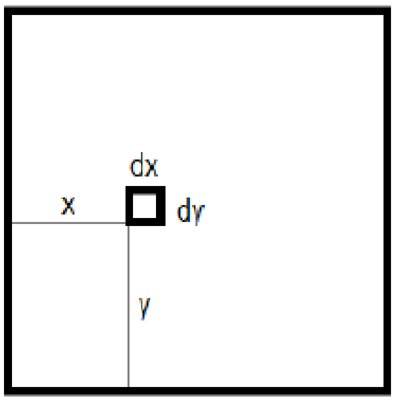
Ph1a - Flipped Section

Problem Set 10 - Solutions

November 7, 2019

1. Moment of inertia of a square

a. So it is easiest to set up the integral for the middle box. We consider an infinitesimal square with sides dx and dy, located at position (x, y):



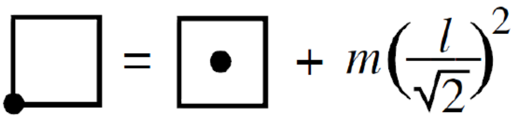
The density of the sheet is M/l^2, so the mass of the infinitesimal square is $dm = Mdxdy/l^2$. The distance to an infinitesimal square is $\sqrt{x^2 + y^2}$. Thus we can set up the integral for moment of inertia:

$$I = \int\int r^2 dm = \int_0^l \int_0^l (x^2 + y^2) \frac{M}{l^2} dxdy = \frac{2}{3} Ml^2 \tag{1}$$

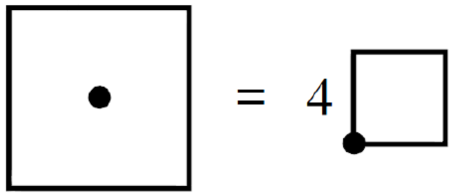
We can also find the moment of inertia of the leftmost square (square rotating about center), by using the parallel axis theorem:

$$I_{\text{COM}} = I' - M\frac{l}{\sqrt{2}} = \frac{2Ml^2}{3} - \frac{Ml^2}{2} = \frac{Ml^2}{6} \tag{2}$$

b. We can determine the relationship from the parallel axis theorem:

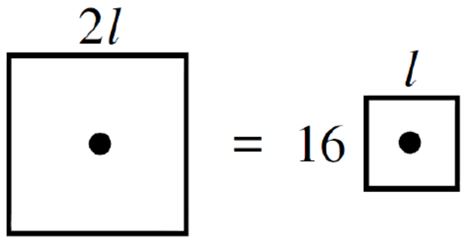


c. Turning the large box around its center is equivalent to turn 4 small ones around their corners:



d. We can use a scaling relationship. Mass scales by a factor of 4, length doubles, so Ml^2 scales by a factor of 16.

e. Just solve the system of equations, treating the figures as symbols representing the corresponding moments of inertia. We find $Ml^2/6$



2. Sliding to a roll

a. Let the ball travel to the right. Define all linear quantities to be positive to the right, and all angular quantities to be positive clockwise, as shown in the figure. (Then, for example, the friction force F_f is negative.) The friction force slows down the translational motion and speeds up the rotational motion, according to (looking at torque around the COM):

$$F_f = ma$$
$$-F_f R = I\alpha$$

Eliminating F_f and using $I = \eta m R^2$ gives $a = -\eta R\alpha$. Integrating this over time, up to the time when the ball stops slipping, gives

$$\Delta V = -\eta R \Delta\omega \tag{3}$$

(This is the same statement as the impulse equation). Using $\Delta V = V_f - V_0$ and $\Delta\omega = \omega_f - \omega_0 = \omega_f$, and also $\omega_f = V_f/R$ (the non-slipping condition), we find:

$$V_f = \frac{V_0}{1+\eta} \tag{4}$$

independent of how F_f depends on position. (For that matter F_f could even depend on time or speed. The relation $a = -\eta R\alpha$ would still be true at all times)

The loss in kinetic energy is given by (using the relation $\omega_f = V_f/R$).

$$\begin{aligned}\Delta KE &= \frac{1}{2}mV_0^2 - (\frac{1}{2}mV_f^2 + \frac{1}{2}I\omega_f^2) \\ &= \frac{1}{2}mV_0^2\left(1 - \frac{1}{(1+\eta)^2} - \frac{\eta}{(1+\eta)^2}\right) \\ &= \frac{1}{2}mV_0^2\frac{\eta}{1+\eta}\end{aligned} \tag{5}$$

For $\eta \to 0$, no energy is lost, which makes sense. And for $\eta \to \infty$, all energy is lost, which also makes sense (this case is essentially like a sliding block which can't rotate).

b. Lets find t. The friction force is $F_f = -\mu mg$. So $F = ma$ gives $-\mu g = a$ (so a is constant). Therefore, $\Delta V = at = -\mu gt$. But our equation for V_f says that $\Delta V = V_f - V_0 = -V_0 \frac{\eta}{1+\eta}$. So we find:

$$t = \frac{\eta}{\mu(1+\eta)} \frac{V_0}{g} \tag{6}$$

For $\eta \to 0$, we have $t \to 0$, which makes sense. And for $\eta \to \infty$, we have $t \to V_0/(\mu g)$ which is exactly the time a sliding block would take to stop. Now let's find d. We have $d = V_0 t + (1/2)at^2$. Using $a = -\mu g$, and plugging in t from above gives:

$$d = \frac{V_0^2}{g} \frac{\eta(2+\eta)}{2\mu(1+\eta)^2} \tag{7}$$

The two extreme cases for η check here. To calculate the work done by friction, one might be tempted to take the product $F_f d$. But the result doesn't look much like the loss in kinetic energy calculated before. What's wrong with this? The error is that the friction force does not act over a distance d. To find the distance over which F_f acts, we must find how far the surface of the ball moves relative to the ground.

The relative speed of the point of contact and the ground is $V_{\text{rel}} = V(t) - R\omega(t) = (V_0 + at) - R\alpha t$. Using $a = -\eta R\alpha$ and $a = -\mu g$, this becomes

$$V_{\text{rel}} = V_0 - \frac{1+\eta}{\eta} \mu g t \tag{8}$$

Integrating over time, this yields:

$$d_{\text{rel}} = \int V_{\text{rel}} dt = \frac{V_0^2 \eta}{2\mu g(1+\eta)} \tag{9}$$

The work done by friction is $F_f d_{\text{rel}} = \mu m g d_{\text{rel}}$, which does indeed give the expected ΔKE.

3. Rolling uphill and downhill

a. Let's say the wheel is initially rolling to the right. We have initial energy $\frac{1}{2}mv_0^2 + \frac{1}{2}I\omega_0^2$ and final energy $mgh + \frac{1}{2}I\omega_0^2$ since it is still spinning when it reaches the maximum height since the incline is frictionless. Hence

$$h = \frac{1}{2}v_0^2/g.$$

As it comes down, it is still spinning clockwise, and $v = \omega R$ does not hold.

b. As it comes down and reaches the surface, it has speed v_0 to the left and also the contact point is moving with speed ωR toward the left relative to the center of mass. The contact point moves with speed $v_0 + \omega R$ with respect to the ground, leftward. Hence, friction $f = \mu mg$ acts toward the right, and this exerts a counterclockwise (ccw) torque

$$\tau = Rf = I\alpha = \eta m R^2 \alpha$$

leading to

$$\alpha = Rf/(\eta m R^2) = f/(\eta m R).$$

Angular velocity (measured ccw) is then

$$\omega(t) = -\omega_0 + \alpha t = -\omega_0 + ft/(\eta m R)$$

Velocity (measured toward the right) is given by:

$$v(t) = v_0 - at = v_0 - (f/m)t$$

since we have $ma = f$ from friction.

The final state is reached when $v(T) = \omega(T)R$.

We know that, initially $\omega_0 = v_0/R$.

$$v_0 - (f/m)T = -\omega_0 R + fT/(\eta m) = -v_0 + fT/(\eta m)$$

$$2v_0 = (f/m)T(1 + \frac{1}{\eta})$$

so $(f/m)T = 2v_0 \frac{\eta}{1+\eta}$.

Then,

$$v(T) = v_0 - (f/m)T = v_0 - 2v_0 \frac{\eta}{1+\eta} = v_0 \frac{1-\eta}{1+\eta}.$$

The time it takes, T, is:

$$T = 2\frac{v_0 m}{f} \frac{\eta}{1+\eta}.$$