Ph1a - Flipped Section

Problem Set 13 - Solutions

November 18, 2019

1. Acceleration in polar coordinates

a. Start with $\vec{r} = r\hat{r}$. We also note that, from $\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y}$ and $\hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y}$ that $\dot{\hat{r}} = \dot{\theta}\hat{\theta}$ and $\dot{\hat{\theta}} = -\dot{\theta}\hat{r}$. Therefore,

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}.$$

b.

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + r\dot{\theta}(-\dot{\theta}\hat{r}) = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}.$$

c. Just note that

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{1}{r}(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

which is the $\hat{\theta}$ component of $\vec{a}$.

This vanishes when $r^2\frac{d\theta}{dt}$ is a constant. As you move from a position (r, θ) to $(r + dr, \theta + d\theta)$, the area swept out is $r^2 d\theta$ (terms of order $drd\theta$ are much smaller and can be ignored), so $r^2\frac{d\theta}{dt}$ is indeed the rate at which you sweep out an area.

2. Bead on a spoke

a. Since the bead moves at constant speed u along the spoke of the wheel, $r = ut$, $\dot{r} = u$, and we have $\dot{\theta} = \omega$. Hence,

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} = u\hat{\mathbf{r}} + ut\omega\hat{\theta}.$$

In cartesian coordinates,

$$v_x = v_r\cos\theta - v_\theta\sin\theta$$
$$v_y = v_r\sin\theta + v_\theta\cos\theta.$$

Since $v_r = u$, $v_\theta = r\omega = ut\omega$, $\theta = \omega t$, the velocity in cartesian coordinates is

$$\mathbf{v} = (u\cos\omega t - ut\omega\sin\omega t)\hat{\mathbf{i}} + (u\sin\omega t + ut\omega\cos\omega t)\hat{\mathbf{j}}.$$

This is an example where the velocity looks much simpler in polar coordinates compared to the expression in cartesian coordinates.

b. The acceleration of the bead is

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = -ut\omega^2\hat{\mathbf{r}} + 2u\omega\hat{\theta}.$$

3. Radial motion without radial acceleration

a. The acceleration of the particle is

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = (\beta^2 r_0 e^{\beta t} - r_0 e^{\beta t}\omega^2)\hat{\mathbf{r}} + 2\beta r_0 \omega e^{\beta t}\hat{\theta}.$$

For the radial component of acceleration to vanish, $(\beta^2 r_0 e^{\beta t} - r_0 e^{\beta t}\omega^2) = 0$, and $\beta = \pm\omega$.

b. The radial component of the acceleration is $(\ddot{r} - r\dot{\theta}^2)$ and contains not only the $\ddot{r}$ but also the $-r\dot{\theta}^2)$ term. Hence, although it may seem surprising that a particle has zero acceleration while executing motion that has $r = r_0 e^{\beta t}$, the issue in thinking about it this way is that we are misled by the special case of cartesian coordinates. In polar coordinates,$v_r \neq \int a_r dt$ because $\int a_r dt$ does not take into account the fact that the unit vectors $\hat{\mathbf{r}}$ and $\hat{\theta}$ are functions of time.

(Also, think about circular motion, in which case there is no radial movement, $r =$ constant. Even though you are not moving radially, there is radial acceleration inward due to the centripetal force. If you had no radial acceleration, you would fly out, which is the case shown in this problem.)