Ph1a - Flipped Section

Problem Set 5 - Solutions

October 21, 2019

1. Some more astrophysics

Consider the limiting case when the Crab Pulsar(of mass M, radius R) is just about to disintegrate. Then, the centrifugal force on a test body (of mass m) at the equator of the Crab is just smaller than the gravitational force,

$$\frac{mv^2}{R} = mR\omega^2 \leq \frac{GmM}{R^2} \,,$$

which gives us,

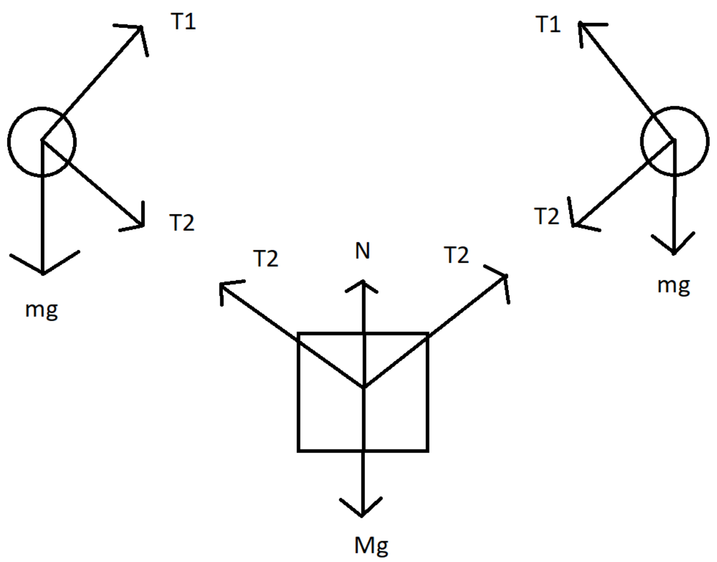
$$\frac{M}{R^3} \geq \frac{\omega^2}{G} \,.$$

In the above equation, v is the tangential speed of the test body and G is the Universal Gravitational Constant. Hence, the minimum density of the pulsar is,

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} \geq \frac{3\omega^2}{4\pi G} = \frac{3(2\pi \times 30)^2}{4\pi \times 6.67 \times 10^{-11}} \approx 1.4 \times 10^{14}\ \mathrm{kg/m^3}.$$

2. Spinning Masses

a.



b. Analyzing the y-component of force for the m masses, we have the equation:

$$\frac{T_1}{\sqrt{2}} = mg + \frac{T_2}{\sqrt{2}}. \tag{1}$$

Analyzing the x-component of m:

$$m\omega^2 \frac{L}{\sqrt{2}} = \frac{T_1}{\sqrt{2}} + \frac{T_2}{\sqrt{2}}, \tag{2}$$

and finally, the y-component of M (the x-component vanishes trivially):

$$F_N + 2\frac{T_2}{\sqrt{2}} = Mg. \tag{3}$$

We want to solve for ω when $F_N = 0$. Then equation (3) tells us that $T_2 = Mg/\sqrt{2}$. Moreover, one can combine equations (1) and (2) to solve $T_2 = \frac{1}{2}m\omega^2 L - mg/\sqrt{2}$. Then:

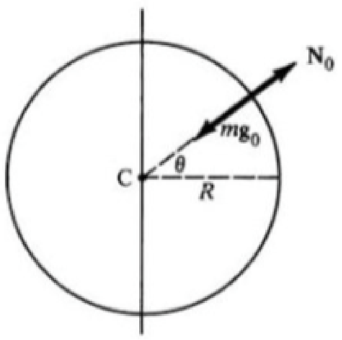
$$Mg/\sqrt{2} = \frac{1}{2}m\omega^2 L - mg/\sqrt{2}, \tag{4}$$

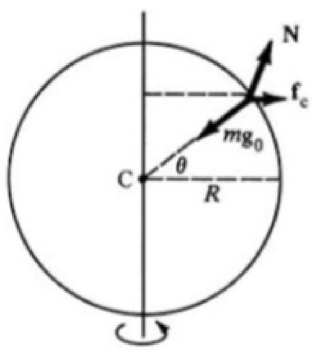
and thus finally:

$$\omega^2 = \sqrt{2}\frac{g}{L}\left(1 + \frac{M}{m}\right) \tag{5}$$

3. Where do I weigh the least!

Let us analyze the weight of an object of mass m, measured on a weighing scale situated at latitude θ on the earth's surface. If the earth did not rotate, the only forces on the object would be the gravitational attraction $m\vec{g_0}$ of the earth toward its center, and the push $\vec{N_0}$ of the scale, which would be equal and opposite to $m\vec{g_0}$ as shown in fig (a). On the rotating earth, the true gravitational force $m\vec{g_0}$ still points directly toward the center C. But now the centrifugal force $\vec{f_c} = (m\omega^2 r)\hat{r}$ also acts on the object. This force is directed outward perpendicular to the earth's axis of rotation, not outward from its center, as shown in the figure (b).





In terms of the earth's radius R its magnitude is therefore $m\omega^2 R\cos\theta$. The equilibrium condition for the object is,

$$\sum \vec{F} = \vec{0} = m\vec{g_0} + (m\omega^2 R\cos\theta)\hat{r} + \vec{N}\ ,$$

where $\vec{N}$, the push of the scale, is slightly modified from $\vec{N_0}$ as a result of the centrifugal force (the shift indicated in the figure is greatly exaggerated). The negative of $\vec{N}$ is what we call the weight of the object and label it as $m\vec{g}$. Thus, we have,

$$m\vec{g} \ = \ m\vec{g_0} + (m\omega^2 R\cos\theta)\hat{r}$$

Applying the Law of Cosines, we obtain,

$$g^2 = g_0^2 + (\omega_2 R\cos\theta)^2 - 2g_0\omega^2 R\cos^2(\theta)\ .$$

Numerically $\omega^2 R\cos\theta$ is much smaller than g or g_0 , so the second term on the right can be dropped, leaving,

$$g^2 \ = \ g_0^2\left(1 - \frac{2\omega^2}{g_0}R\cos^2(\theta)\right)\ .$$

Taking the positive square root on both sides, and using the fact that $\frac{2\omega^2}{g_0} R \cos^2(\theta)$ is smaller than unity, we can retain only till ω^2 order to obtain,

$$g \approx g_0 - R\omega^2 \cos^2(\theta) \,.$$

4. Roller coasters

Let θ be the angle which describes the roller coaster's position on the circle (so when it's at the ground, $\theta = 3\pi/2$, and when it's at the top of the loop, $\theta = \pi/2$.

In the frame of the car, the centrifugal force is mu^2/R radially outward. There is gravity mg which is always downward; and then some force T tangential to the car, and a normal force N which is radially inward.

Balancing forces in the radial direction, we get

$$N + mg \sin\theta = mu^2/R.$$

In the tangential direction, we get

$$T = mg \cos\theta.$$

Therefore, the total force is

$$\sqrt{N^2 + T^2} = \sqrt{m^2 g^2 + m^2 u^2/R^4 - 2m^2 g u^2 \sin\theta/R}.$$