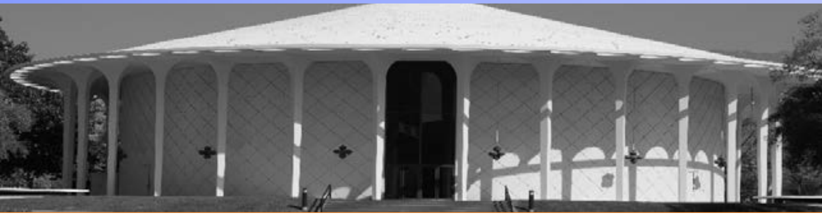

# Rigorous Uncertainty Quantification with Focus on Material Uncertainty

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USACM Specialty Conference on
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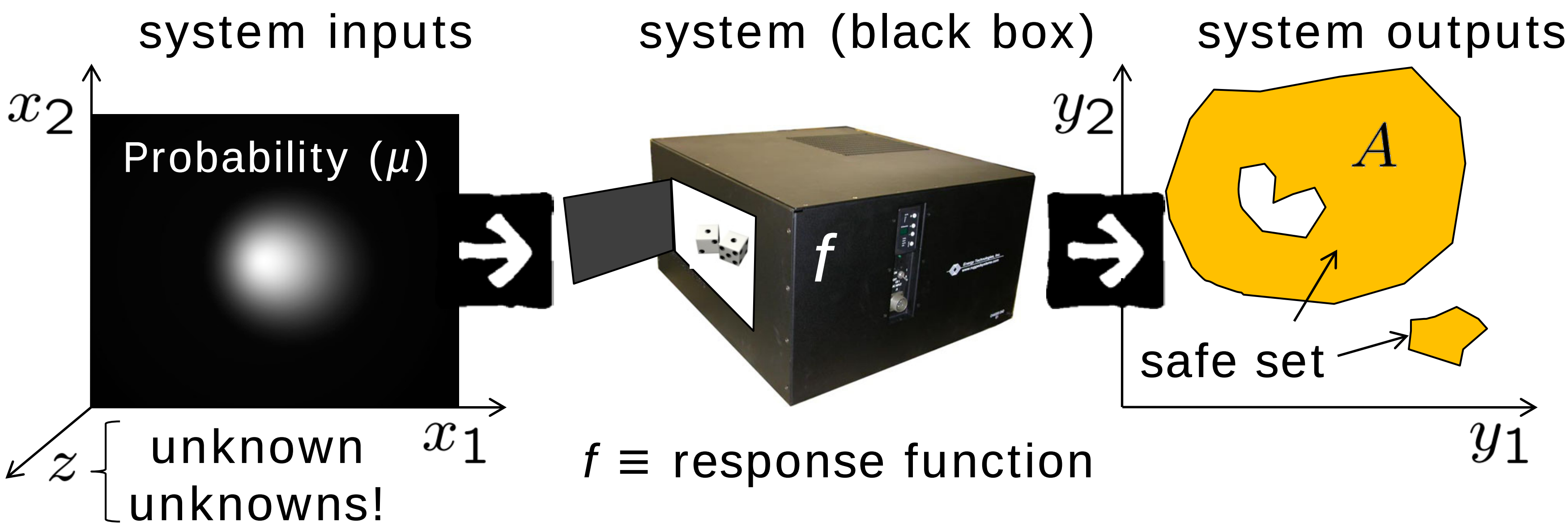
Johns Hopkins University
Baltimore, MD, January 17-18, 2019

# UQ and safe design



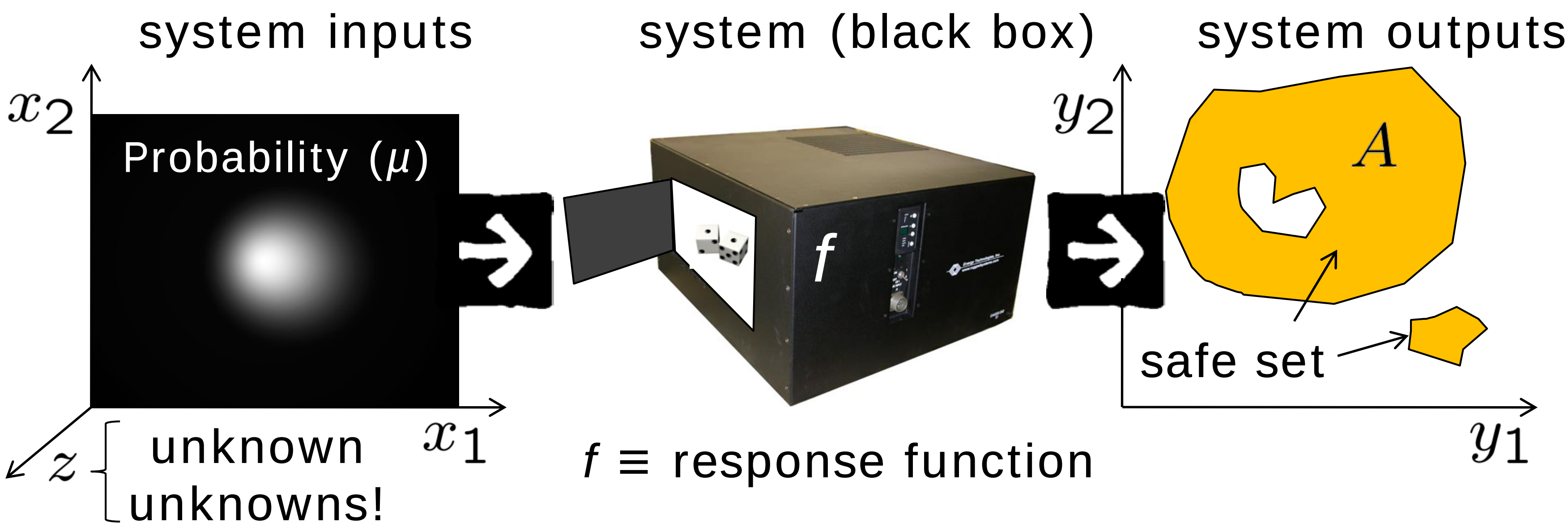

- Safe design: PoF of the system below tolerance,

$$\mathbb{P}[\,\text{failure}\,] = \mathbb{P}[Y \notin A] \leq \epsilon$$


- Exact probability of failure:

$$\mathbb{P}[\,\text{failure}\,] = \int \left\{ \begin{array}{ll} 0, & \text{if } f(x) \in A \\ 1, & \text{if } f(x) \notin A \end{array} \right\} d\mu(x)$$

# UQ and safe design

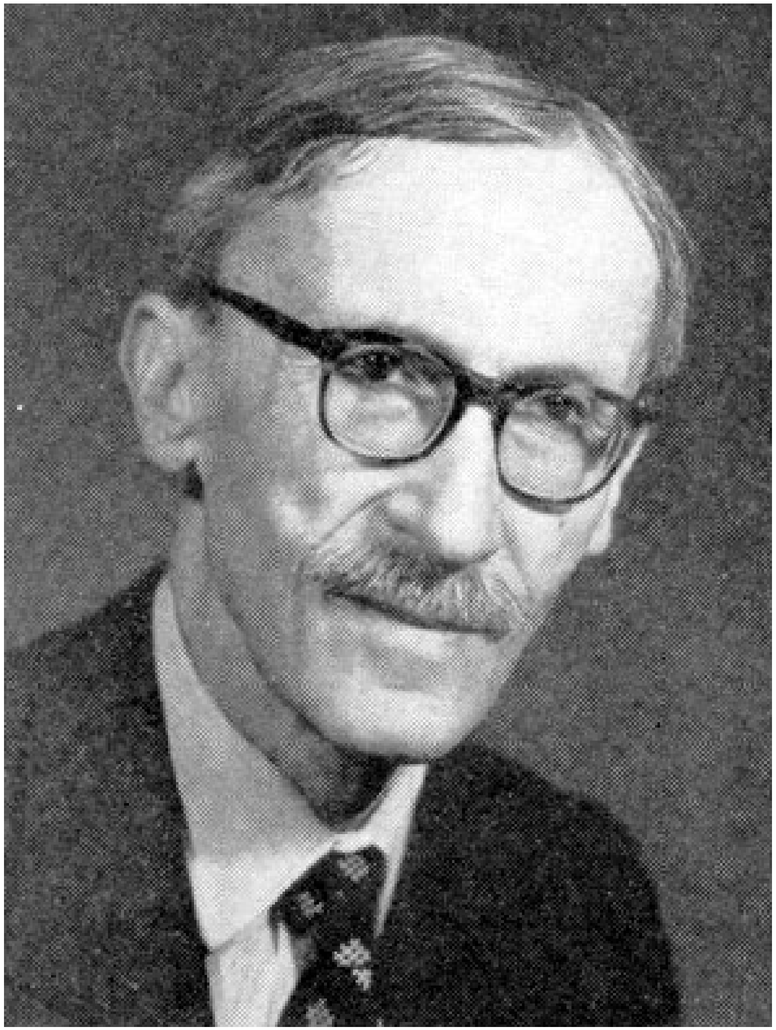



- Conservative design: **Upper bound** on the PoF of the system below tolerance,

$$\mathbb{P}[\,\text{failure}\,] = \mathbb{P}[Y \notin A] \leq \text{upper bound} \leq \epsilon$$

- Objective: Obtain tight (optimal?) PoF **upper bounds** from all known information about the system. . .

# Concentration of Measure PoF bounds



Paul Pierre Levy (1886-1971)

- *CoM* (Levy, 1951): Functions over high-dimensional spaces with small local oscillations in each variable are almost constant
- Example of CoM: *Law of large numbers*
- *Blessing of dimensionality!*
- CoM gives rise to a class of probability-of-failure inequalities that can be used for rigorous UQ and *conservative design*

# McDiarmid's inequality

## ON THE METHOD OF BOUNDED DIFFERENCES

Colin McDiarmid

(1.2) Lemma: Let $X_1,\ldots,X_n$ be independent random variables, with $X_k$ taking values in a set $A_k$ for each k. Suppose that the (measurable) function $f: \Pi A_k \to \mathbb{R}$ satisfies

(1.3) $$|f(\underline{x}) - f(\underline{x}')| \le c_k$$

$D_k = \min c_k$
($x_k$-diameter)

whenever the vectors $\underline{x}$ and $\underline{x}'$ differ only in the kth co–ordinate. Let Y be the random variable $f[X_1,\ldots,X_n]$. Then for any $t > 0$,

$$P(|Y - E(Y)| \ge t) \le 2\exp\left[-2t^2/\Sigma c_k^2\right].$$

$D^2 = \sum D_k^2$
(total diameter)

McDiarmid, C. (1989) "On the method of bounded differences". In J. Simmons (ed.), Surveys in Combinatorics: *London Math. Soc. Lecture Note Series* **141**. Cambridge University Press.

# McDiarmid's inequality

**Theorem** [McDiarmid] *Suppose that:*

*i)* $\{X_1, \ldots, X_N\}$ *are independent random variables,*

*ii)* $f : E \subset \mathbb{R}^N \to \mathbb{R}$ *integrable,* $Y = f(X_1, \ldots, X_N)$.

*Then, for every* $r \geq 0$

$$\mathbb{P}[|\, Y - \mathbb{E}[Y]\,| \geq r] \leq \exp\left(-2\frac{r^2}{D^2}\right),$$

*where* $D$ *is the diameter of* $f$ *over* $E$.

- Bound does not require distribution of inputs
- Bound depends on two numbers: Function *mean* and function *diameter!*

McDiarmid, C. (1989) "On the method of bounded differences". *London Math. Soc. Lecture Note Series* **141**. Cambridge University Press.

# McDiarmid's inequality and safe design

**Corollary** *A conservative safe-design criterion is:*

$$\underbrace{\mathbb{P}[Y \geq a]}_{\text{Probability of failure}} \leq \underbrace{\exp\left(-2\frac{(a-\mathbb{E}[Y])_+^2}{D^2}\right)}_{\text{Upper bound}} \leq \underbrace{\epsilon,}_{\text{Failure tolerance}}$$

- Equivalent statement (confidence factor CF):

$$\text{CF} \equiv \frac{M}{U} \equiv \frac{(a-\mathbb{E}[Y])_+}{D} \geq \sqrt{\log\sqrt{\frac{1}{\epsilon}}} \Rightarrow \text{safe design!}$$

- Rigorous definition of design margin (*M*)
- Rigorous definition of uncertainty (*U* = *D*)

Michael Ortiz


Lucas, L., Owhadi, H. and Ortiz, M., *CMAME*, **197** (2008) 4591–4609.

# Extension to empirical mean

**Theorem** [Lucas, Owhadi, MO] *With probability* $1 - \epsilon'$,

$$\mathbb{P}[Y \geq a] \leq \exp\left(-2\frac{(a - \langle Y \rangle - \alpha)_+^2}{D^2}\right),$$

*where* $\langle Y \rangle = \frac{1}{m}\sum_{i=1}^{m} Y_i$ *and* $\alpha = D\, m^{-\frac{1}{2}}(-\log \epsilon')^{\frac{1}{2}}$.

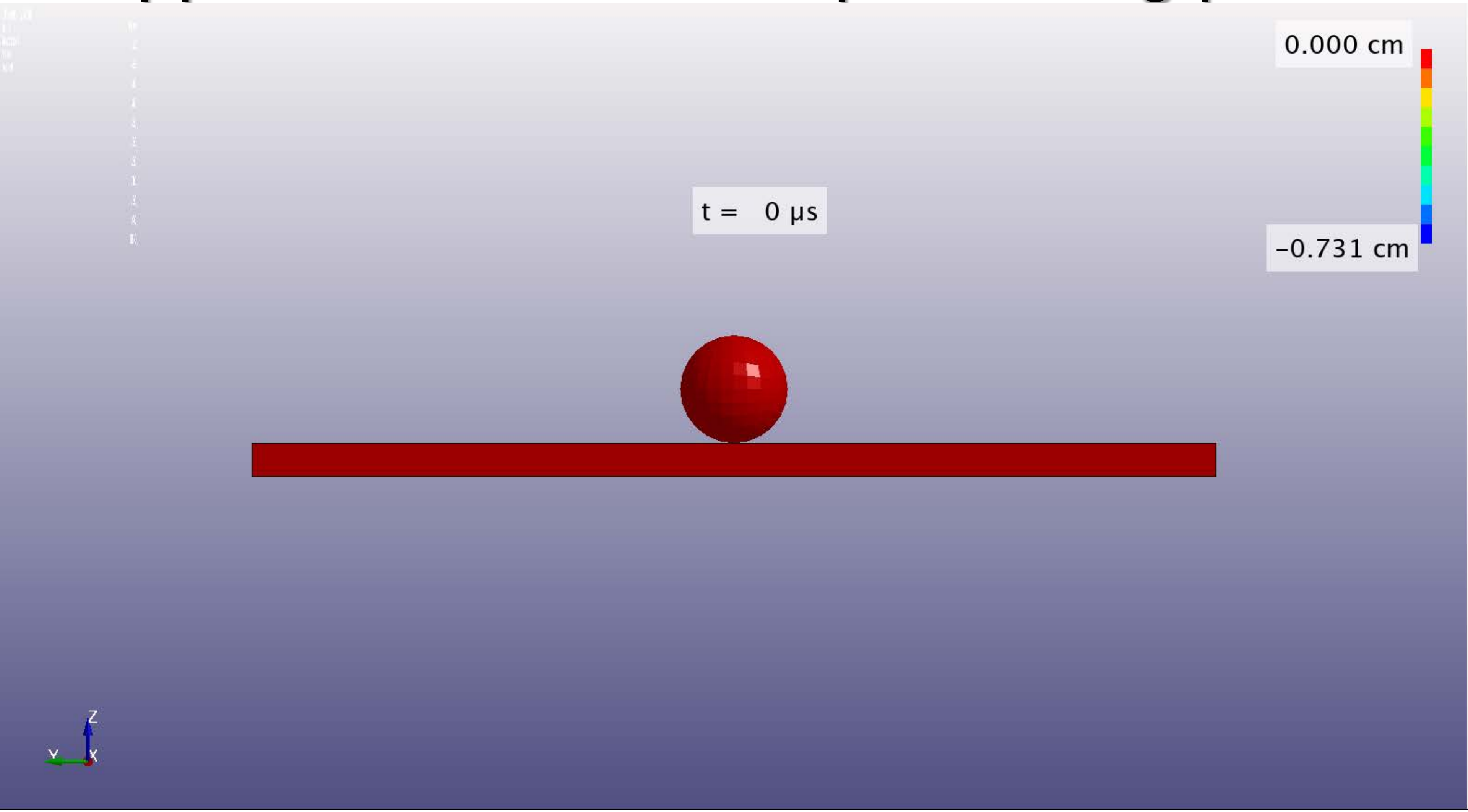
- Equivalent statement (confidence factor CF):

$$\text{CF} \equiv \frac{M}{U} \equiv \frac{(a - \langle Y \rangle - \alpha)_+}{D} \geq \sqrt{\log \sqrt{\frac{1}{\epsilon}}} \Rightarrow \text{safe!}$$

- Use of empirical mean results in margin hit! ($\alpha$)
- Uncertainty remains unchanged ($U = D$)

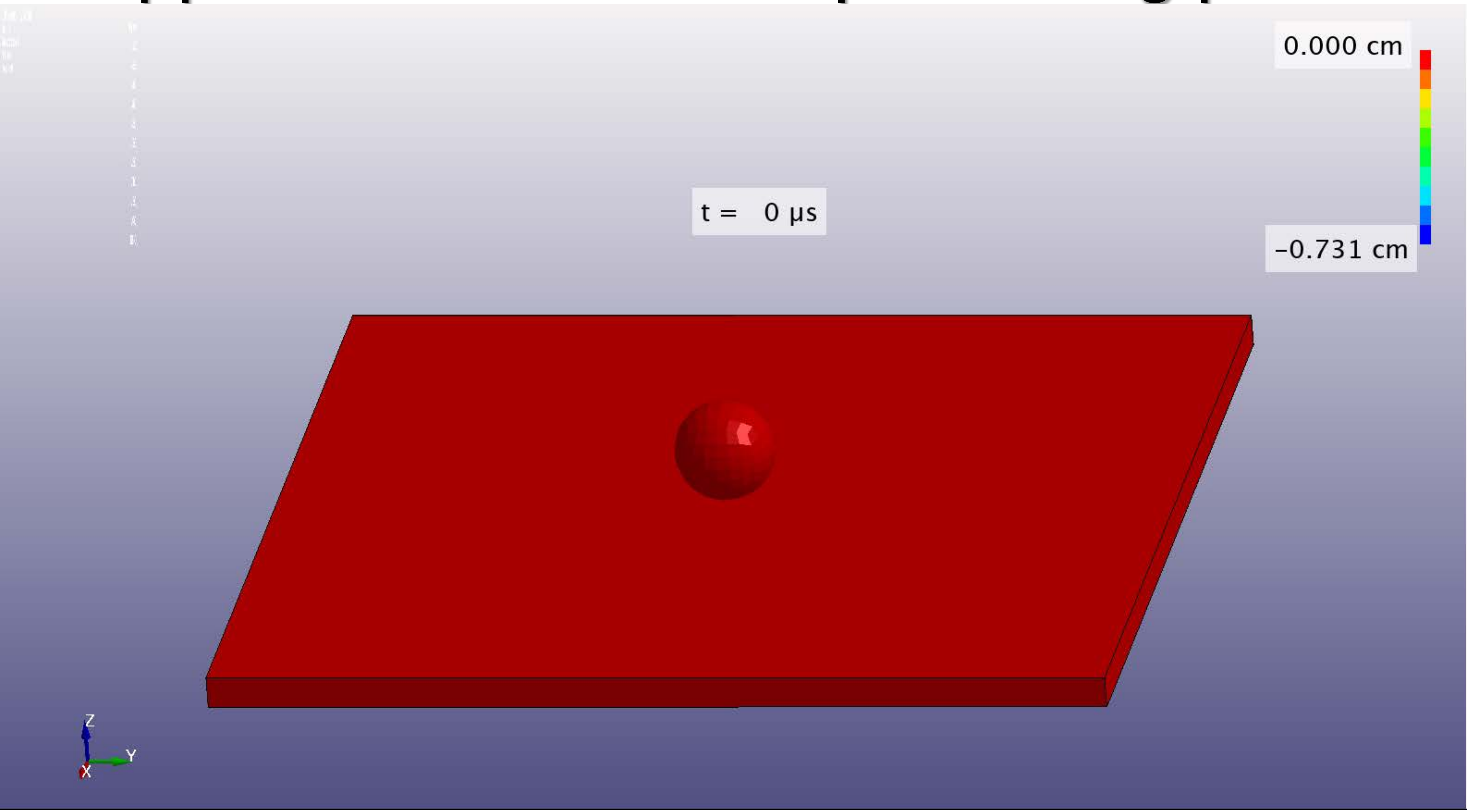
Lucas, L., Owhadi, H. and Ortiz, M., *CMAME*, **197** (2008) 4591–4609.

# Application: Ballistic impact of Mg plates



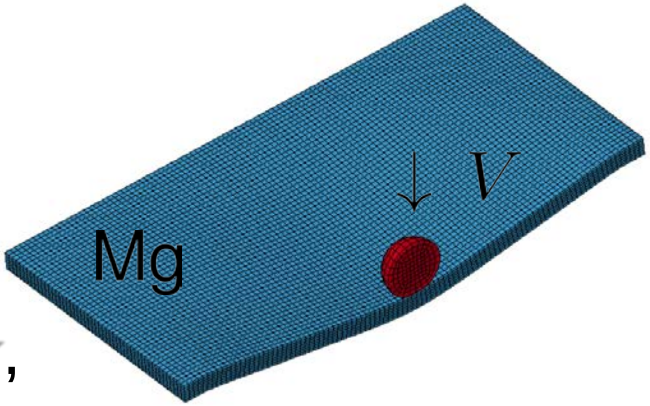

LS-Dyna simulation of spherical 8 gram Pb projectile striking a 10cm x 10cm x 0.35 cm Mg plate at 150 m/s

# Application: Ballistic impact of Mg plates




LS-Dyna simulation of spherical 8 gram Pb projectile striking a 10cm x 10cm x 0.35 cm Mg plate at 150 m/s

# Application: Ballistic impact of Mg plates



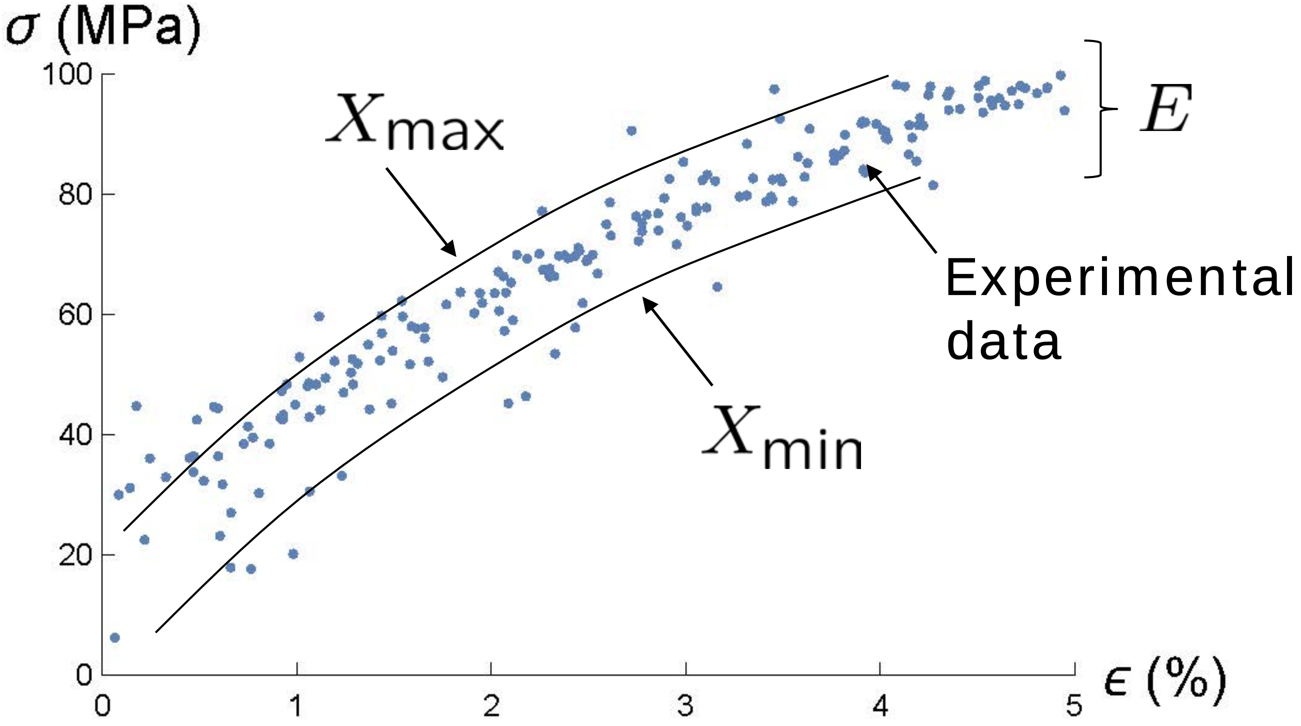

- Objective: Safe design of *protective Mg plates* against (sub)ballistic threats
- Material model: *Johnson-Cook*,

$$\sigma = (A + B\,\varepsilon_p^n)(1 + C \ln \dot{\varepsilon}^*)(1 - T^{*m})$$

- Design criterion: *Indentation* < *allowable*
- Assumption: *Material behavior is the main source of uncertainty*, all other parameters are deterministic (projectile mass, impact velocity...)
- Uncertain parameters: A, B, n, C, m (at all material points in the plate)
- Solvers: *LS-Dyna*, *Dakota* 4.0 (Sandia)

# Application: Ballistic impact of Mg plates

- Johnson-Cook inputs $X = (A, B, n, C, m) \in E$
- Find *parameter ranges* that include a given *percentile* $1 - \epsilon''$ of the experimental data




Michael Ortiz
USACM 01/19

# Application: Ballistic impact of Mg plates

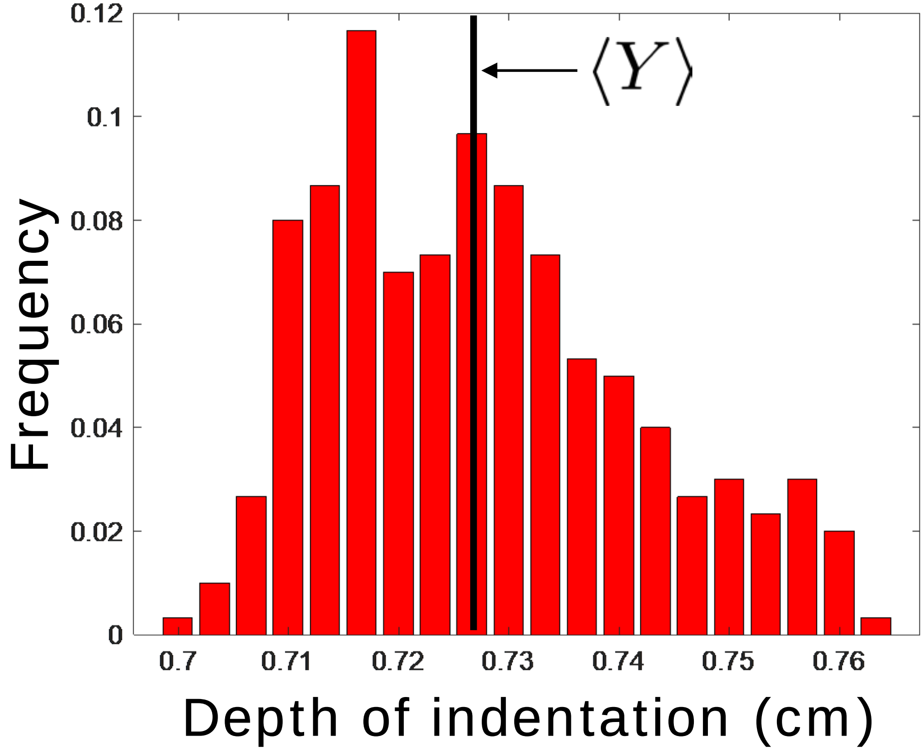
- Johnson-Cook inputs $X = (A, B, n, C, m) \in E$
- Find *parameter ranges* that include a given *percentile* $1 - \epsilon''$ of the experimental data

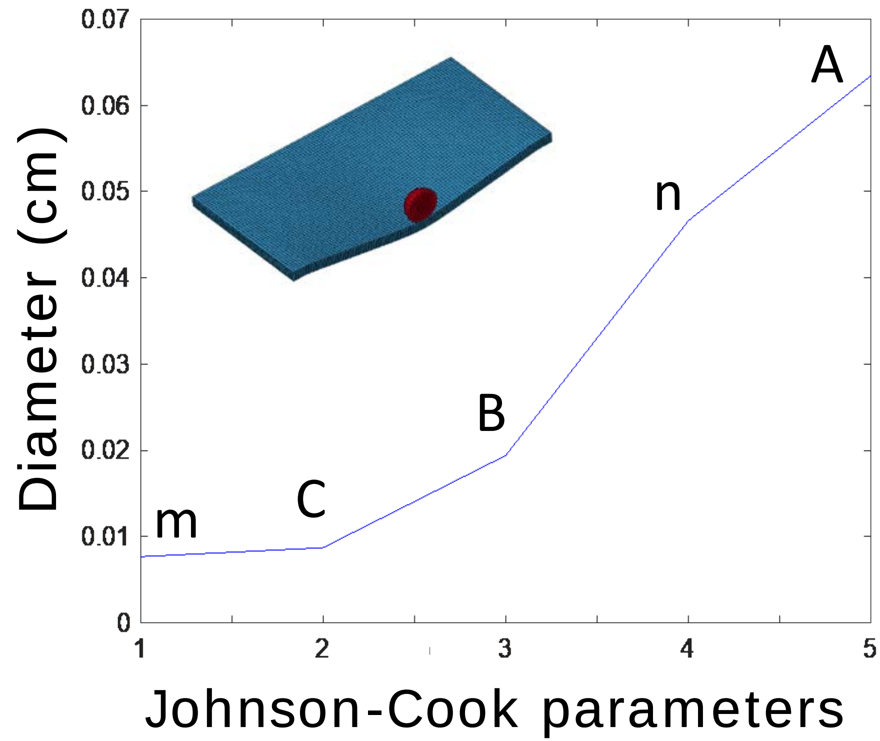
| Parameter | Estimate | Lower 95% | Upper 95% | Uncertainty |
|---|---|---|---|---|
| A [MPa] | 225.171 | 200.372 | 249.970 | +/- 11.01 % |
| B [MPa] | 168.346 | 150.682 | 186.010 | +/- 10.49 % |
| n | 0.242 | 0.160 | 0.324 | +/- 33.88 % |
| C | 0.013 | 0.012 | 0.014 | +/- 7.69 % |
| m | 1.550 | 1.523 | 1.577 | +/- 1.74 % |
| | $R^2$ | 0.896 | $\epsilon'' = 0.05$ | |

D. Hasenpouth, 2010,
Tensile High Strain Rate Behavior of AZ31B Mg Alloy Sheet,
MS thesis, University of Waterloo.

# Application: Ballistic impact of Mg plates

$$\sigma = (A + B\,\varepsilon_p^n)(1 + C \ln \dot{\varepsilon}^*)(1 - T^{*m})$$

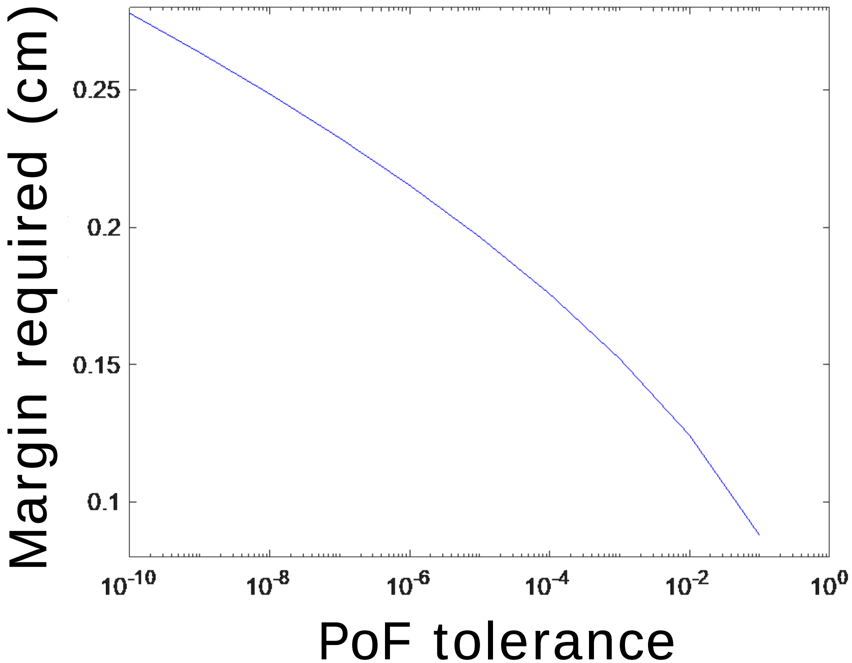


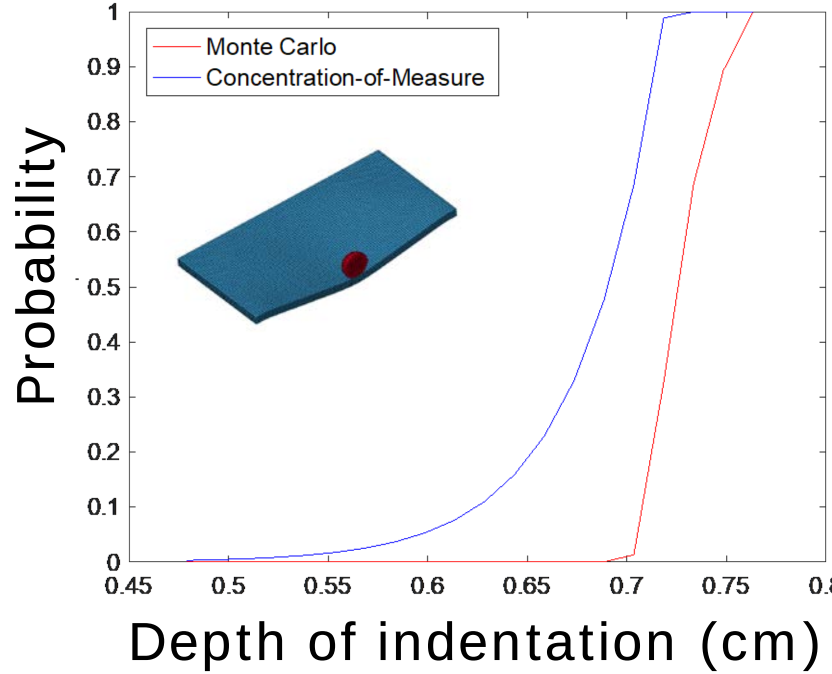




| Sample size (m) | 300 |
|---|---|
| Empirical mean (<Y>) | 0.72761 cm |
| Total diameter (D) | 0.81912 mm |

# Application: Ballistic impact of Mg plates

$$\sigma = (A + B\,\varepsilon_p^n)(1 + C\ln\dot{\varepsilon}^*)(1 - T^{*m})$$




- *Margin requirement* increases (decreases) with uncertainty, sampling confidence (sample size)
- CoM does indeed supply an *upper bound on PoF*, *conservative safe-design criterion*

# Concluding remarks

- Concentration of Measure (CoM) bounds supply computable, practical, *rigorous upper bounds* on probability of failure (PoF) of complex systems
- CoM PoF bounds result in *conservative designs*
- CoM Uncertainty Quantification (UQ) is *non-intrusive*, can be implemented as a *wrapper* around *standard solvers* (e.g., LS-Dyna...)
- *Uncertainties in material behavior* can be *managed effectively and safely through UQ*
- Outlook: Going forward,
  - *Parametric studies (velocity, mass, thickness...)*
  - *Tighter PoF bounds: Optimal UQ[1] (best bounds)*
  - *Machine learning of data sets, optimal ranges*

[1]Owhadi, H. *et al*., "Optimal UQ", *SIAM Review*, **55**(2) (2013) 271-345.

# Concluding remarks

Thank you!