Ph1a - Flipped Section

Problem Set 14 - Solutions

November 21, 2019

1. Elliptic Orbit

Since we are told $m_p >> m_s$, we can effectively take the planet to be stationary at the focus, call it O and consider the motion of the satellite alone. Since gravitational force is central, we know that the orbit(in this elliptical) is planar and angular momentum of the satellite is conserved about O. Now, at both periapsis(closest point in orbit from O) and apoapsis(furthest point in orbit from O), the velocity is perpendicular to the radial vector and the conservation of angular momentum simply gives us,

$$L = m_s r_p v_p = m_s r_a v_a \ . \tag{1}$$

We can also solve for v_p in terms of the constants G, m_p, r_a and r_p as follows. Choose zero for the gravitational potential energy, $U(r \to \infty) = 0$. When the satellite is at the maximum distance from the planet, the mechanical energy is,

$$E_a = \frac{1}{2} m_s v_a^2 - \frac{G m_p m_s}{r_a} \ . \tag{2}$$

When the satellite is at closest approach the energy is,

$$E_p = \frac{1}{2} m_s v_p^2 - \frac{G m_p m_s}{r_p} \ . \tag{3}$$

Since mechanical energy is conserved in central force motion, $E_a = E_p$ and using $v_a = (r_p v_p)/r_a$ from angular momentum conservation, we can solve for v_p,

$$v_p = \sqrt{\frac{2 G m_p r_a}{(r_a + r_p) r_p}} \ , \tag{4}$$

and v_a is,

$$v_a = \sqrt{\frac{2 G m_p r_p}{(r_a + r_p) r_a}} \ . \tag{5}$$

2. Transfer Orbit

a. We know that the energy of a circular orbit is given by:

$$E = K + U = \frac{1}{2} m_s v^2 - G \frac{m_s M_e}{r} \tag{6}$$

and moreover, for a circular orbit, by force considerations:

$$\frac{G M_e m_s}{r^2} = \frac{m_s v^2}{r} \tag{7}$$

i.e. $v^2 = \frac{G M_e}{r}$. So:

$$E = -\frac{G M_e m_s}{2r} = \frac{1}{2} U(r) \tag{8}$$

This result that energy is actually an example of the ”Virial Theorem”, if you are curious.

Thus:

$$\Delta E = -\frac{G M_e m_s}{2} \left(\frac{1}{4R_e} - \frac{1}{2R_e} \right) = \frac{G M_e m_s}{8R_e} \tag{9}$$

b. Denote $v_{A,i}$ as the speed of the object at A before it accelerates to $v_{A,f}$. Morover, $v_{B,i}$ is the speed of the object at B right before it decelerates to $v_{B,f}$. We are basically interested in finding $v_{A,f} - v_{A,i}$ and similarly with point B. We already know from part a that:

$$v_{A,i} = \sqrt{\frac{GM_e}{2R_e}} \tag{10}$$

and

$$v_{B,f} = \sqrt{\frac{GM_e}{4R_e}} \tag{11}$$

So we must determine $v_{A,f}$ and $v_{B,i}$. First of all, since $L = m_s v r$ at the perihelion and aphelion (since $\vec{v} \perp \vec{r}$ at these points), we have by conservation of momentum:

$$m_s v_{A,f} 2R_e = m_s v_{B,i} 4R_e \tag{12}$$

i.e. $v_{A,f} = 2v_{B,i}$. We can also use conservation of energy:

$$\frac{1}{2}m_s v_{A,f}^2 - \frac{GM_e m_s}{2R_e} = \frac{1}{2}m_s v_{B,i}^2 - \frac{GM_e m_s}{4R_e} \tag{13}$$

and thus:

$$v_{A,f}^2 = \frac{2GM_e}{3R_e} \tag{14}$$

and

$$v_{B,i}^2 = \frac{GM_e}{6R_e} \tag{15}$$

and thus we have:

$$\Delta v_A = \sqrt{\frac{GM_e}{R_e}}\left(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}}\right) \tag{16}$$

and

$$\Delta v_B = \sqrt{\frac{GM_e}{R_e}}\left(\sqrt{\frac{1}{4}} - \sqrt{\frac{1}{6}}\right) \tag{17}$$

3. Central Force Proportional to Distance Cubed

a. We know that the potential energy associated with a conservative force is given by (assuming our zero-energy reference point is at $r = 0$):

$$U(r) = -\int_0^r \vec{F}(\vec{r'}) \cdot d\vec{r'} = -\int_0^r (-br'^3)dr' = br^4/4 \tag{18}$$

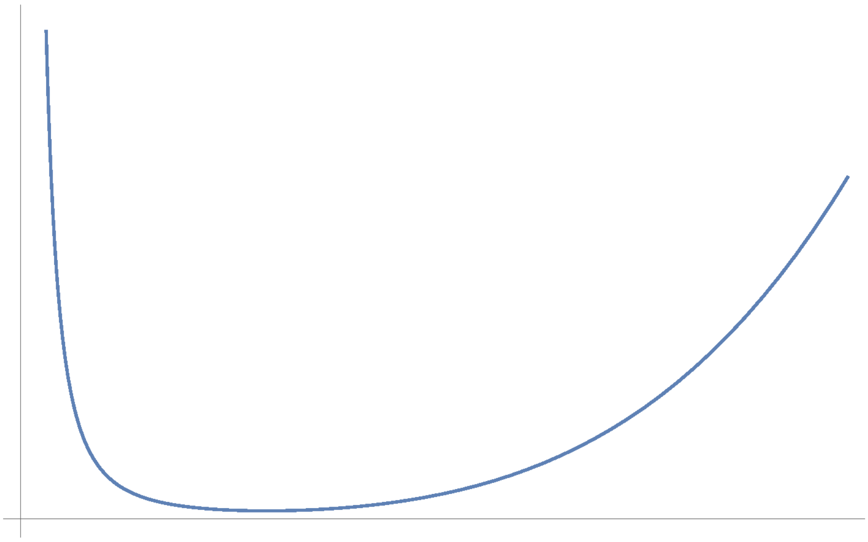
b. Energy is given by:

$$E = \frac{br^4}{4} + \frac{1}{2}m(\dot{\theta}^2 r^2 + \dot{r}^2) = \frac{br^4}{4} + \frac{1}{2}m(\dot{\theta}^2 r^2 + \dot{r}^2) = \frac{br^4}{4} + \frac{L^2}{2mr^2} + \frac{1}{2}m\dot{r}^2 \tag{19}$$

c. Our effective potential is now:

$$U_{\text{eff}}(r) = \frac{br^4}{4} + \frac{L^2}{2mr^2} \tag{20}$$

A sketch can be obtained qualitatively by considering the small r and large r asymptotics. At small r, the r^{-2} term dominates, and at large r, the r^4 term dominates



d. For a circular orbit, the energy must only accommodate exactly one value of the radius. Thus a circular orbit occurs at the minimum $U(r)$.

e. Note that our kinetic energy is simply $\frac{1}{2}m\dot{r}^2$. This implies that at the perihelion and aphelion, the kinetic energy is zero, i.e. at these points we have entirely potential energy. Then if our orbit interpolates between r_0 and $2r_0$, we know that $U_{\text{eff}}(r_0) = U_{\text{eff}}(2r_0)$. Then:

$$\frac{L^2}{2mr_0^2} + \frac{br_0^4}{4} = \frac{L^2}{8mr_0^2} + 4br_0^4 \tag{21}$$

and it is easy to solve from here that:

$$r_0^6 = \frac{L^2}{10mb} \tag{22}$$