Ph1a - Flipped Section

Solutions

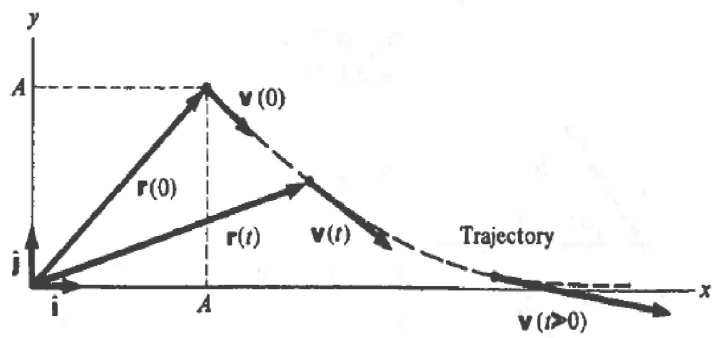
October 7, 2019

1. Sketching trajectories

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = A(\alpha e^{\alpha t}\ \hat{\mathbf{i}} - \alpha e^{-\alpha t}\ \hat{\mathbf{j}}) = v_x\ \hat{\mathbf{i}} + v_y\ \hat{\mathbf{j}}$

Therefore, the magnitude of the velocity, $v = (v_x^2 + v_y^2)^{1/2} = A\ \alpha\ (e^{2\alpha t} + e^{-2\alpha t})^{1/2}$.

In order to sketch the motion, it helps to look at limiting cases. At $t = 0$, $\mathbf{r}(0) = A(\hat{\mathbf{i}} + \hat{\mathbf{j}})$, and $\mathbf{v}(0) = \alpha A(\hat{\mathbf{i}} - \hat{\mathbf{j}})$. As $t \to \infty$, $e^{\alpha t} \to \infty$ and $e^{-\alpha t} \to \infty$. In this limit, $\mathbf{r} \to Ae^{\alpha t}\hat{\mathbf{i}}$, which is a vector along the x axis, and $\mathbf{v} \to \alpha Ae^{\alpha t}\hat{\mathbf{i}}$; the speed increases without limit. The graph would be:



2. Swing! Shoot!

This is a problem of projectile motion in two dimensions under the influence of gravity. Set up your coordinate axis $x - y$ such that the origin coincides with the rock when the string is cut and x - axis is horizontal and y is vertical. The initial speed is $u = 5$ m/s and it makes an angle $\theta = 45°$ with the horizontal as can be seen with some easy geometry. Now, the position of the rock can be written as a function of time,

$$x(t) = (u\cos\theta)t\ , \tag{1}$$

and

$$y(t) = (u\sin\theta)t - \frac{1}{2}gt^2\ , \tag{2}$$

where g is the magnitude of the acceleration due to gravity.

Now, we want to find the x - position of the rock when it is at the same height as when it got cut from the string, *i.e.* $y = 0$, and solving for which from Eq. (2) we get the value of time(s) as,

$$t = 0 \ \text{ or } \ \mathrm{t} = \frac{2\mathrm{u}\sin(\theta)}{\mathrm{g}}\ . \tag{3}$$

We chose the $t > 0$ solution since that is when the rock reaches the same height as when it left the circular motion and for this time, we substitute back in Eq. (1) to get the horizontal distance travelled, which is nothing but,

$$d = (u\cos\theta)\frac{2u\sin(\theta)}{g} = \frac{u^2\sin(2\theta)}{g} , \tag{4}$$

and substituting for the values given, we get,

$$d = \frac{25}{g}m . \tag{5}$$

3. Slanted Snowfall

In the reference frame of the car, the snow has a horizontal component of 60 km/hr (due to Galilean relativity), and still has a vertical component of 3 m/s. Note that 60 km/hr is 50/3 m/s. The angle with respect to the vertical will be given by $\tan\theta = v_x/v_y$ so that $\theta = \arctan(50/9)$.

4. Projectiles and the Firefighter

This is again a rather simple problem of projectile motion in two dimensions under the influence of gravity. Set up your coordinate axis $x - y$ such that the origin coincides with the firefighter's position and $+x$ - axis is horizontal(parallel to the ground and pointing towards the building) and y is vertical. The initial speed is u and it makes an angle, say θ with the horizontal. Now, the position of the rock can be written as a function of time,

$$x(t) = (u\cos\theta)t , \tag{6}$$

and

$$y(t) = (u\sin\theta)t - \frac{1}{2}gt^2 , \tag{7}$$

where g is the magnitude of the acceleration due to gravity.

Now we want to eliminate time from Eq. (6) which can be done as follows,

$$t = \frac{x(t)}{u\cos\theta} . \tag{8}$$

For our case of interest, we want to water stream from the hose to reach the fire in the building which is at a height $y = h$ and a distance $x = d$ away from the firefighter. Thus, it takes a time t_0 for the water to reach the fire,

$$t_0 = \frac{d}{u\cos\theta} , \tag{9}$$

and we can now substitute this in Eq. (7) with $y = h$ to get an equation in θ,

$$h = tan(\theta)d - \frac{1}{2}g\left(\frac{d^2}{u^2\cos^2\theta}\right) . \tag{10}$$

Now getting the quadratic equation in $\tan\theta$ is a matter of simple trigonometric manipulation. Identify $\sec^2\theta = (1 + \tan^2\theta)$ and we obtain,

$$\frac{gd^2}{2u^2}\tan^2\theta - (d)\tan\theta + \left(h + \frac{gd^2}{2u^2}\right) = 0 . \tag{11}$$

5. Canoeing

The crux of the problem is adding vectors. In the absence of a current, Tim would go upstream at some angle θ and speed v. To this vector pointing upstream at some angle θ, we add the vector of the current flowing downstream, with magnitude u. Suppose that the river is flowing in the y-direction, and the x-direction is the direction going across the river. In order for the resultant velocity to be in the x-direction, we know that the y-component of $\vec{v}$ must be equal in magnitude to u. Then we have that $u = v\sin\theta$. This immediately implies that $\theta = \arcsin(u/v)$.

Moreover, we know the resulting speed in the x-direction going across the river will be $v\cos\theta$. But recall $\sin\theta = u/v$, so that $\cos\theta = \sqrt{1-u^2/v^2}$ (since $\cos^2\theta + sin^2\theta = 1$). We know that $v_x = d/t$, so $t = \frac{d}{v\sqrt{1-u^2/v^2}}$

BONUS QUESTION

a. Let B stand for the butterfly and C stand for the boy. Then,

$$x_C = vt, \qquad x_B = x_{B_0} + \frac{1}{2}at^2.$$

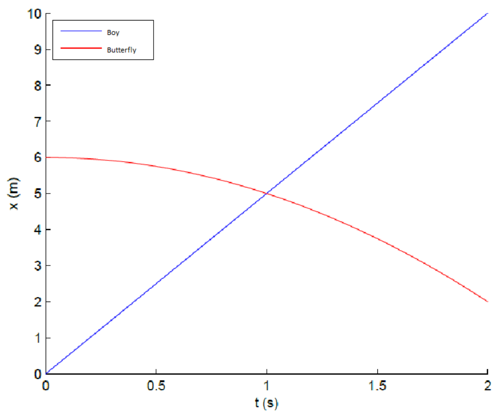
They meet when $x_C = x_B$,

$$vt = x_{B_0} + \frac{1}{2}at^2 \implies \frac{1}{2}at^2 - vt + x_0 = 0.$$

Therefore, the time at which they meet is given by

$$t = \frac{v \pm \sqrt{v^2 - 2ax_0}}{a} = 1 \text{ s}.$$

At this time, $x_C = x_B = vt = 5$ m.



b. The same equations apply but a is now a positive quantity. From the equation for the time at which they meet we see that there are no solutions for t when

$$v^2 - 2ax_0 < 0.$$

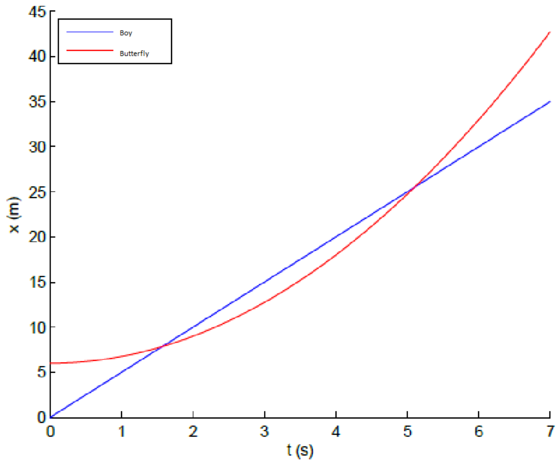
Therefore, $a_{\max}$ occurs when we set this to zero,

$$a_{\max} = \frac{v^2}{2x_0} = 2.1 \text{ m/s}^2$$

When $a = a_{\max}$, the time at which they meet is

$$t = \frac{v}{a} = 2.4 \text{ s}.$$

When $a < a_{\max}$, there are two solutions to the time at which they meet and the boy can ctach up to the butterfly twice.


Boy
Butterfly
x (m)
t (s)
0
5
10
15
20
25
30
35
40
45
0
1
2
3
4
5
6
7