Ph1a - Flipped Section

Problem Set 4 - Solutions

October 17, 2019

1. Hands on Astrophysics

For the motion of Earth around the Sun,

$$\frac{mv^2}{r} = \frac{GmM_\odot}{r^2} \, ,$$

where m is the Earth's mass and $M_\odot$ is the Sun's mass and we have taken r to be the Earth-Sun distance and v is the tangential Earth's speed around the Sun.

For the motion of Sun around the galactic center,

$$\frac{M_\odot V^2}{R} = \frac{GMM_\odot}{R^2} \, ,$$

where M is the galactic mass and we the Sun is at a distance R from the galaxy's center and V is the Sun's speed. Combining the two equations given above, we get,

$$M = \frac{RV^2}{G} \; = \frac{R}{r}\left(\frac{V}{v}\right)^2 M_\odot \, .$$

Using $V = 2\pi R/T$ and $v = 2\pi r/t$, where T and t are the revolution periods of the Sun and Earth respectively, we have,

$$M = \left(\frac{R}{r}\right)^3 \left(\frac{t}{T}\right)^2 M_\odot \, ,$$

and, from the given data, we obtain,

$$M \;\approx\; 1.53 \times 10^{11} \, M_\odot$$

2. Driving on a curved road

The car can be thought of as a block on an inclined plane; there is gravity acting downward, a normal force from the plane (the road), and friction. The net force should be mv^2/R in the x-direction. The tricky question is: which way does friction act?

We want to maximize our speed. This means that the centripetal acceleration mv^2/R is as large as possible. In order to achieve this, we want as much force in the x-direction as possible. The way to achieve this is if friction acted downward along the slope of the incline.

Then, in the y-direction, we have

$$N\cos\theta - f\sin\theta - mg = 0$$

In the x-direction, we have

$$N\sin\theta + f\cos\theta = mv^2/R$$

The y equation gives us $N = \frac{mg}{\cos\theta - \mu\sin\theta}$. (We want to maximize N as well, so for fixed θ, we want the maximum possible coefficient of friction, which is μ.)

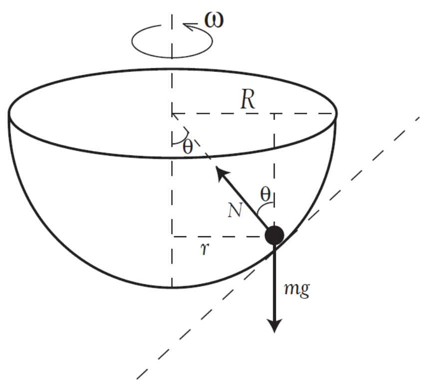
Then,

$$mv^2/R = \frac{mg}{\cos\theta - \mu\sin\theta}(\sin\theta + \mu\cos\theta)$$

and we can solve for v.

3. Marble in a bowl

a. The inner surface of the bowl exerts a normal force N on the marble. The vertical component of that force balances the marble's weight, while the horizontal component must correspond to the centripetal force necessary to keep the marble in its circular orbit.



Balancing the forces at equilibrium gives:

$$X : N\sin\theta = m\omega^2 r$$

$$Y : N\cos\theta = mg$$

By dividing these two equations, we get the angle of the normal force:

$$\tan\theta = \frac{\omega^2 r}{g},$$

$$\theta = \arctan\frac{w^2 r}{g}.$$

By examining the triangle in the original diagram, we see that:

$$\sin\theta = \frac{r}{R},$$

$$N = \frac{m\omega^2 r}{\sin\theta},$$

$$N = m\omega^2 R.$$

b. Again by looking at the triangle, we can get another expression for $\tan\theta$, in addition to the one from part (a).

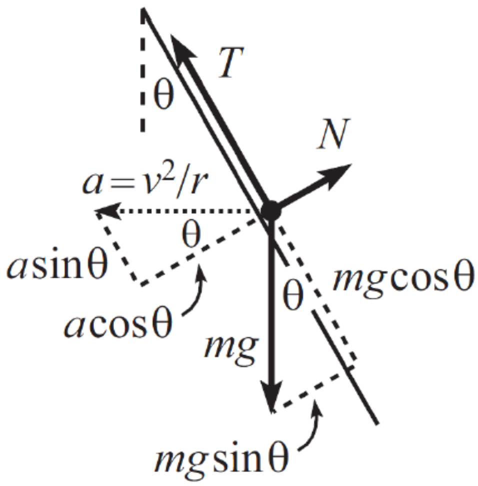
$$\tan\theta = \frac{r}{\sqrt{R^2 - r^2}} = \frac{\omega^2 r}{g}$$

$$\frac{1}{R^2 - r^2} = \frac{\omega^4}{g^2}$$

$$r = \sqrt{R^2 - \frac{g^2}{\omega^4}}$$

c. When $\omega < \sqrt{\frac{g}{R}}$, the above answer for r is imaginary and therefore physically meaningless. Notice that in solving for r in part (b), we divided both sides of the equation by r. This automatically threw out the solution $r = 0$. For small enough ω, $r = 0$ is the correct solution because the marble will naturally sit at the bottom of the bowl.

4. Circling around a Cone



a. The free-body diagram is shown above. The net force produces the horizontal centripetal acceleration of $a = v^2/r$, where $r = l\sin\theta$. Let's work with axes parallel and perpendicular to the cone. Horizontal and vertical axes would work fine too, but things would be a little messier because the tension T and the normal force N would each appear in both of the $F = ma$ equations, so we would have to solve a system of equations.

The $F = ma$ equation along the cone is (using $a = v^2/l$):

$$T - mg\cos\theta = m\left(\frac{v^2}{l\sin\theta}\right)\sin\theta \longrightarrow T = mg\cos\theta + \frac{mv^2}{l}$$

LIMITS: If $\theta \to 0$, then $T \to mg + mv^2/l$. We will find below that if contact with the cone is to be maintained, then v must be essentially zero in this case. So we simply have $T \to mg$, which makes sense because the mass is hanging straight down. If $\theta \to \pi/2$ then $T \to mv^2/l$, which makes sense because the mass is moving in a circle on a horizontal table.

b. The $F = ma$ equation perpendicular to the cone is:

$$mg\sin\theta - N = m\left(\frac{v^2}{l\sin\theta}\right)\cos\theta \longrightarrow N = mg\sin\theta - \frac{mv^2}{l\tan\theta}$$

LIMITS: If $\theta \to 0$, then $N \to 0 - mv^2/l\tan\theta$. Again, we will find below that v must be essentially zero in this case, so we have $N \to 0$, which makes sense because the cone is vertical. If $\theta \to \pi/2$ then $N \to mg$, which makes sense because, as above, the mass is moving in a circle on a horizontal table.

c. The mass stays in contact with the cone if $N \geq 0$. Using the equation from b), this implies that:

$$mg\sin\theta \geq \frac{mv^2}{l\tan\theta} \longrightarrow v \leq \sqrt{gl\sin\theta\tan\theta} \equiv v_{\max}$$

If v equals $v_{\max}$ then the mass is barely in contact with the cone. If the cone were removed in this case, the mass would maintaining the same circular motion.

LIMITS: If $\theta \to 0$ then $v_{\max} \to 0$ and if $\theta \to \pi/2$ then $v_{\max} \to \infty$. These limits make intuitive sense.