2}{2\mu_k mg} \ = 2.04 \text{ m} \ .$$

4. Oscillatory Motion

a. Differentiating with respect to time, we obtain:

$$\dot{x} = -x_0\omega_1\sin(\omega_1 t), \qquad \ddot{x} = -x_0\omega_1^2\cos(\omega_1 t),$$
$$\dot{y} = y_0\omega_2\cos(\omega_2 t), \qquad \ddot{y} = -y_0\omega_2^2\sin(\omega_2 t).$$

Newton's second law gives:

$$\vec{F} = m(\ddot{x}\ \hat{\imath} + \ddot{y}\ \hat{\jmath}) = -m(x_0\omega_1^2\cos(\omega_1 t)\ \hat{\imath} + y_0\omega_2^2\sin(\omega_2 t)\ \hat{\jmath}) = -m(\omega_1^2 x\ \hat{\imath} + \omega_2^2 y\ \hat{\jmath}).$$

The x and y components of the force are therefore:

$$F_x = -m\omega_1^2 x,$$
$$F_y = -m\omega_2^2 y.$$

b.

$$\text{Work} = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^{t'} \left(F_x \cdot \frac{dx}{dt} dt + F_y \frac{dy}{dt} dt \right)$$

$$= \int_0^{t'} dt(-m)(x_0\omega_1^2 \cos(\omega_1 t)(-x_0\omega_1 \sin(\omega_1 t)) + y_0\omega_2^2 \sin(\omega_2 t)(y_0\omega_2 \cos(\omega_2 t)))$$

$$= mx_0^2\omega_1^3 \int_0^{t'} dt(\cos(\omega_1 t) \sin(\omega_1 t)) - my_0^2 w_2^3 \int_0^{t'} dt(\cos(\omega_2 t) \sin(\omega_2 t))$$

$$= \frac{1}{2} mx_0^2\omega_1^3 \int_0^{t'} dt \sin(2\omega_1 t) - \frac{1}{2} my_0^2 w_2^3 \int_0^{t'} dt \sin(2\omega_2 t)$$

$$= \frac{1}{4} mx_0^2\omega_1^2(-\cos(2\omega_1 t') + 1) - \frac{1}{4} my_0^2 w_2^2(-\cos(2\omega_2 t') + 1)$$

$$= \frac{1}{4} mx_0^2\omega_1^2(2 - 2\cos^2(\omega_2 t')) - \frac{1}{4} my_0^2 w_2^2(2\sin^2(\omega_2 t'))$$

$$= \frac{1}{2} m\omega_1^2 x_0^2 - \frac{1}{2} m\omega_1^2 x(t')^2 - \frac{1}{2} m\omega_2^2 y(t')^2.$$

c. We know that $\vec{F} = -\nabla V$ and thus $F_x = -\frac{\partial V}{\partial x}$, $F_y = -\frac{\partial V}{\partial y}$, we obtain the potential energy:

$$V = \frac{1}{2} m(\omega_1^2 x^2 + \omega_2^2 y^2).$$

$$\Delta V = \frac{1}{2} m(\omega_1^2 x(t')^2 + \omega_2^2 y(t')^2) - \frac{1}{2} m(\omega_1^2 x(0)^2 + \omega_2^2 y(0)^2)$$
$$= \frac{1}{2} m(\omega_1^2 x(t')^2 + \omega_2^2 y(t')^2) - \frac{1}{2} m\omega_1^2 x_0^2.$$

As expected for a conservative force, the change in potential energy is negative of the work.

d. The kinetic energy of the particle is:

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m(x_0^2\omega_1^2 \sin^2(\omega_1 t) + y_0^2\omega_2^2 \cos^2(\omega_2 t)).$$

The total energy is then:

$$\begin{aligned} E &= T + V \\ &= \frac{1}{2}(x_0^2\omega_1^2 \sin^2(\omega_1 t) + y_0^2\omega_2^2 \cos^2(\omega_2 t) + x_0^2\omega_1^2 \cos^2(\omega_1 t) + y_0^2\omega_2^2 \sin^2(\omega_2 t)) \\ &= \frac{1}{2} m(x_0^2\omega_1^2 + y_0^2\omega_2^2) \\ &= \text{ constant.} \end{aligned}$$