Ph1a - Flipped Section

Problem Set 12 - Solutions

November 14, 2019

1. Oscillating and Rolling

a.The energy of the system at some arbitrary time is

$$E = \frac{1}{2}Mv_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2 + \frac{1}{2}kx^2$$

where $I = \frac{1}{2}MR^2$ is the moment of inertia of a solid disk about its CM (and therefore that of a solid cylinder). Moreover, rolling without slipping gives us the condition $\omega = v_{\text{cm}}/R$. So:

$$E = \frac{1}{2}Mv_{cm}^2 + \frac{1}{4}MR^2(v_{\text{cm}}/R)^2 + \frac{1}{2}kx^2 = \frac{3}{4}Mv_{\text{cm}}^2 + \frac{1}{2}kx^2$$

b. By conservation of energy:

$$0 = \frac{dE}{dt} = \frac{6}{4}Mv_{\text{cm}}\frac{dv_{\text{cm}}}{dt} + kx\frac{dx}{dt} = \frac{3}{2}M\frac{dx}{dt}\frac{d^2x}{dt^2} + kx\frac{dx}{dt}$$

and thus we have the equation:

$$\frac{d^2x}{dt^2} = -\frac{2k}{3M}x$$

which is of course our equation for simple harmonic motion with $\omega_0 = \sqrt{\frac{2k}{3M}}$ and thus the period is:

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{3M}{2k}}$$

c. The force equation for the system is

$$ma = m\frac{d^2x}{dt^2} = -kx - f$$

where f is the frictional force which opposes motion.

(Since the disk is rolling without slipping, the contact point is always at rest with respect to the ground. Hence we are concerned with static friction, not kinetic friction. Kinetic friction would oppose the velocity, but static friction opposes the direction in which the force acts.)

Torque is produced due to the frictional force. The equation is

$$\tau = fR = I\alpha.$$

d. Using $a = R\alpha$,

$$f = \frac{I\alpha}{R} = \frac{Ia}{R^2} = \frac{1}{2}ma.$$

Substituting for the frictional force in the force equation, we get

$$ma = -kx - \frac{1}{2}ma.$$

Thus, equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{2k}{3M}x,$$

and we can find the time period as we did in part (b).

2. Oscillating Bar

a) We want to find an equation of motion for $x(t)$. First of all, let us find the net force on the bar if the bar is displaced a distance x from the center of the rollers (as in the figure). The net force on the bar in the x direction will give us:

$$m\ddot{x} = F_{f1} - F_{f2} = \mu(F_{N1} - F_{N2}) \tag{1}$$

since we know that $F_f = \mu F_N$. So now we must find the normal forces. To do this, we will consider the fact that the net torque on the bar is zero (it is only accelerating linearly and not rotating so torque is zero).

Let's use roller one (the one on the left) as our reference point. Then the torque gives us:

$$F_{N2}D = mg(D/2 + x) \tag{2}$$

We also know from simply summing the forces in the y-direction that:

$$F_{N1} + F_{N2} = mg \tag{3}$$

So from these two equations, it is trivial to see that

$$F_{N1} = mg(1/2 - x/D) \tag{4}$$

$$F_{N2} = mg(1/2 + x/D) \tag{5}$$

Then, going back to equation (1), we find:

$$m\ddot{x} = \mu mg(-2x/D) \tag{6}$$

and thus:

$$\ddot{x} = -\frac{2\mu g}{D}x \tag{7}$$

which we know is simple harmonic motion with frequency $\omega_0 = \sqrt{\frac{2\mu g}{D}}$. Our general solution is then:

$$x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t) \tag{8}$$

Our "starting from rest" condition implies that $A = 0$ and our $x(0) = x_0$ condition implies that $B = x_0$. Then:

$$x(t) = x_0 \cos\left(\sqrt{\frac{2\mu g}{D}}t\right) \tag{9}$$

b) Now we have the rollers going in the opposite direction. Now we have $m\ddot{x} = F_{f2} - F_{f1}$, i.e. the sign has flipped on the right hand side. All of our other force/torque analysis from a) remains the same, so:

$$\ddot{x} = +\frac{2\mu g}{D}x \tag{10}$$

which has the general solution:

$$x(t) = x(t) = A\exp(\omega_0 t) + B\exp(-\omega_0 t) \tag{11}$$

The "starting from rest" condition gives us that $A - B = 0$ and the $x(0) = x_0$ condition gives us that $A + B = x_0$, so we find $A = B = x_0/2$. Then:

$$x(t) = x_0\cosh(\omega_0 t) \tag{12}$$

3. Mass on a spring hitting another mass

a. Using conservation of energy, the amplitude of oscillatory motion d, and spring constant k, one gets

$$\frac{1}{2}kd^2 = \frac{1}{2}mv_i^2 + \frac{1}{2}k\left(-\frac{d}{2}\right)^2$$

which, when rearranged, gives

$$v_i = \frac{d}{2}\sqrt{\frac{3k}{m}}.$$

b. Using conservation of momentum,

$$mv_i = 2mv_f,$$

so

$$v_f = \frac{d}{4}\sqrt{\frac{3k}{m}}.$$

c. For the new system,

$$E = \frac{1}{2}(2m)v^2 + \frac{1}{2}kx^2.$$

Since energy is conserved, $\frac{dE}{dt} = 0$, i.e.

$$0 = 2mv\frac{dv}{dt} + kx\frac{dx}{dt}.$$

From this relation, we get

$$\frac{d^2x}{dt^2} = -\frac{k}{2m}x.$$

We can now identify that $\omega = \sqrt{\frac{k}{2m}}$.

In order to determine the amplitude, consider energy conservation for the new system,

$$\frac{1}{2}kA^2 = \frac{1}{2}(2m)v_f^2 + \frac{1}{2}k\left(-\frac{d}{2}\right)^2$$

which gives

$$A = d\sqrt{\frac{5}{8}}.$$

At $t = 0$, $x = -d/2$. Substituting this in the equation for x gives

$$-\frac{d}{2} = d\sqrt{\frac{5}{8}}\cos\delta$$

and we get

$$\delta = \cos^{-1}\left(\sqrt{\frac{2}{5}}\right).$$