Ph1a - Flipped Section Problem Set 3

Solutions

October 14, 2019

1. Leaning Ladder

a. See solutions for problem sheet 2.

b. The situation is mostly the same except that we have a friction $f_2 \leq \mu_2 N_2$, acting at the point of contact between the ladder and the wall, upwards.

The ground has friction $f_1 \leq \mu_1 N_1$.

We want to find the minimal θ that makes the configuration stable, and at that point the frictions would be maximized, so we use $f_1 = \mu_1 N_1$ and $f_2 = \mu_2 N_2$.

Balancing the forces, we get:

(for x)

$$\mu_1 N_1 = N_2$$

(for y)

$$\mu_2 N_2 + N_1 = mg.$$

Balancing torques around the contact point between the ladder and the ground, we get

$$N_2 d \sin\theta + \mu_2 N_2 d \cos\theta = mg\frac{d}{2}\cos\theta$$

From which we get

$$N_2 d \tan\theta = mg\frac{d}{2} - \mu_2 N_2 d.$$

From the force equations, $(\mu_1\mu_2 + 1)N_1 = mg$ and $N_2 = mg\mu_1/(1 + \mu_1\mu_2)$,so

$$\tan\theta = \frac{1}{2}(1 + \mu_1\mu_2)/\mu_1 - \mu_2 = \frac{1}{2\mu_1} - \frac{1}{2}\mu_2.$$

So we need

$$\theta \geq \arctan\left(\frac{1}{2\mu_1} - \frac{1}{2}\mu_2\right).$$

2. Three Clowns

See solutions for problem sheet 2.

3. Inclined Trough

The normal force exerted on the block should be $mg\cos\theta$ in the component perpendicular to the direction of motion. This is a sum of the normal forces from the two sides. Call each of them N. $2N/\sqrt{2} = mg\cos\theta$, so $N = \frac{1}{\sqrt{2}} mg\cos\theta$. This cause a friction $\frac{1}{\sqrt{2}}\mu mg\cos\theta$; but we have two walls, so the total friction is

$$f = \sqrt{2}\ \mu\ m\ g\cos\theta,$$

and $a = mg\sin\theta - f$.

4. Inclined plane with finite mass

First, note that when $M \to \infty$, this is just an inclined plane problem: the large block doesn't move, $A = 0$. When $M \to 0$, the large block doesn't oppose the motion of the small block in any way, and the small block just falls freely.

(Convention: a_x and A are positive toward the right; a_y is positive toward up.)

Consider the small block: it has normal force N from the large block, and also gravity acting on it. The equations of motion are: x: $N\sin\theta = ma_x$ and y: $N\cos\theta - mg = ma_y$.

For the large block, which has a normal force, equal and opposite to the N above, acting on it, the equation of motion in the x-direction gives

$$-N\sin\theta = mA.$$

We require that the small block remains on the slope of the large block (before it falls off, at least). With respect to the outside observer, the the position of the small block after time t is given by

$$(\frac{1}{2}a_x t^2, \frac{1}{2}a_y t^2)$$

if the position at $t = 0$ is $(0, 0)$.

The position of the large block after time t is given by

$$(\frac{1}{2}At^2, 0)$$

since it only moves in the x-direction.

This means that, in the frame of the large block, the position of the small block would be given by

$$(\frac{1}{2}(a_x - A)t^2, \frac{1}{2}a_y t^2).$$

(Note that $a_x - A > a_x$ since A should be negative.)

From the point of view of someone on the large block, all we see is that the small block slides down on the inclined plane, and it slides down with the x and y acceleration given above. From this point of view it is clear that the above vector, which describes the position of the small block with respect to the large inclined plane, should always satisfy the condition that $\tan\theta = -(\frac{1}{2}a_y t^2)/(\frac{1}{2}(a_x - A)t^2)$ (Note that $a_y t^2$ is negative.)

This gives:

$$-\frac{1}{2}a_y t^2 = \tan\theta(a_x - A)\frac{1}{2}t^2.$$

So we get $\tan\theta(a_x - A) = -a_y = -\frac{mg - N\cos\theta}{m}$,

$$-\tan\theta A = -\frac{mg - N\cos\theta}{m} - N\sin\theta/m$$

and $N = -mA/\sin\theta$ so we can solve for A.

5. Springs

a. The block can move in the horizontal direction. Since, the spring force is the only horizontal force acting on the block, the equation of motion is

$$Ma = M\frac{d^2x}{dt^2} = -kx,$$

or

$$\frac{d^2x}{dt^2} + \frac{k}{M}x = 0,$$

where x is the displacement from the equilibrium position. It is convenient to define

$$\omega = \sqrt{\frac{k}{M}},$$

and rewrite the equation of motion as

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

The general solution to this differential equation (the simple harmonic oscillator equation) is

$$x = A\cos\omega t + B\sin\omega t.$$

Equivalently,

$$x = C\cos(\omega t + \phi).$$

The constants A and B or C and ϕ are determined by any initial conditions that may be specified in an oscillator problem as we will see in part b.

(If this is new to you, that is OK. We will learn more about harmonic oscillators at a later point in this course. To see why this solution works, note that taking the derivative of $\sin\omega t$ or $\cos\omega t$ twice with respect to t returns $-\omega^2\sin\omega t$ or $-\omega^2\cos\omega t$ respectively. Therefore these functions are solutions to $\frac{d^2x}{dt^2} = -\omega^2 x$.)

b. Let x be the axis along the direction of motion with the origin at the unstretched position. The position of the piston is given by

$$x(t) = A\sin\omega t + B\cos\omega t,$$

where $\omega = \sqrt{\frac{k}{m+M}}$. This equation is true up to the time the marble and piston lose contact. The velocity is

$$v(t) = \frac{dx(t)}{dt} = \omega A\cos\omega t - \omega B\sin\omega t.$$

The solution has two arbitrary constants, A and B, and to evaluate them we need two pieces of information. We know that at $t = 0$, when the spring is released, the position and velocity are given by

$$x(0) = -L,$$
$$v(0) = 0.$$

Using these values, we find

$$-L = A\sin 0 + B\cos 0 = B,$$

and

$$0 = \omega A\cos 0 - \omega B\sin 0 = \omega A.$$

Hence

$$B = -L, \qquad A = 0.$$

Then, from the time of release until the time when the marble leaves the piston, the motion is described by the eqautions

$$x(t) = -L\cos\omega t, \qquad v(t) = \omega L \sin\omega t.$$

The piston can only push, not pull, on the marble and when the piston begins to slow down, the marble and piston lose contact. At the point the marble will continue to move at a constant velocity. The time t_m at which the velocity reaches a maximum is given by

$$\omega t = \frac{\pi}{2}.$$

Therefore

$$x(t_m) = -L\cos\frac{\pi}{2} = 0,$$

and the final speed of the marble is

$$v_{max} = v(t_m) = \omega L \sin\frac{\pi}{2} = \sqrt{\frac{k}{m+M}}L.$$