Variational models of dynamic fracture

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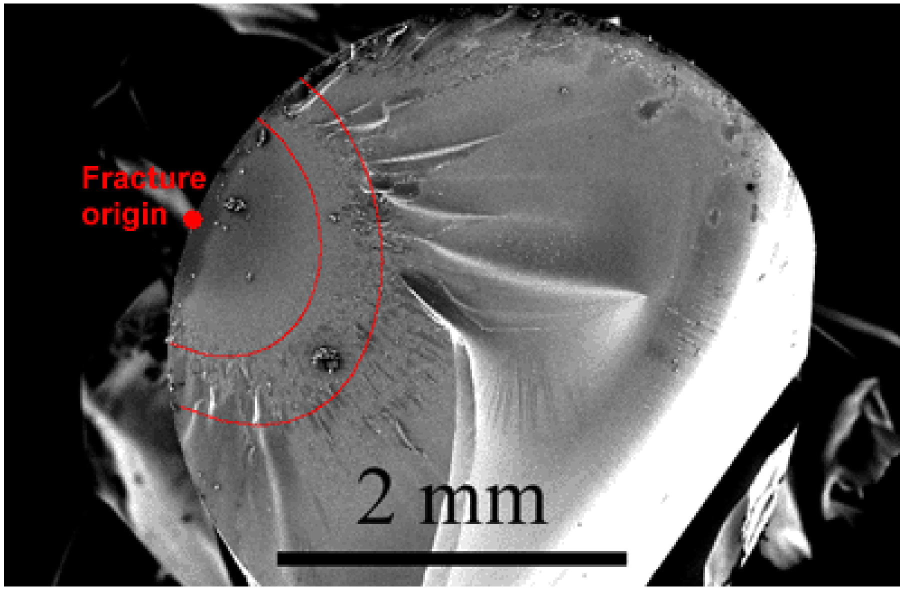
Mini-Workshop on Mathematical Models, Analysis and Numerical Methods for Dynamic Fracture
Mathematisches Forschungsinstitut,
Oberwolfach, April 26, 2011

Overarching objectives

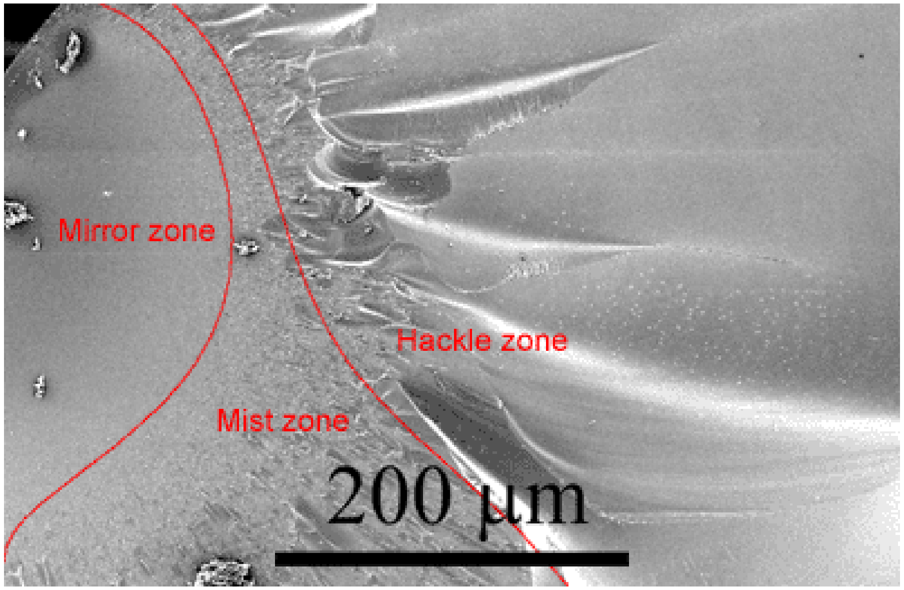
- Re-formulation of physically meaningful, experimentally validated, dynamic fracture models as a ***free-discontinuity*** problems
- ***Existence*** theory:
  - *Sufficient conditions for existence of solutions, approximate solutions, attainment*
  - *Examples of non-existence, ill-posed models*
- ***Effective models***: Surface roughness, fragmentation
- Approximation theory (for relaxed models):
  - *Time discretization*
  - *Spatial discretization (including crack set)*

Dynamic fracture

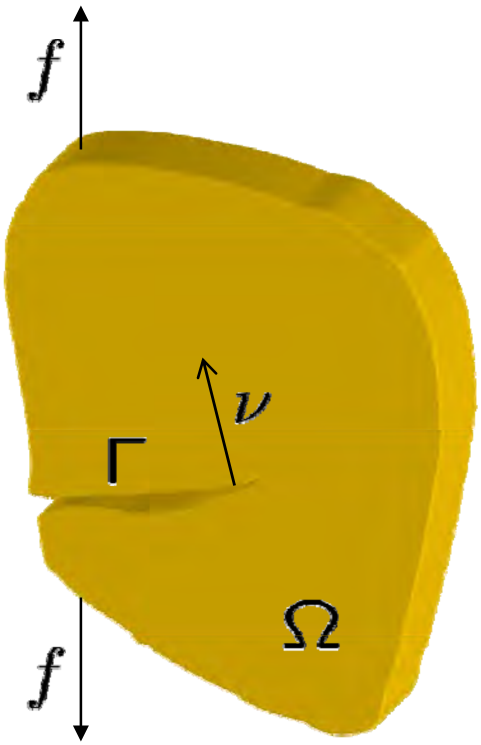


Fracture of soda-lime glass rod

Dept. Materials Science and Metallurgy, University of Cambrige



Irreversible cohesive models of fracture



- Damage measure: $v\mathcal{H}^{n-1}\llcorner\Gamma > 0$
- Displacement gradients, $J_u \subset \Gamma$:

$$Du = \nabla u + [\![u]\!] \otimes \nu_u \mathcal{H}^{n-1}\llcorner J_u$$

- Total energy: $E(u,v) =$

$$\int_{\Omega\backslash J_u} W(\nabla u)\,dx + \int_{J_u} \gamma([\![u]\!], v)\,d\mathcal{H}^{n-1}$$

- Dissipative potential: $\Psi(\dot{v}) = \int_\Gamma \psi(\dot{v})\,d\mathcal{H}^{n-1}$
- Equilibrium: $u \in \text{argmin} E(\cdot, v)$
- Kinetics: $0 \in \partial\Psi(\cdot) + D_2 E([\![u]\!], \cdot)$

SOLVE ME!

Irreversible cohesive models of fracture

- Example: Camacho, G. and MO, *Int. J. Solids & Structures*, **33** (1996) 2899
- Surface energy: Damage + elastic unloading-reloading

$$\gamma(\delta, v) = \phi(v) + \frac{D\phi(v)}{2v}(\delta^2 - v^2)$$

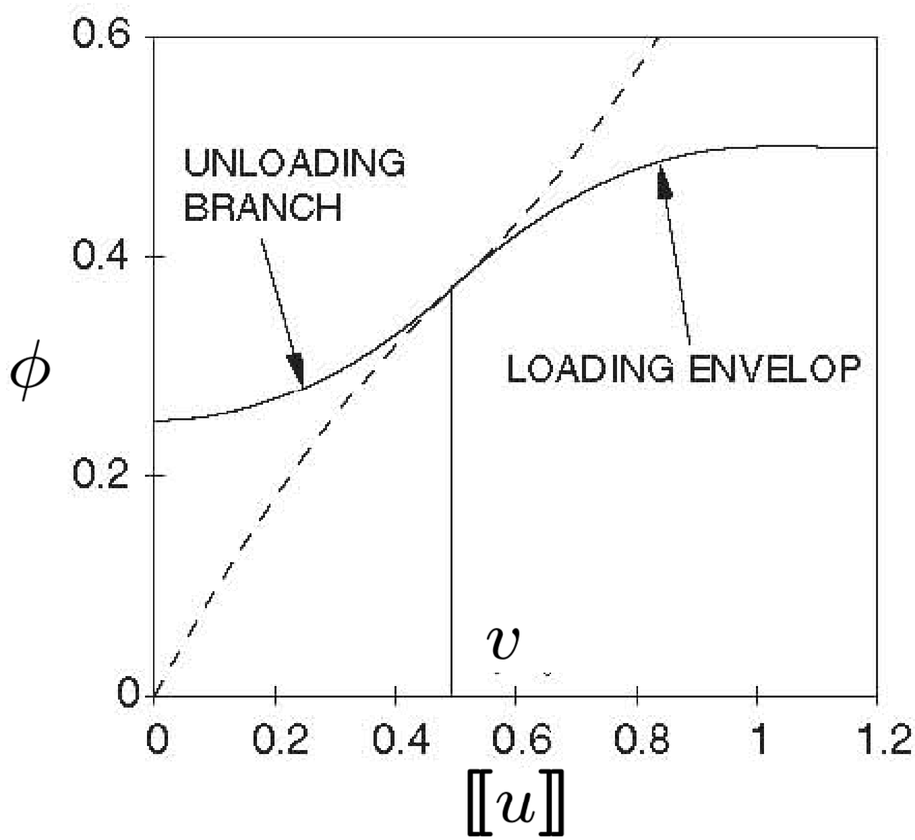
where $\phi(\delta) \equiv$ monotonic envelop

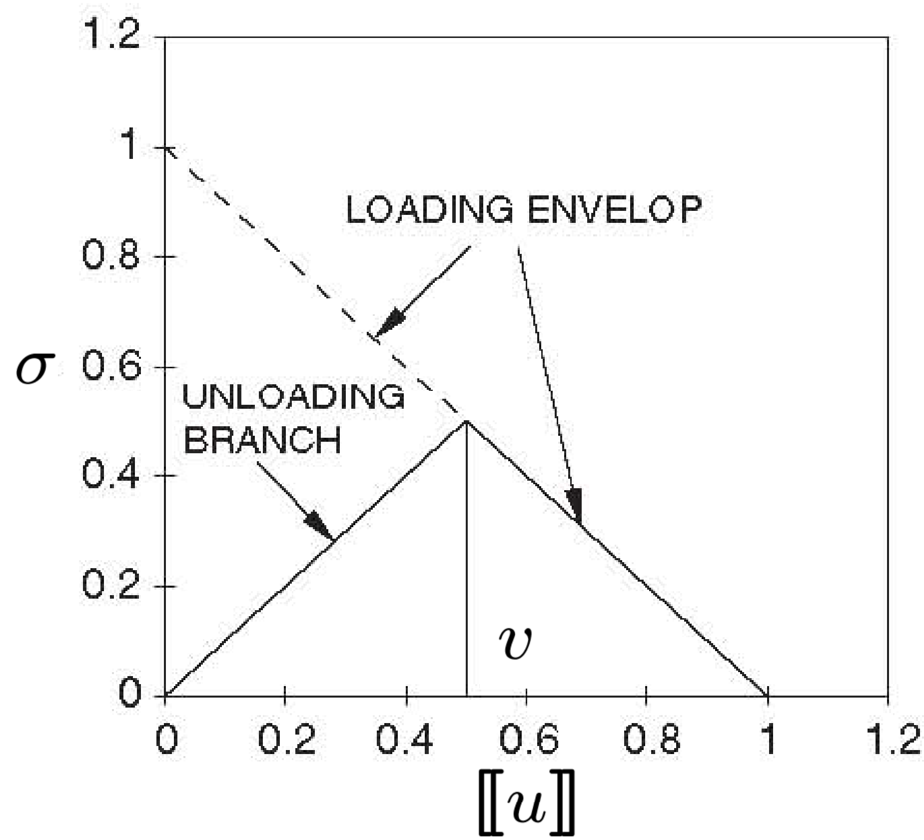
- Rate-independent dissipation potential:

$$\psi(\dot{v}) = \begin{cases} 0, & \text{if } \dot{v} > 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Irreversible cohesive models of fracture

A. Pandolfi et al. / J. Mech. Phys. Solids 54 (2006) 1972 2003





Camacho, G. and MO, *Int. J. Solids & Struct*., **33** (1996) 2899.

Extension to dynamic fracture

- Kinetic energy: $K(\dot{u}) = \int_\Omega \frac{\rho}{2}|\dot{u}|^2\,dx$
- Action: $I(u,v) = \int_0^T [K(\dot{u}(t)) - E(u(t),v(t))]\,dt$
- First variation: $\delta I(u,v,\varphi,(a,b)) =$

$$\int_a^b [\langle \rho\dot{u}(t), \dot{\varphi}(t)\rangle - \langle D_1 E(u(t),v(t)), \varphi(t)\rangle]\,dt$$

- Hamilton's principle of stationary action:

$$\delta I(u,v,\varphi,(a,b)) = 0, \quad \forall (a,b) \subset [0,T]$$

- Kinetics: $0 \in \partial\Psi(\cdot) + DE([\![u]\!],\cdot)$

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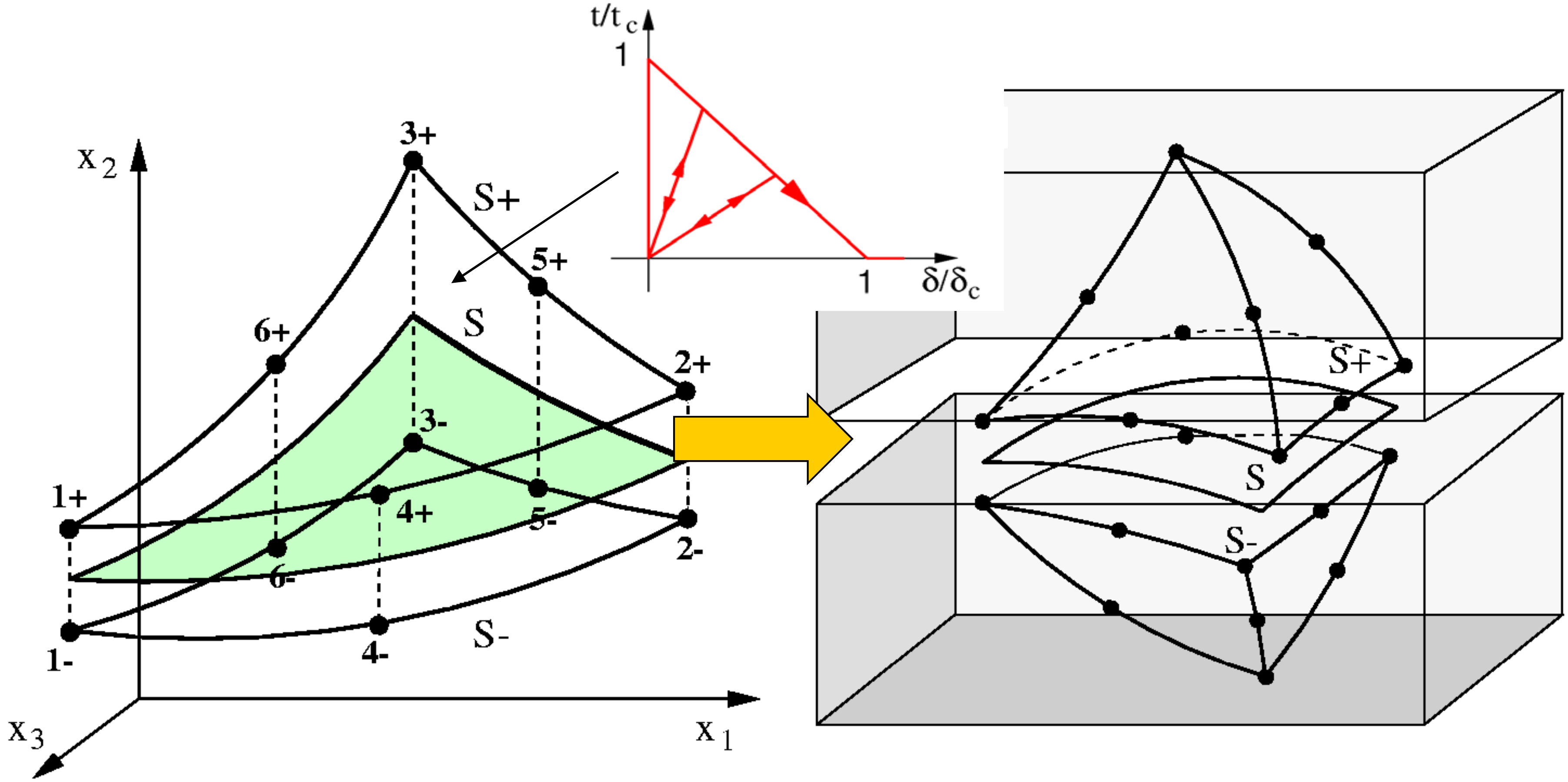
Cohesive fracture - Implementation

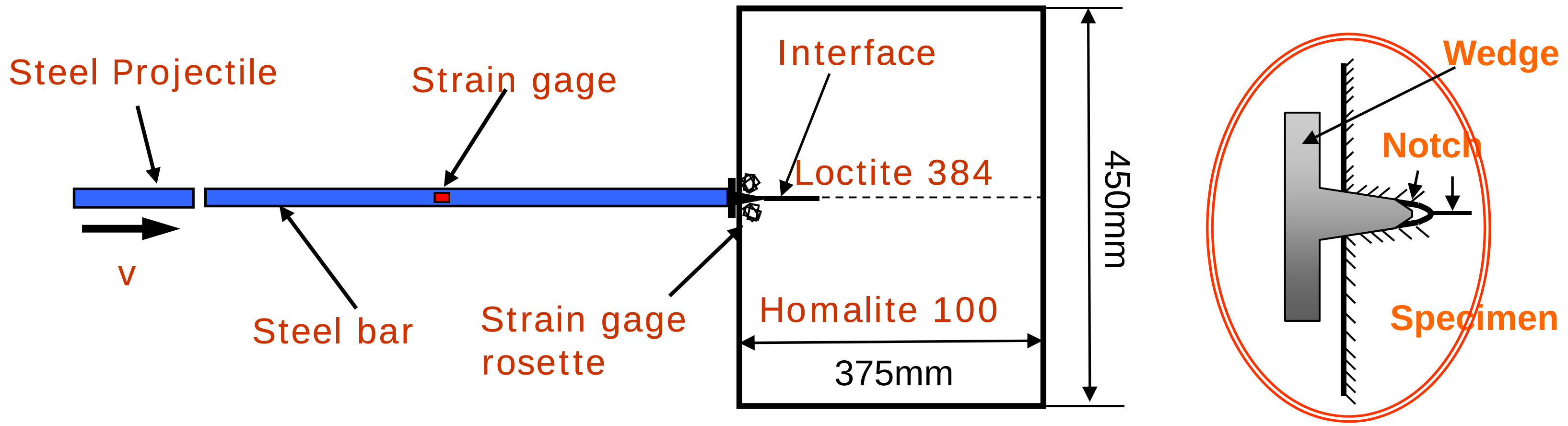


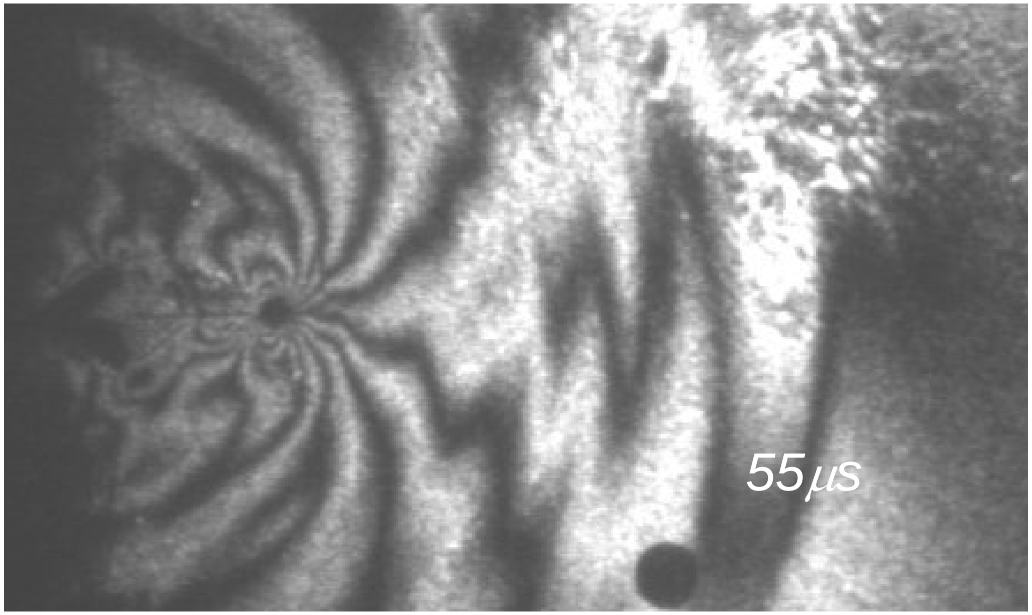
12-node quadratic cohesive elements

Insertion of cohesive element between two volume elements

MO and Pandolfi, A., *IJNME*, **44** (1999) 1267.

Dynamic splitting test – Homalite 100





Chalivendra, V.B. *et al*., *Int. J. Impact Eng.*, **36** (2009) 888

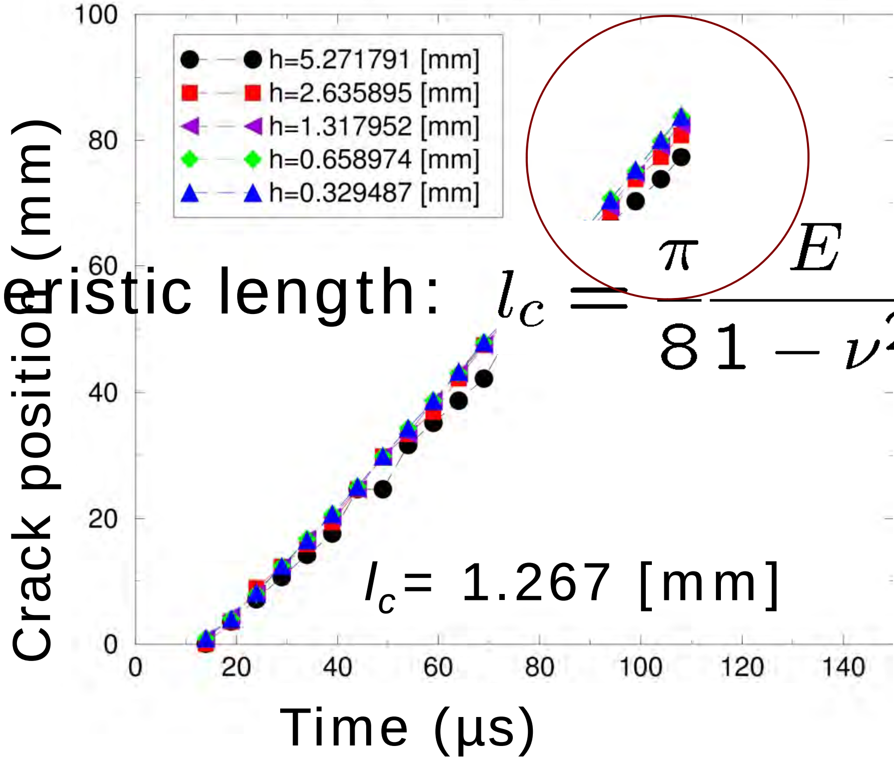
Cohesive elements – Verification

- Homalite-100 characteristic length – l_c= 0.094 [mm]
- Element sizes - *uniform subdivision*

Subdivision level	Element size [mm]	Number elements
0	5.271	32,440
1	2.636	129,760
2	1.318	519,040
3	0.659	2,076,160
4	0.329	8,304,640
5	0.165	33,218,560
6	0.082	132,874,240

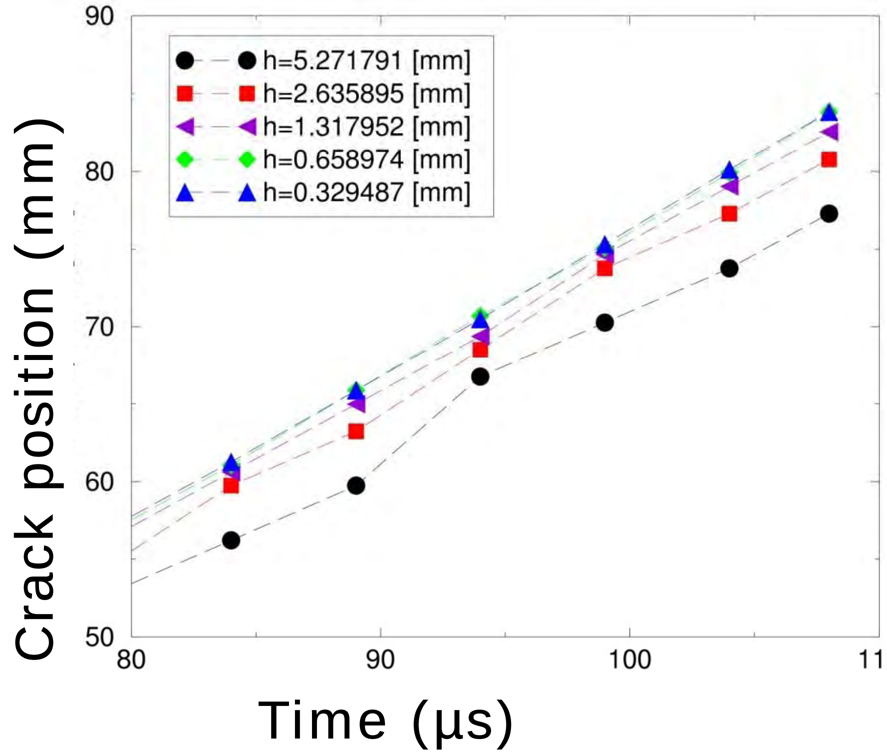
I. Arias *et al*., *CMAME,* **196** (2007) 3833

Cohesive elements – Verification



cteristic length: $l_c = \frac{\pi}{8}\frac{E}{1-\nu^2}$

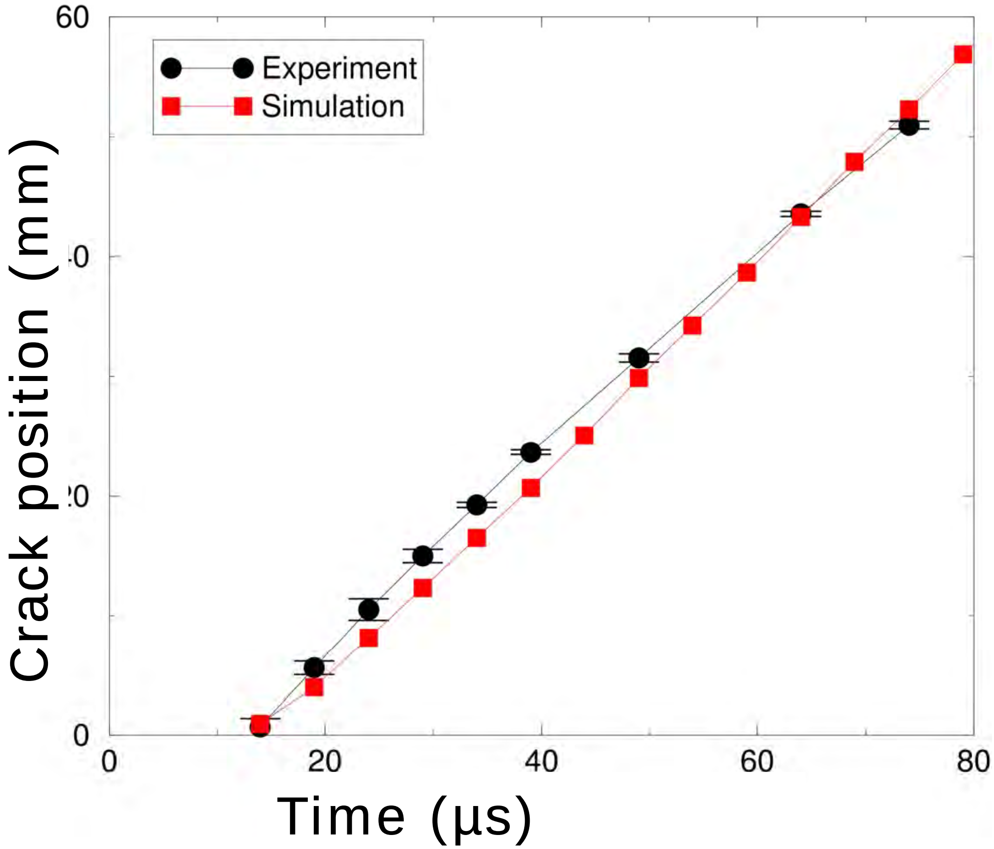
l_c = 1.267 [mm]



- Convergence attained when l_c is resolved!

I. Arias *et al*., *CMAME,* **196** (2007) 3833

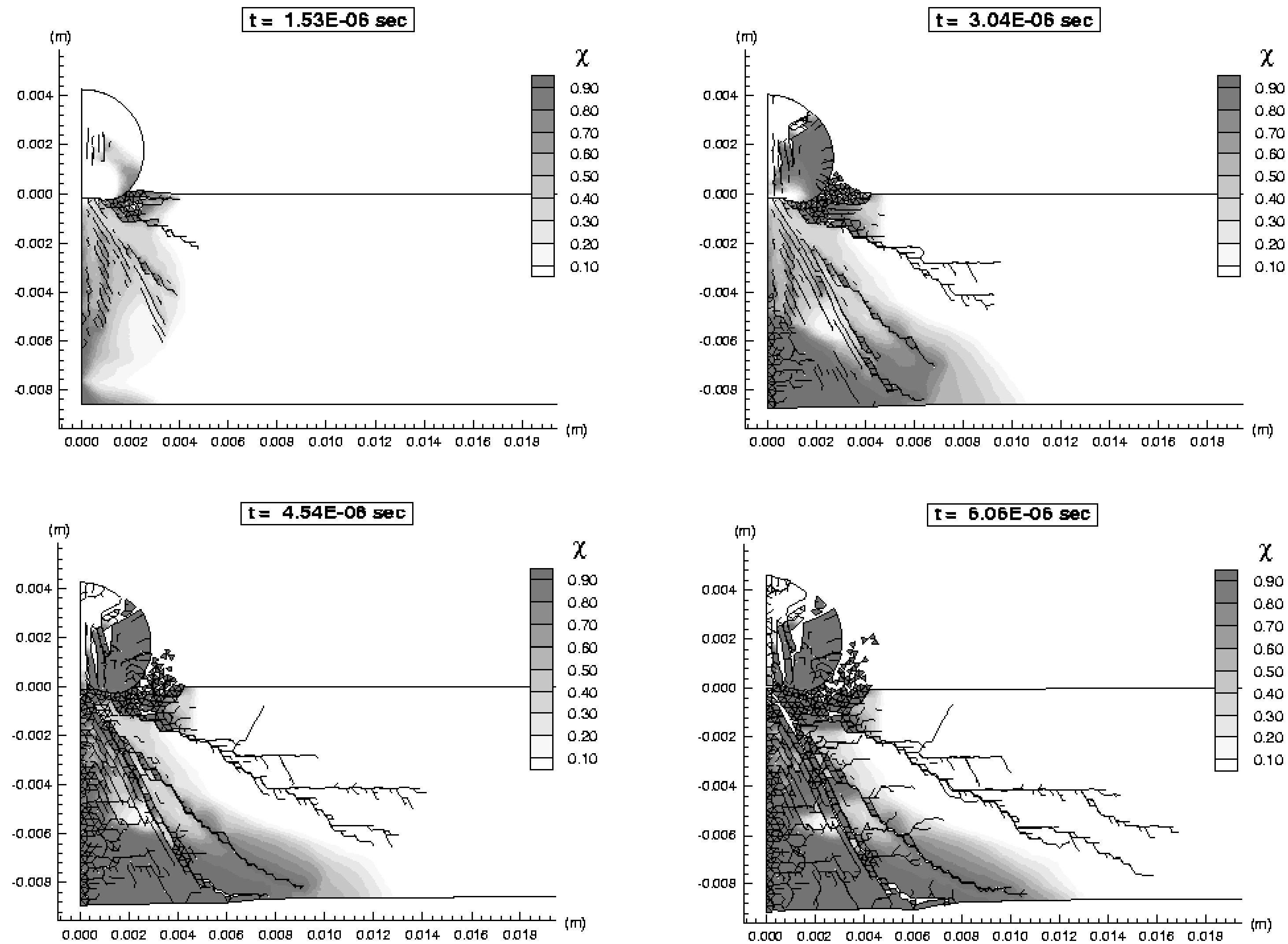
Cohesive elements – Validation



- Experimental velocity 832 [m/s]
- Simulation velocity 893 [m/s]
- Experimental crack initiation time: 13 mm
- Simulation crack initiation time: 13 mm

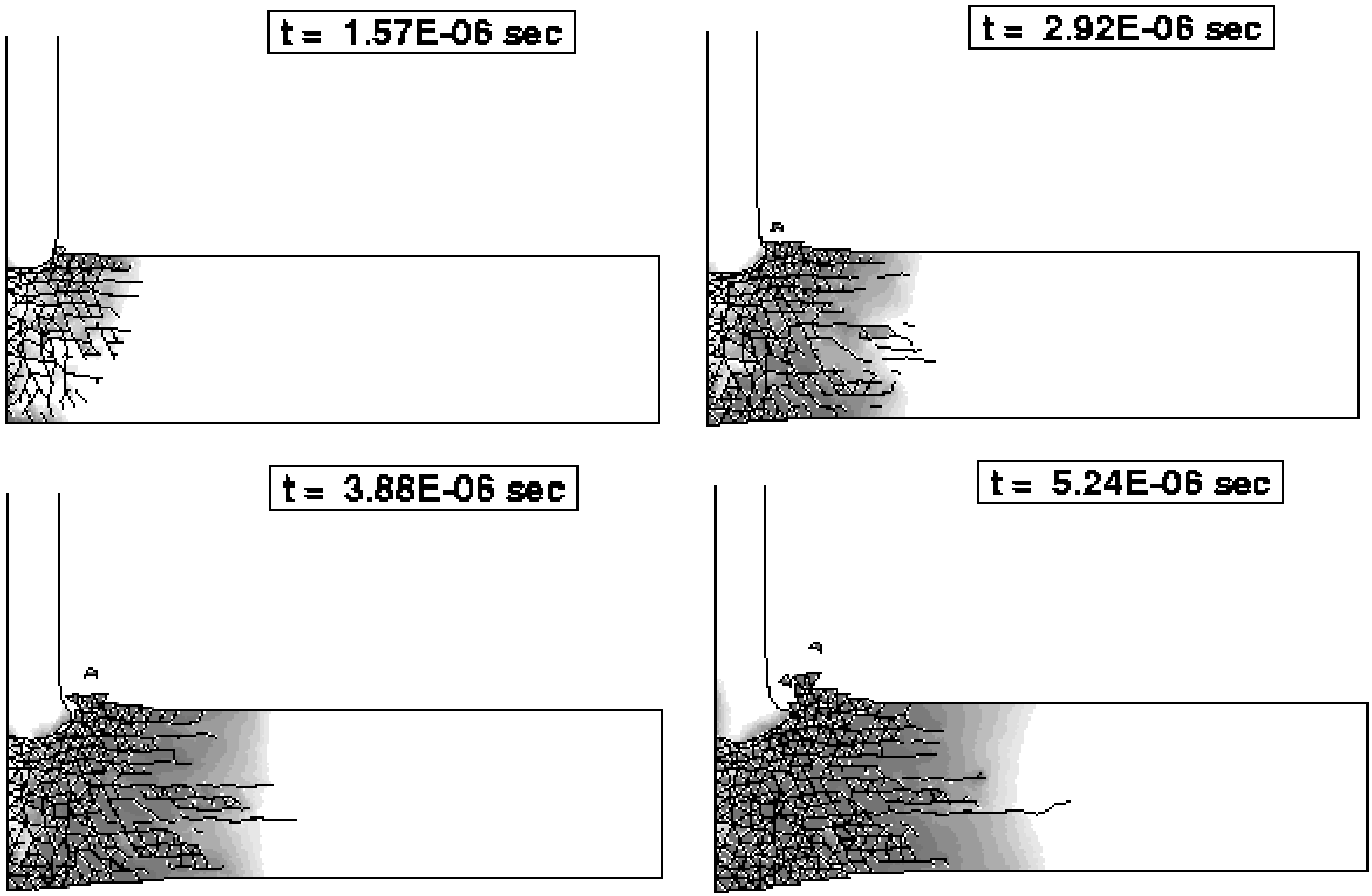
Chalivendra, V.B. *et al*., *Int. J. Impact Eng*., **36** (2009) 888

Steel pellet vs. alumina plate



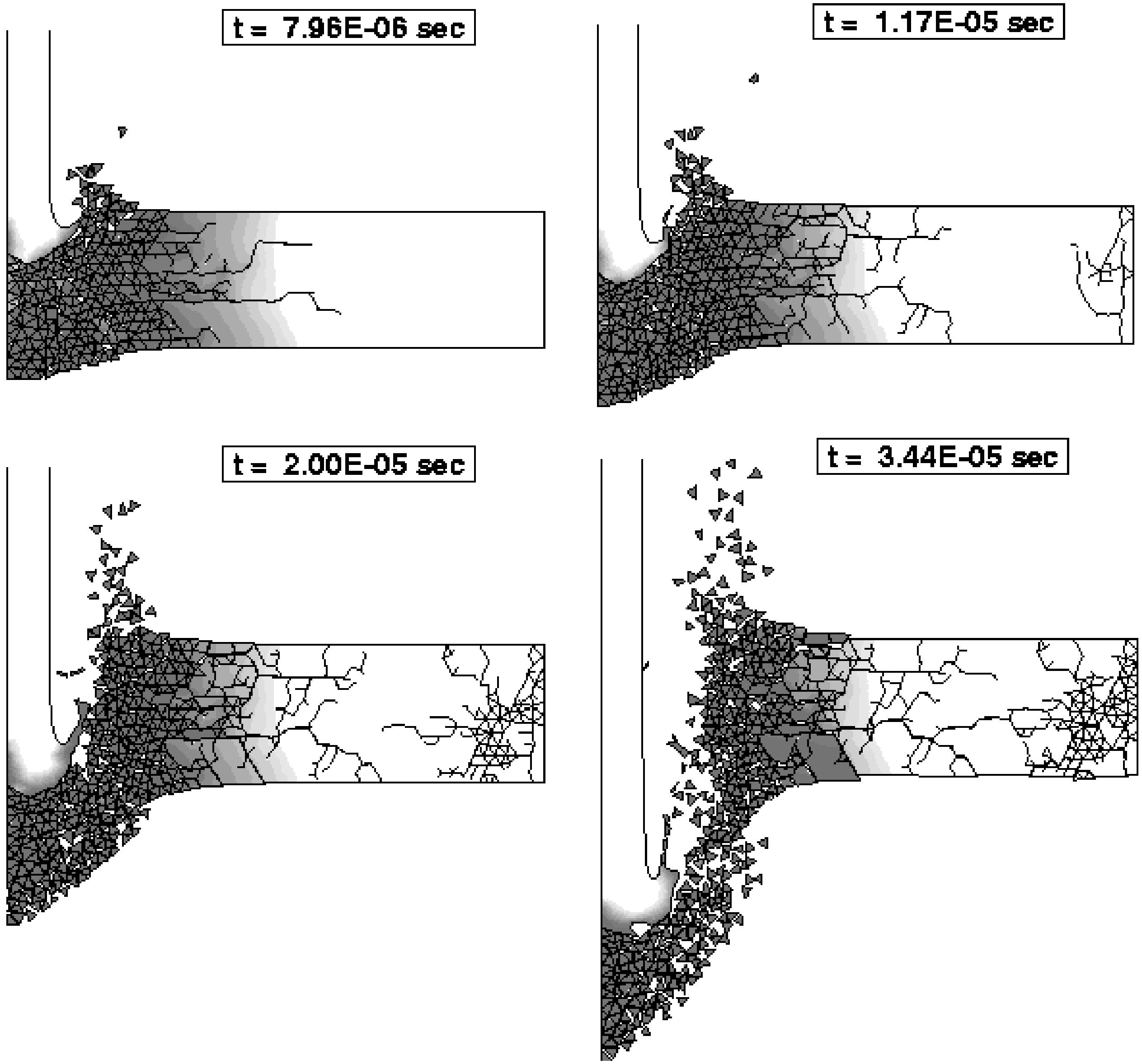
Camacho, G.T. and MO, *Int. J. Solids & Struct.*, **33** (1996) 2899

WHA long rod vs. alumina plate



Camacho, G.T. and MO, *Int. J. Solids & Struct.*, **33** (1996) 2899

WHA long rod vs. alumina plate



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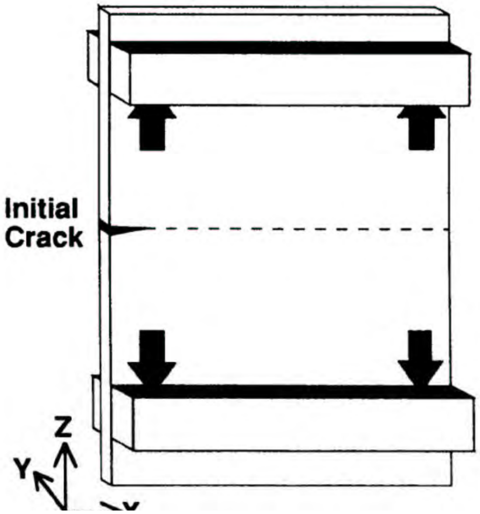
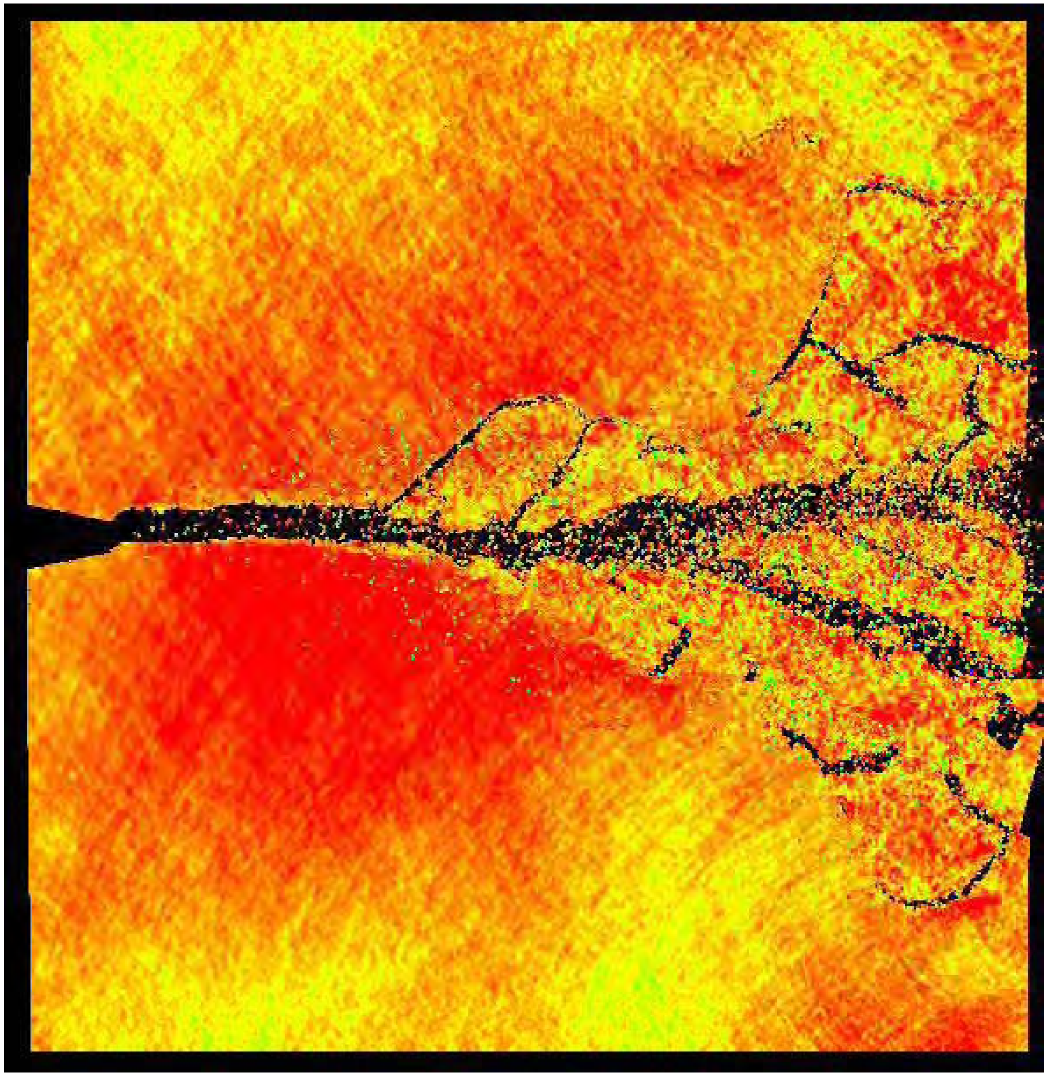
Fracture – Dynamic branching



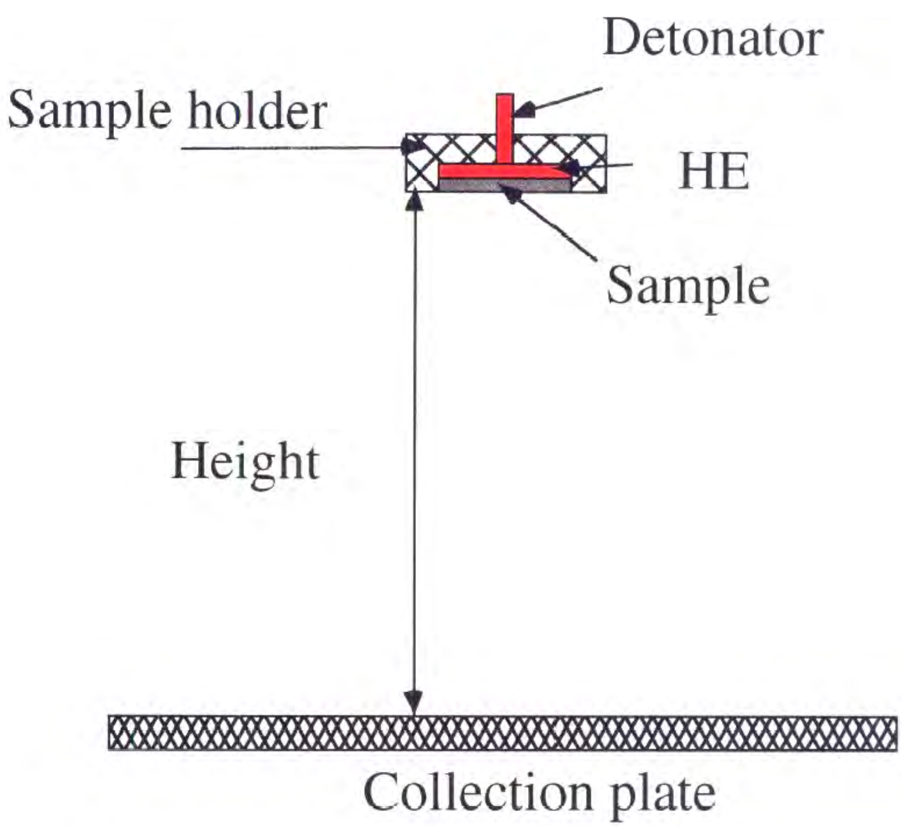
$\dot{\varepsilon} = 4000\ \mathrm{s}^{-1}$

(Fineberg and Sharon, 1992)



I. Arias *et al*., *CMAME,* **196** (2007) 3833

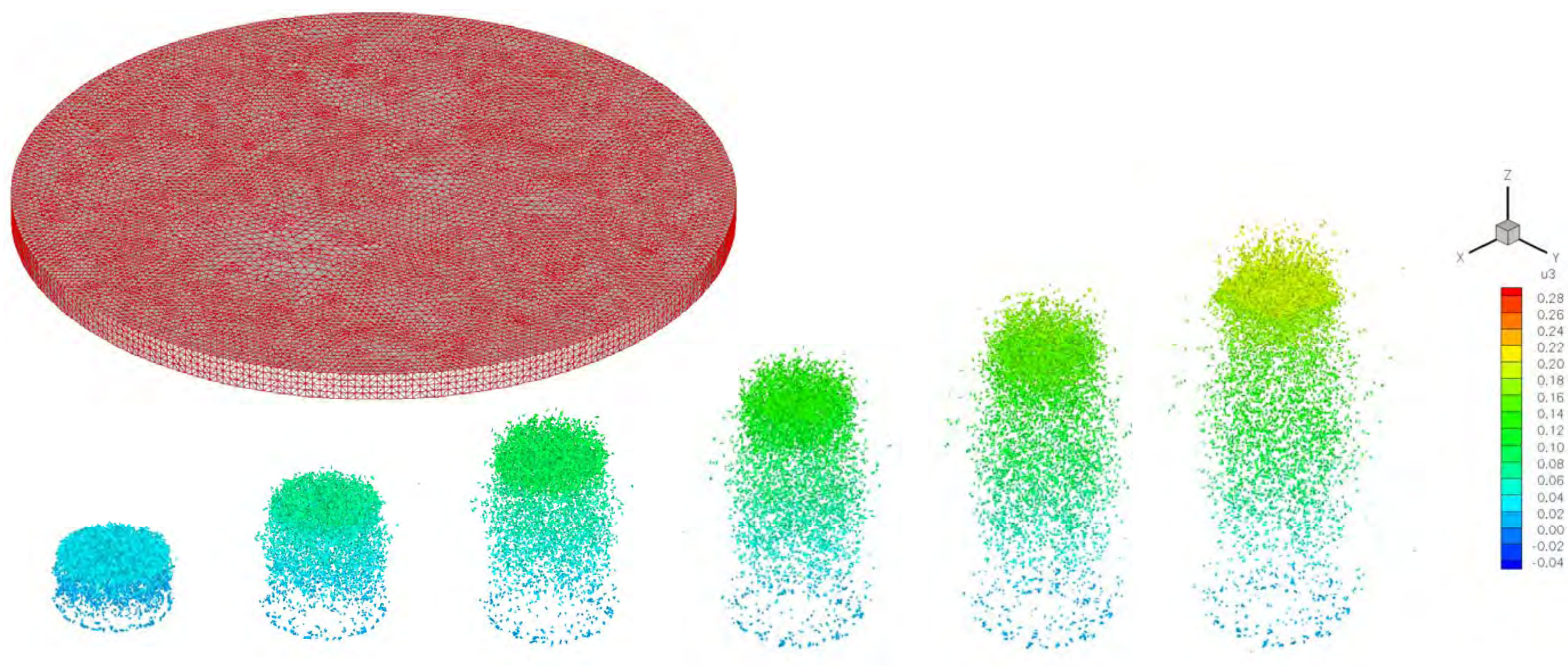
Fracture – Dynamic fragmentation



Collection plate with fragments

(Courtesy Griswold, LLNL, '04)

Fracture – Dynamic fragmentation



100 μs 200 μs 300 μs 400 μs 500 μs 600 μs

Contours of vertical displacement (m) (animation)

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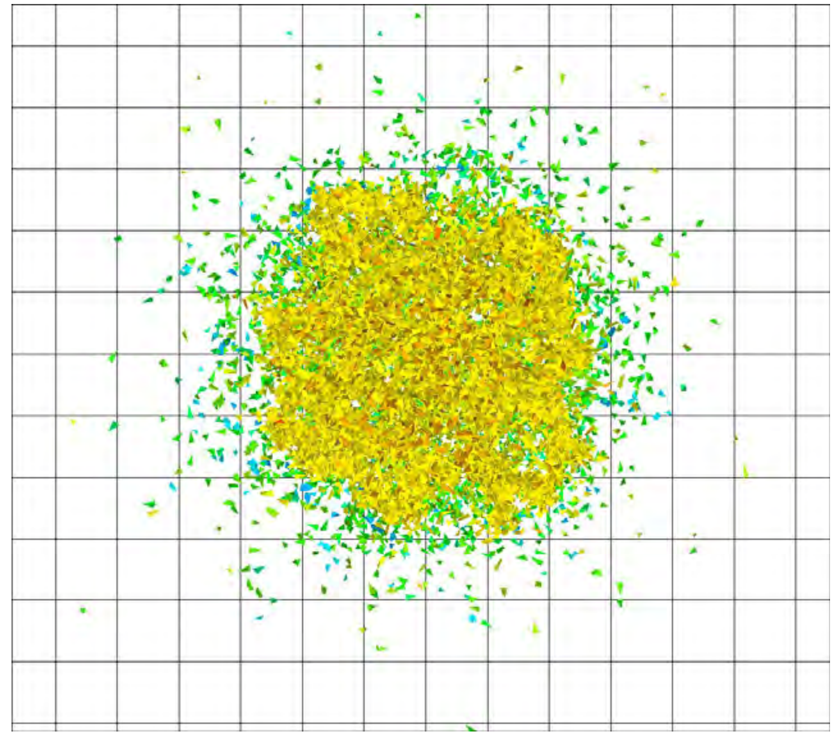
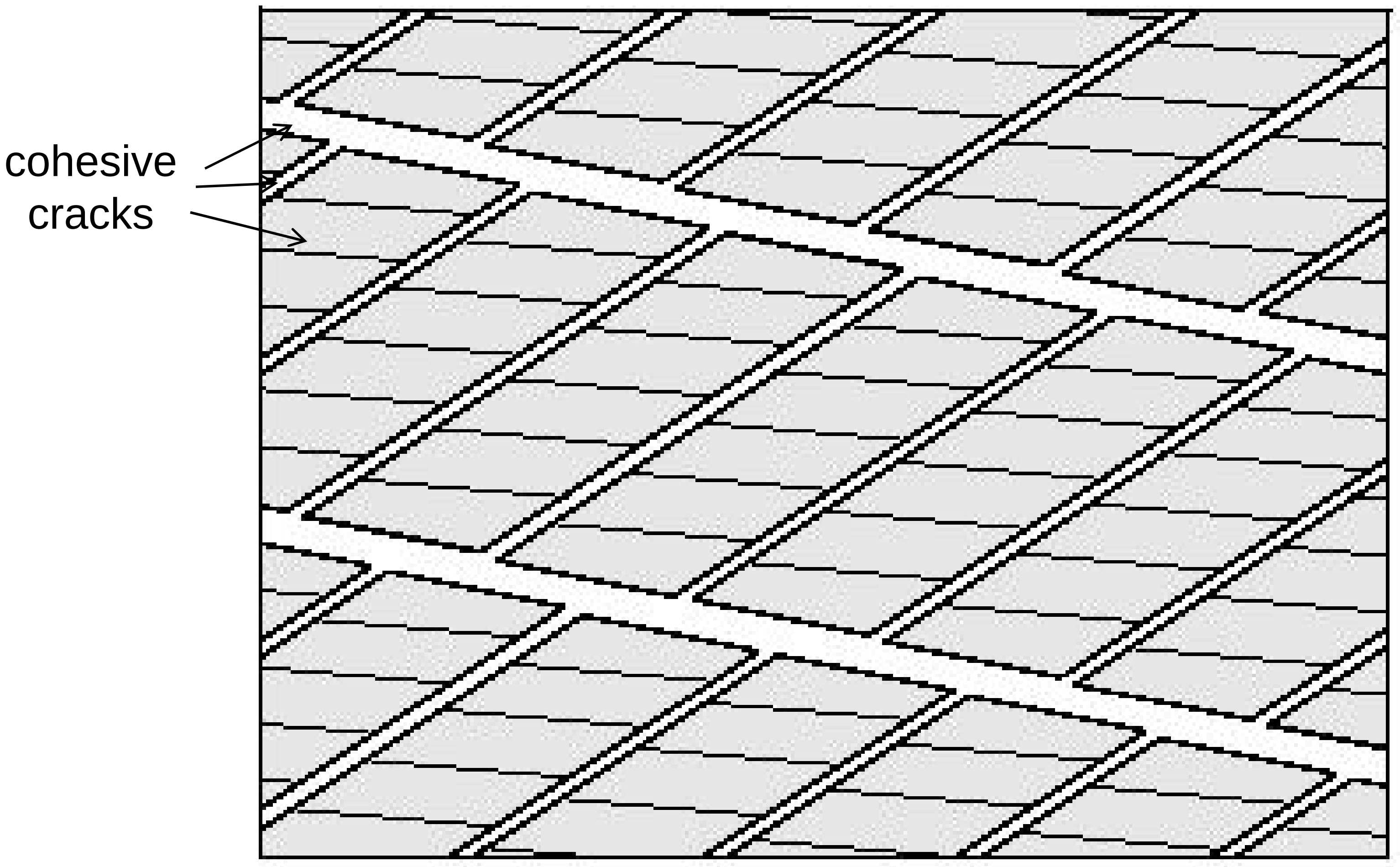
Fracture – Dynamic fragmentation



Experiment Simulation

Vertical view of final configuration (680 ms)
Approximate same size (0.30m x 0.30m)

Weak convergence? Constructions?



Sequential-fault construction

Pandolfi, A., Conti, S. and MO, *JMPS*, **54** (2006) 1972

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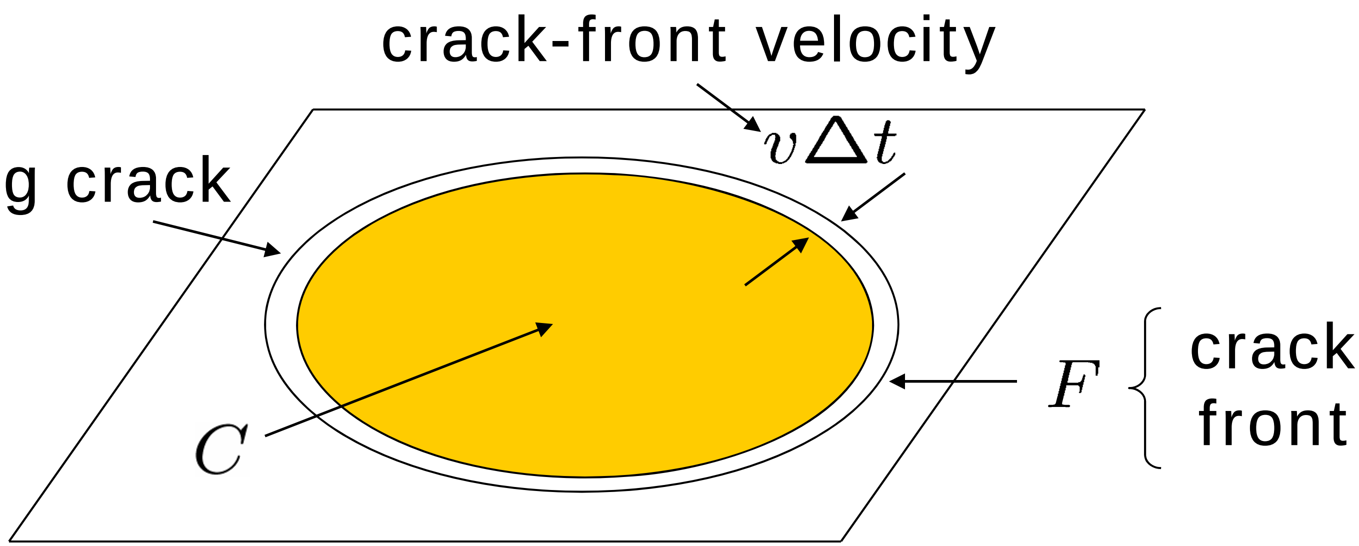
Crack-front models of fracture



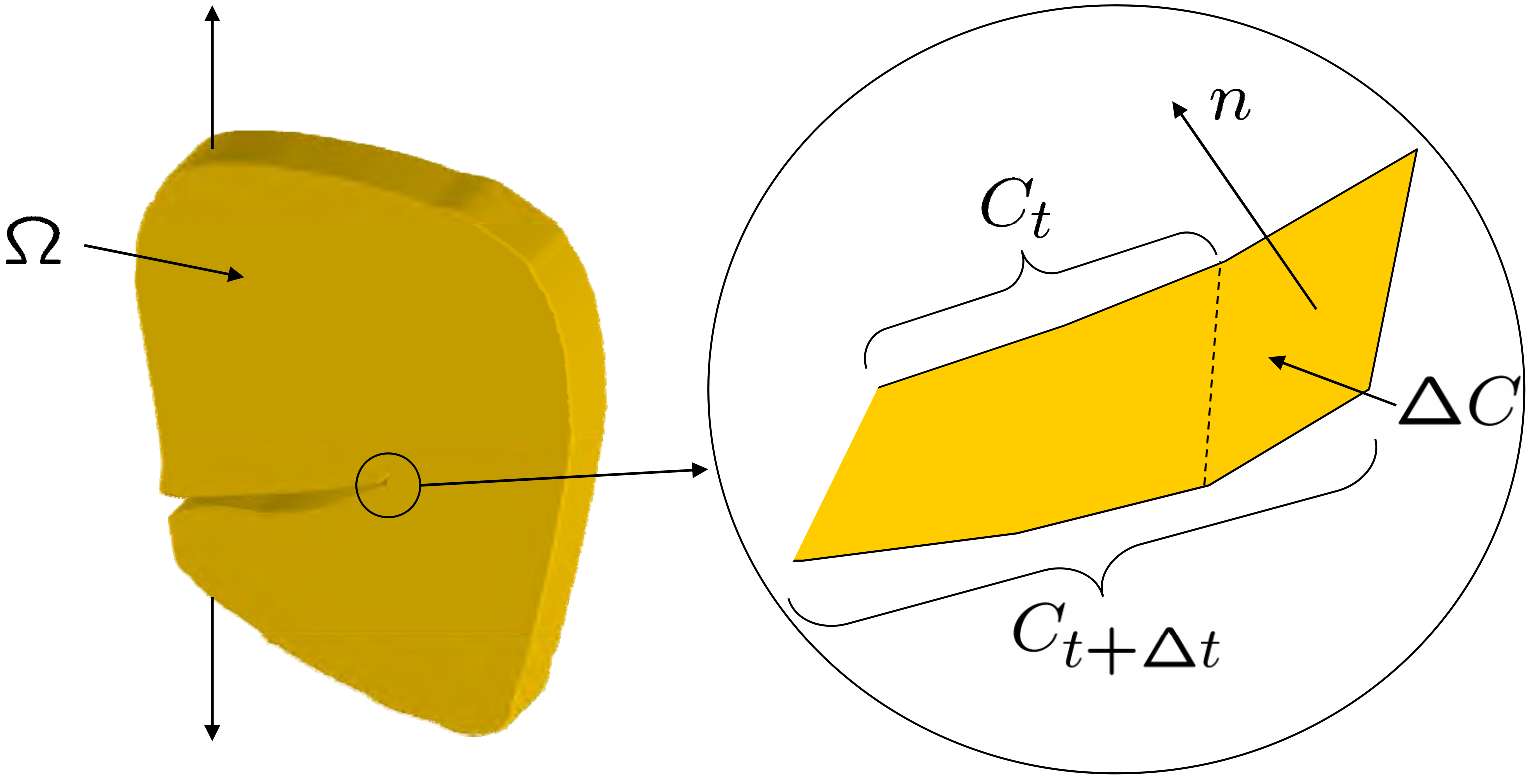
- Crack set: C, $0 < \mathcal{H}^{n-1}(C) < +\infty$.
- Crack-front measure: $\forall \varphi \in C_0^1([0,T])$, $\forall f \in C_0(\Omega)$,

$$\int_0^T \dot{\varphi} \int_{C(t)} f \, d\mathcal{H}^{n-1} dt = - \int_0^T \varphi \int_\Omega f \, d\mu_t(x) dt$$

- Crack front, velocity: $d\mu_t = v_t d\mathcal{H}^{n-2} \lfloor F(t)$.

Larsen, C.L., MO and Richardson, C.J., *ARMA*, **193** (2009) 539

Crack-front models of fracture



- Assume regularity, smoothness . . .
- Load increment, crack extension:

$$-\Delta E = \int_{\Delta C} [DW(\nabla u_t)\, n] \cdot [\![u_{t+\Delta t}]\!] \, d\mathcal{H}^2 + h.o.t.$$

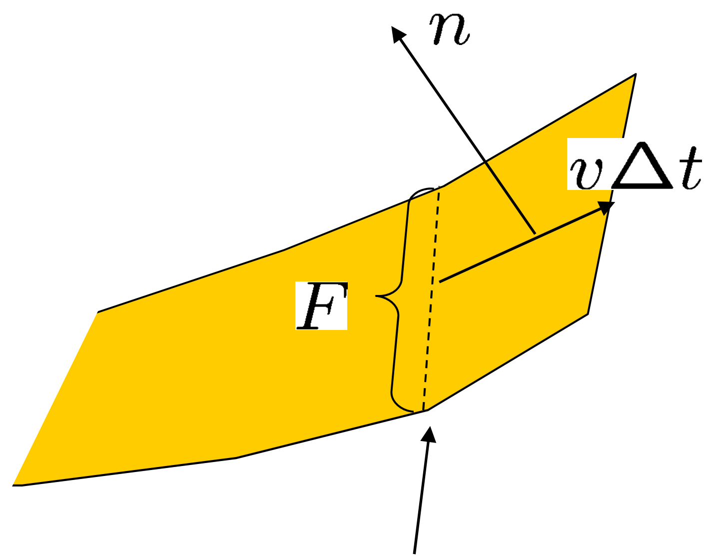
Larsen, C.L., MO and Richardson, C.J., *ARMA*, **193** (2009) 539

Crack-front models of fracture

- Energy-release rate:

$$G = \lim_{\Delta t \to 0} -\frac{\Delta E}{\Delta t}$$

$$= \int_F f(n)\, v \, d\mathcal{H}^1$$



- Driving force:

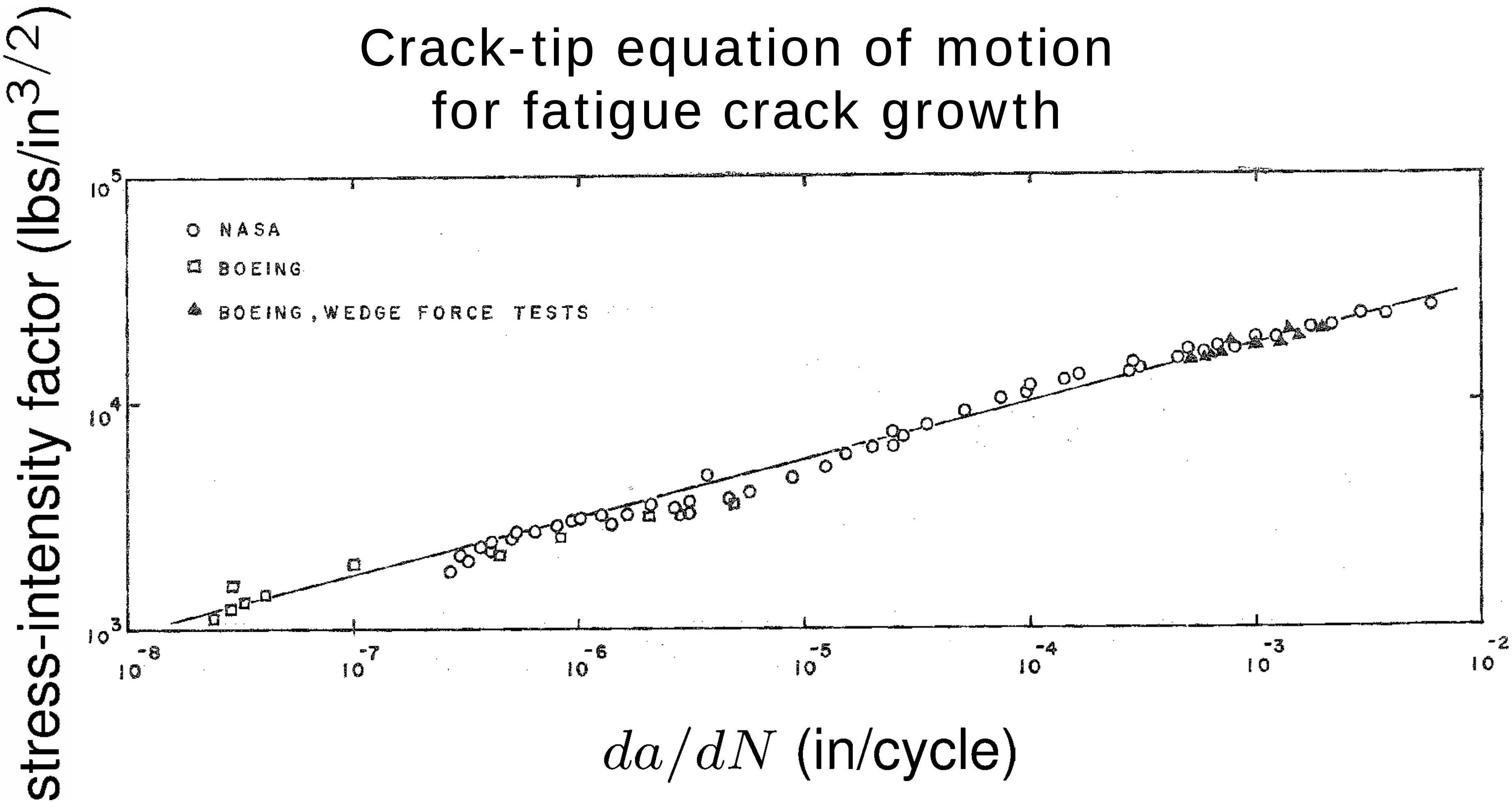
$$f(n) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [DW(\nabla u_t)\, n] \cdot [\![u_{t+\Delta t}]\!]$$

- Crack-tip equation of motion: $f = \partial \psi(v)$
- Dissipation: $\Psi(v) = \int_F \psi(v)\, d\mathcal{H}^1$

Larsen, C.L., MO and Richardson, C.J., *ARMA*, **193** (2009) 539

Crack-front models of fracture - Fatigue

Crack-tip equation of motion
for fatigue crack growth



Crack-growth data for 2024-T3 aluminum alloy
(P. Paris and F. Erdogan, ASME Trans (1963))

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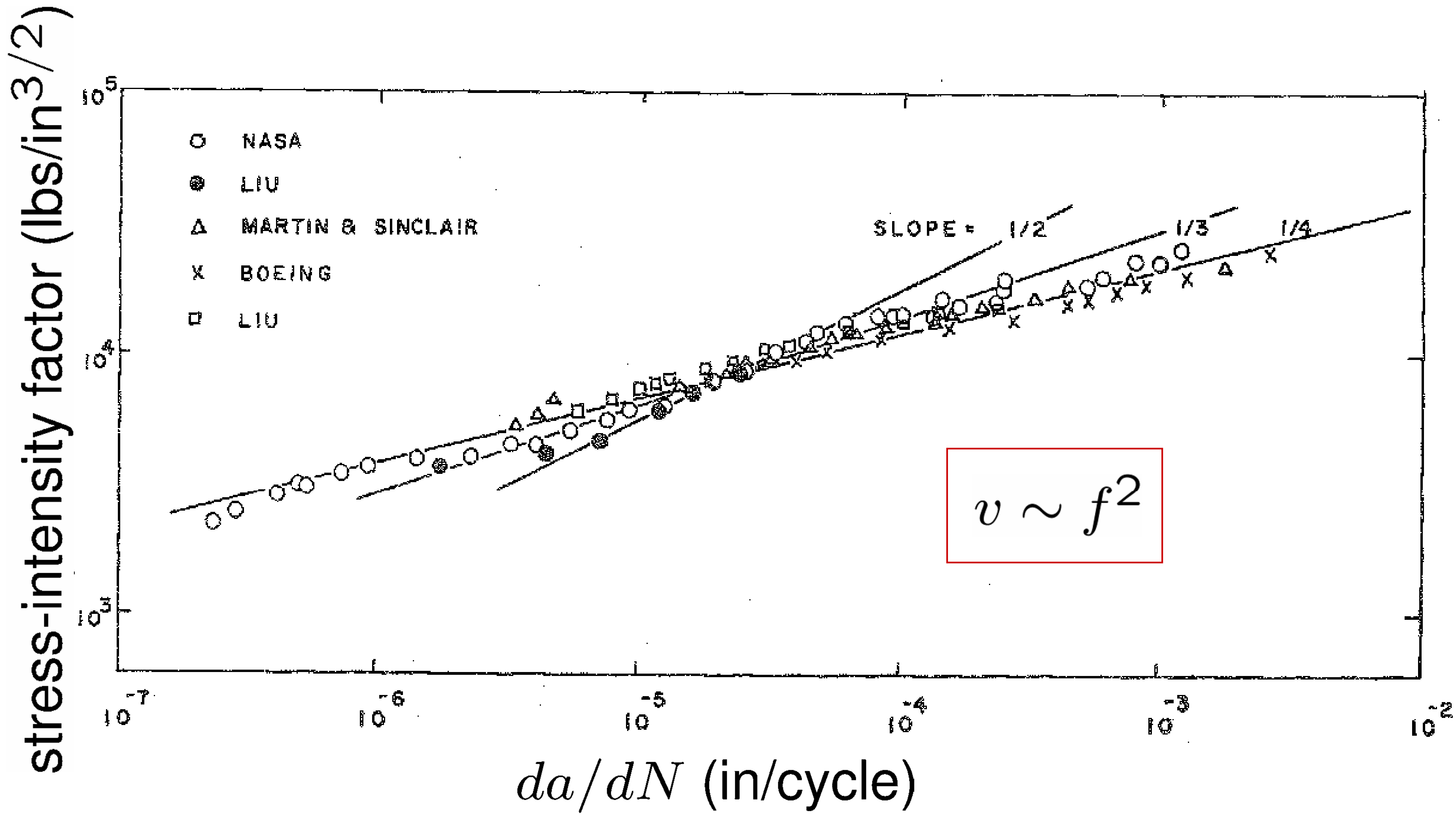
Crack-front models of fracture - Fatigue



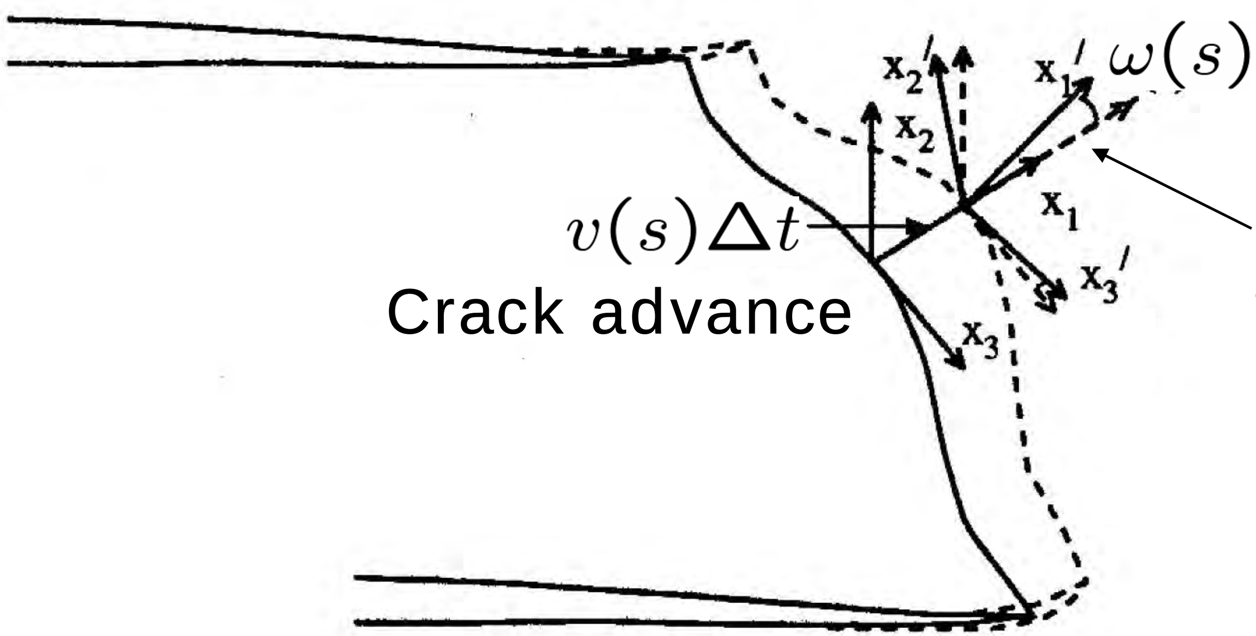
Crack-growth data for 2024-T3 aluminum alloy
(P. Paris and F. Erdogan, ASME Trans (1963)

The rate problem of LEFM

- Rate problem: $\inf_{v,n} \int_F [\psi(v) - f(n)\, v]\, d\mathcal{H}^1$

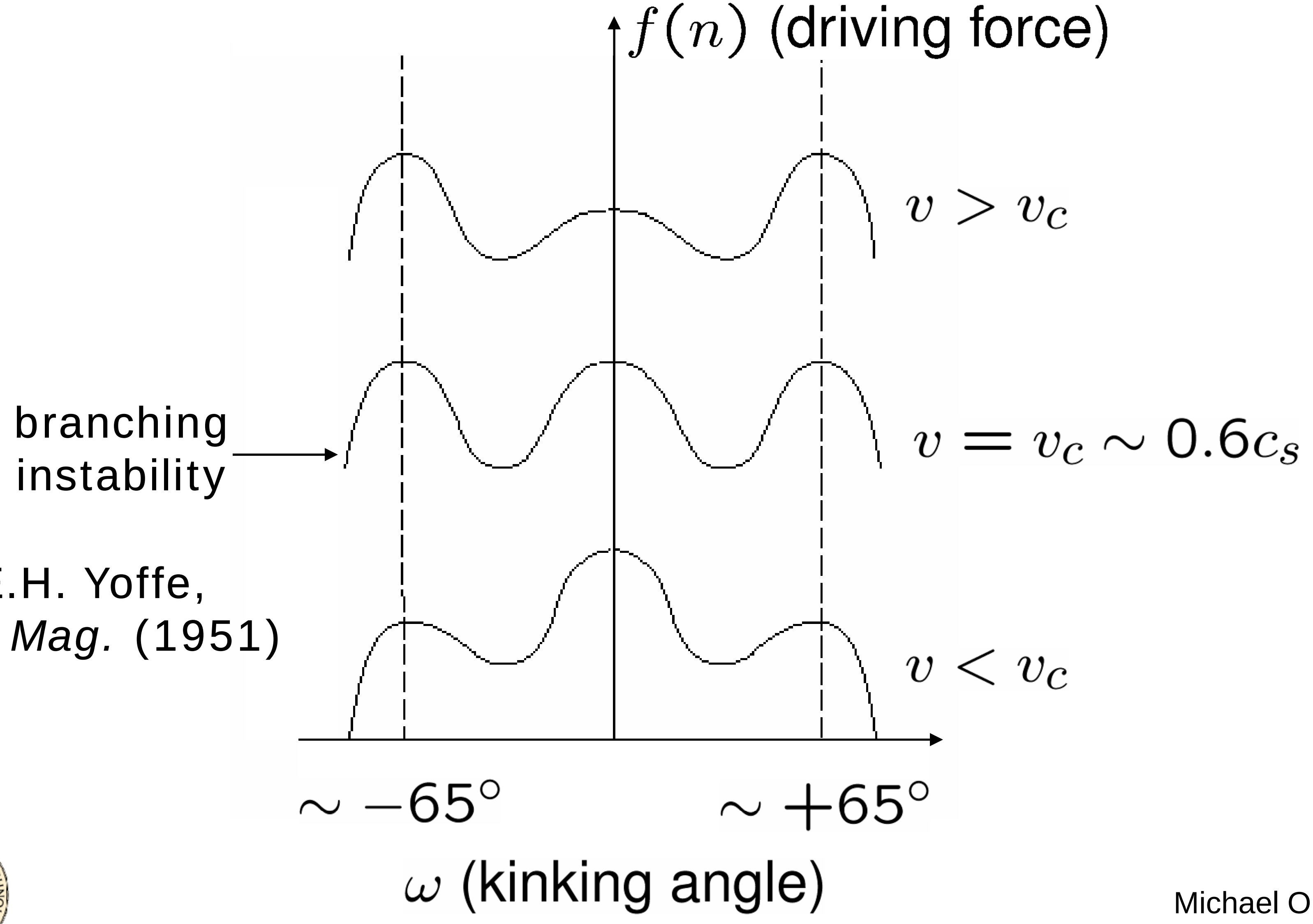
$$\Rightarrow \left\{ \begin{array}{l} \partial\psi(v) = f(n) \\ \partial\psi^*(f(n)) = 0 \end{array} \right\} \longrightarrow (v, n)$$

(maximum driving force)



Larsen, C.L., MO and Richardson, C.J., *ARMA*, **193** (2009) 539

Crack-front models of fracture - Dynamics

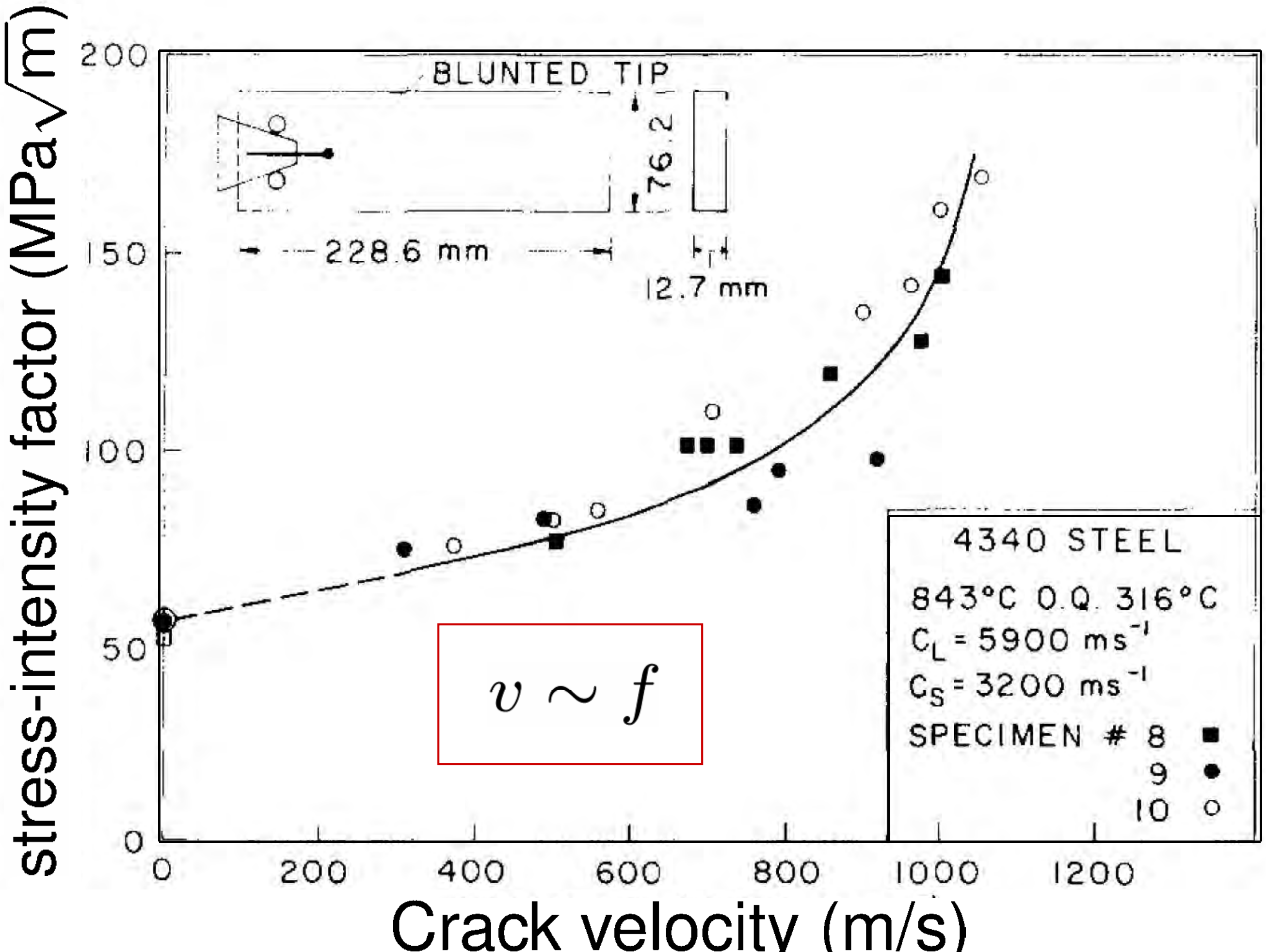


E.H. Yoffe, *Phil. Mag.* (1951)

Larsen, C.L., MO and Richardson, C.J., *ARMA*, **193** (2009) 539

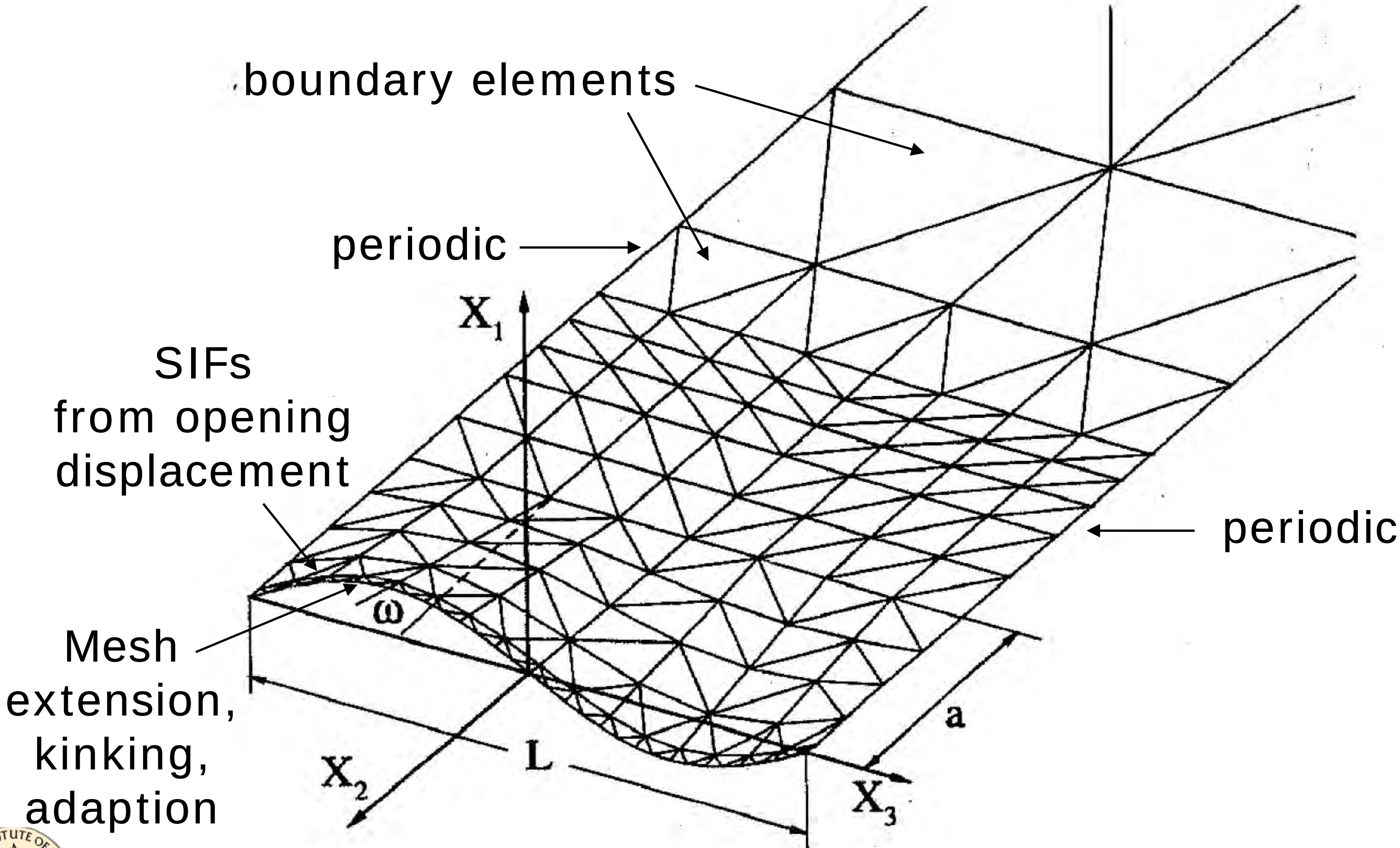
Crack-front models of fracture - Dynamic

Dynamic crack-tip equation of motion



Rosakis, Duffy and Freund, *JMPS* (1984)

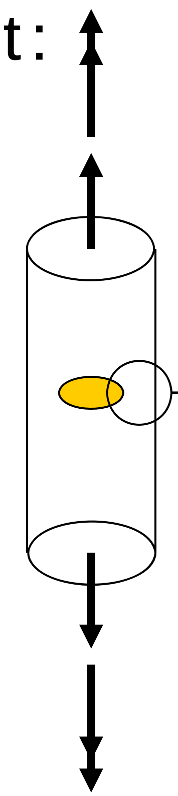
LEFM rate problem – Numerical analysis

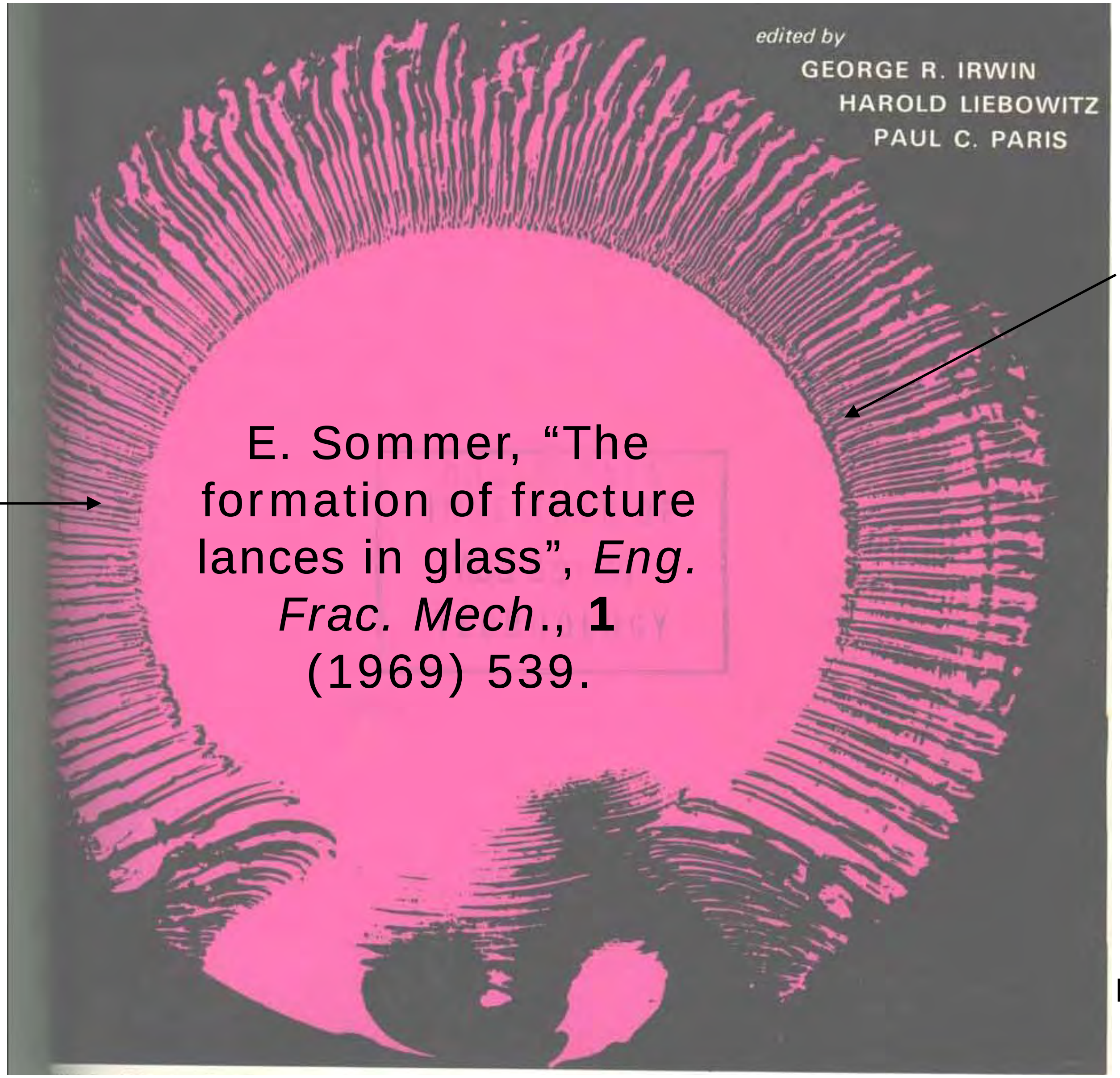


Xu, G. *et al*., *Int. J. Solids & Struct.,* **31** *(*1994) 2167

LEFM rate problem – Numerical analysis

tension-torsion test:

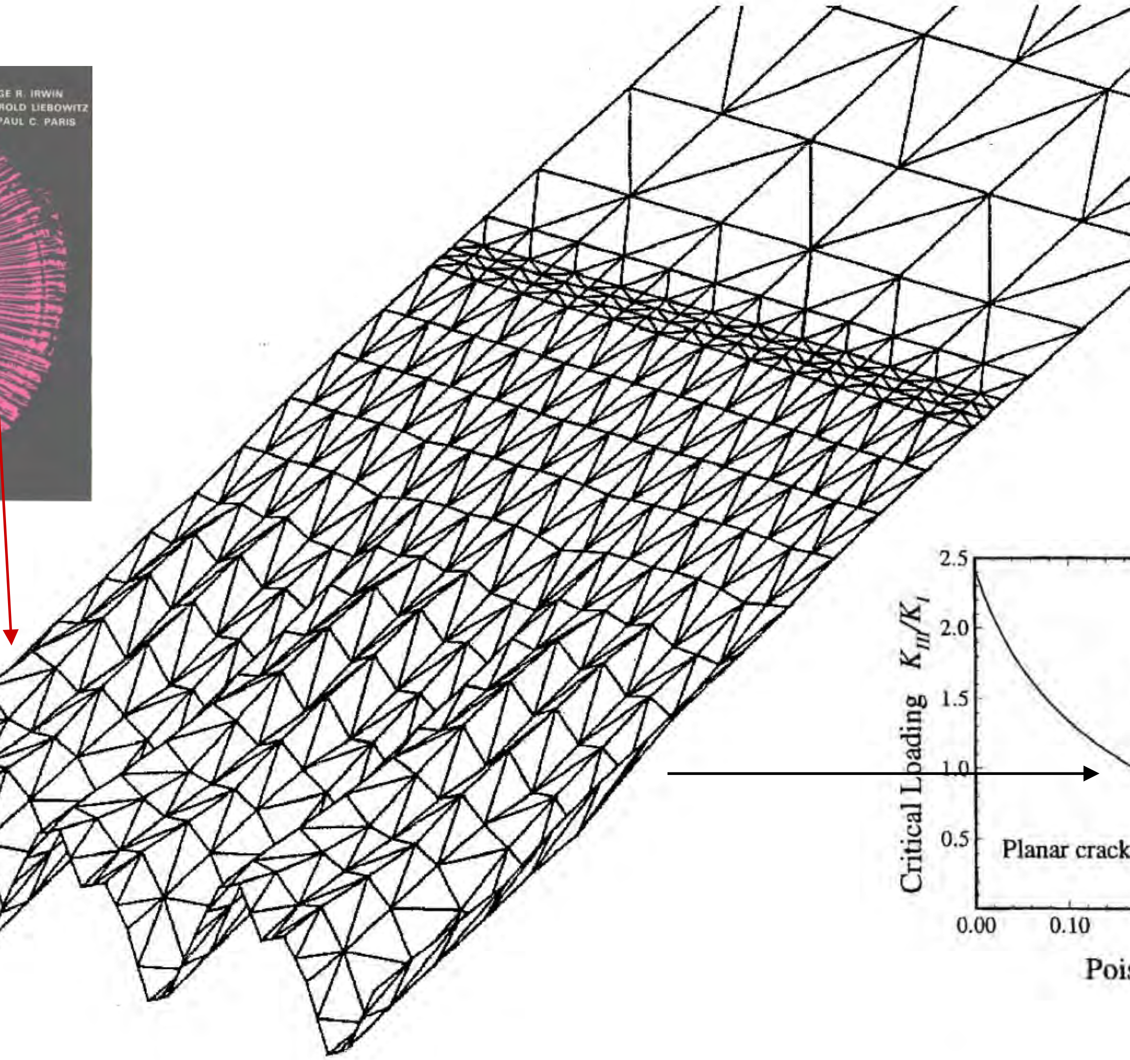


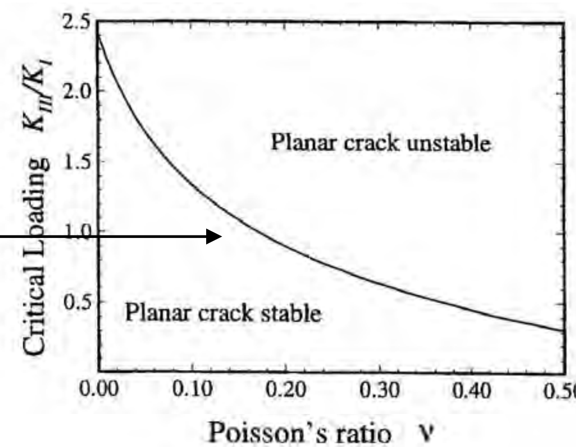


lances

E. Sommer, "The formation of fracture lances in glass", *Eng. Frac. Mech*., **1** (1969) 539.

LEFM rate problem – Numerical analysis





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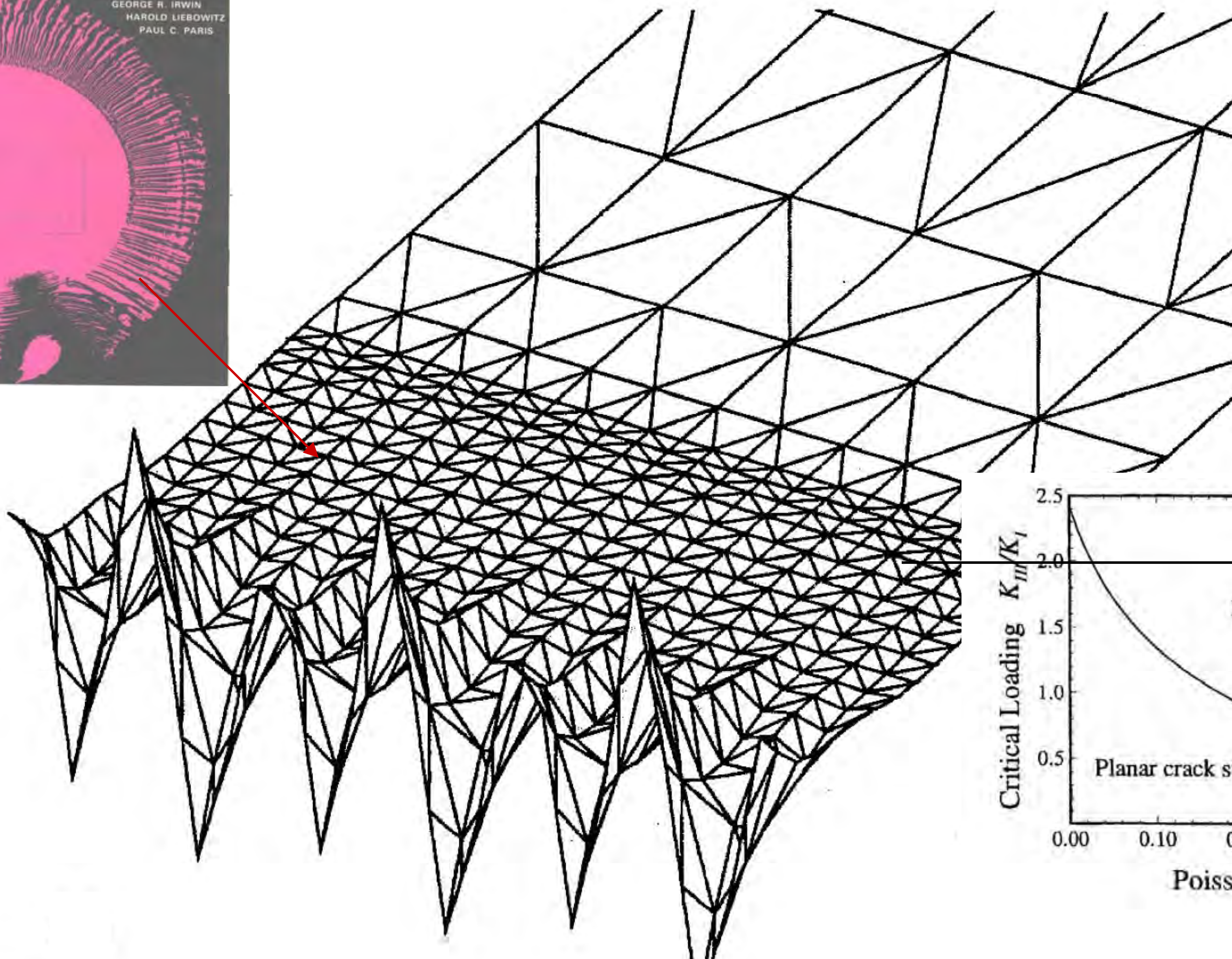
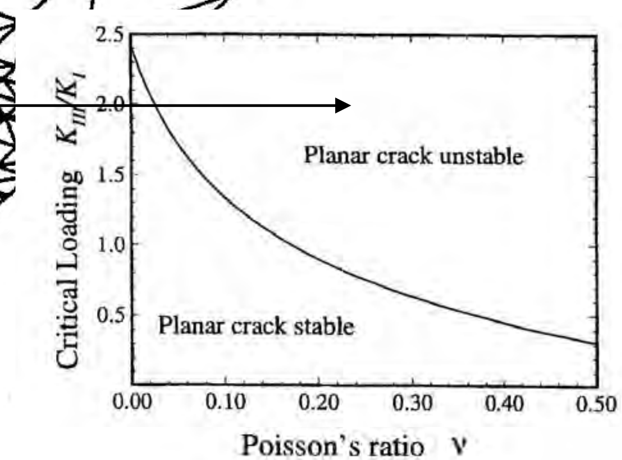
LEFM rate problem – Numerical analysis





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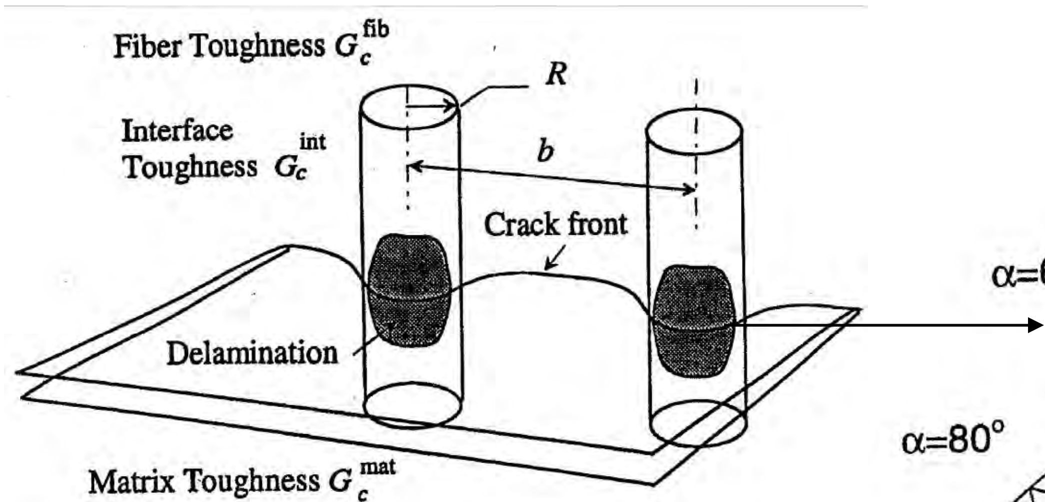
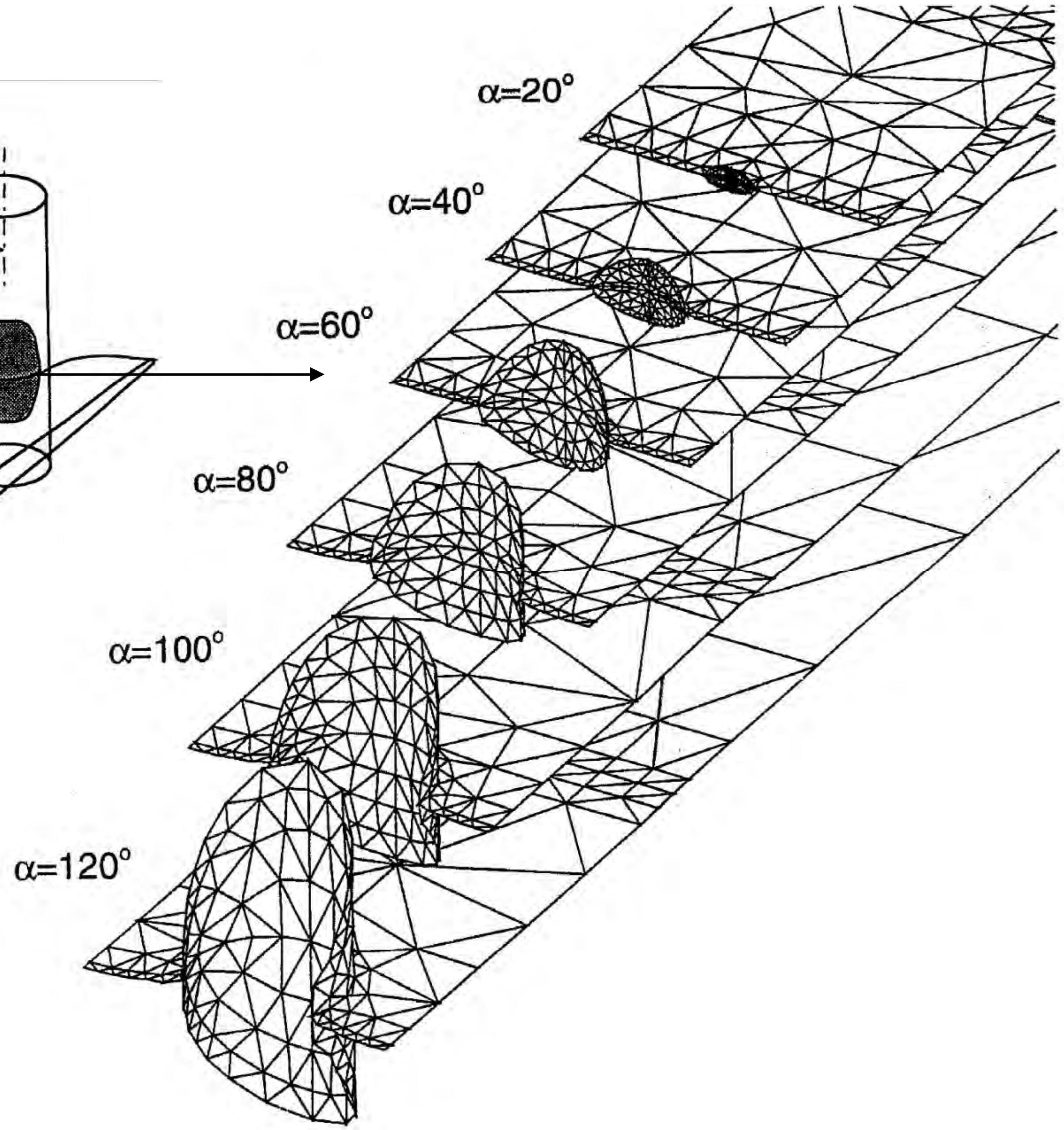
LEFM rate problem – Numerical analysis

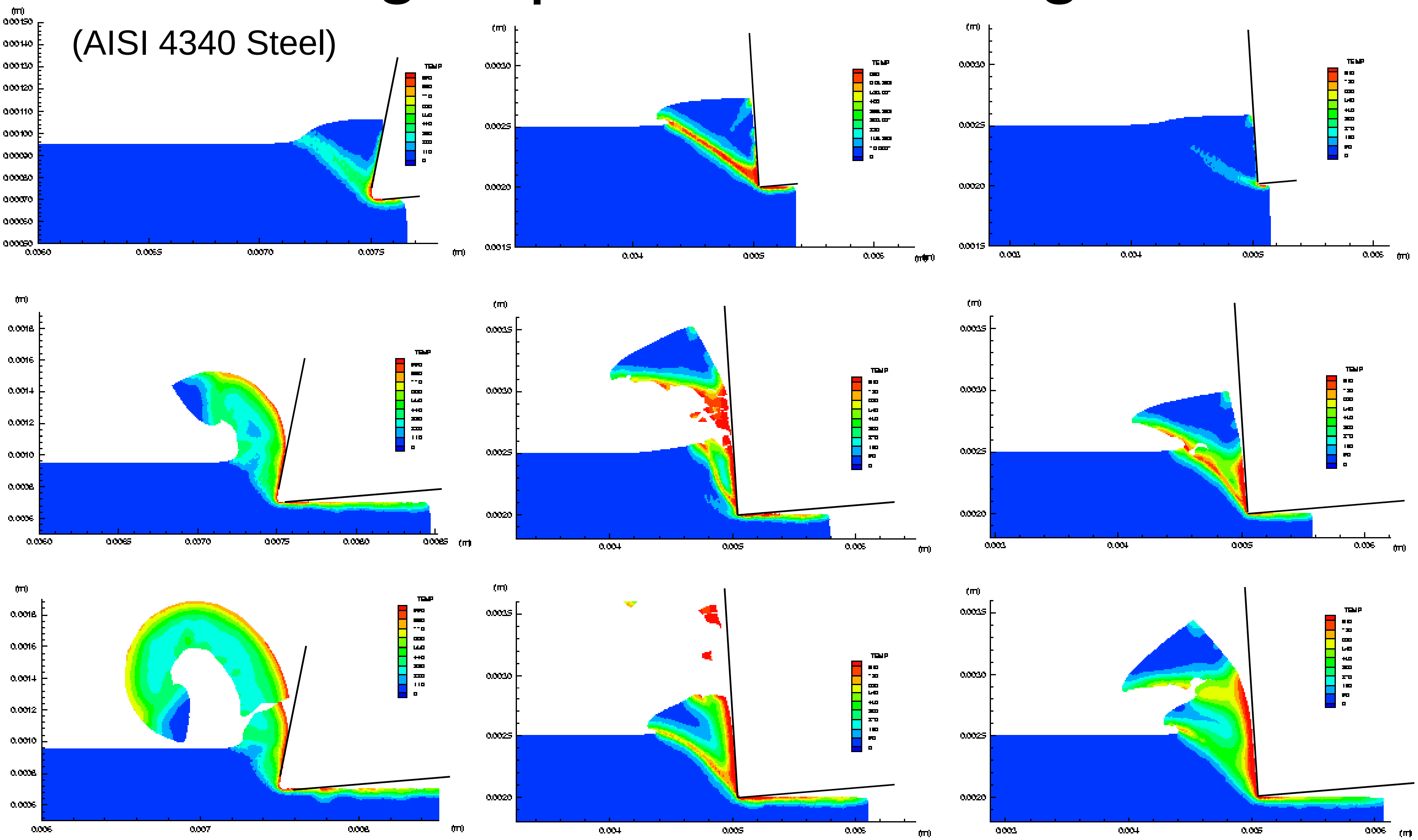




Fiber debonding in composites

Xu, G., Bower, A.F., and MO, *JMPS,* **46** (1998) 1815

High-speed machining



Cutting speed = 20 m/s | Cutting speed = 20 m/s | Cutting speed = 10 m/s

Simulation of high-speed machining

Marusich, T.D. and MO, *IJNME,* **38** *(*1995) 3675

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Eigendeformations and fracture

- $\beta \equiv$ eigendeformation field
- Total energy: Elastic + fracture energy, $\epsilon > 0$

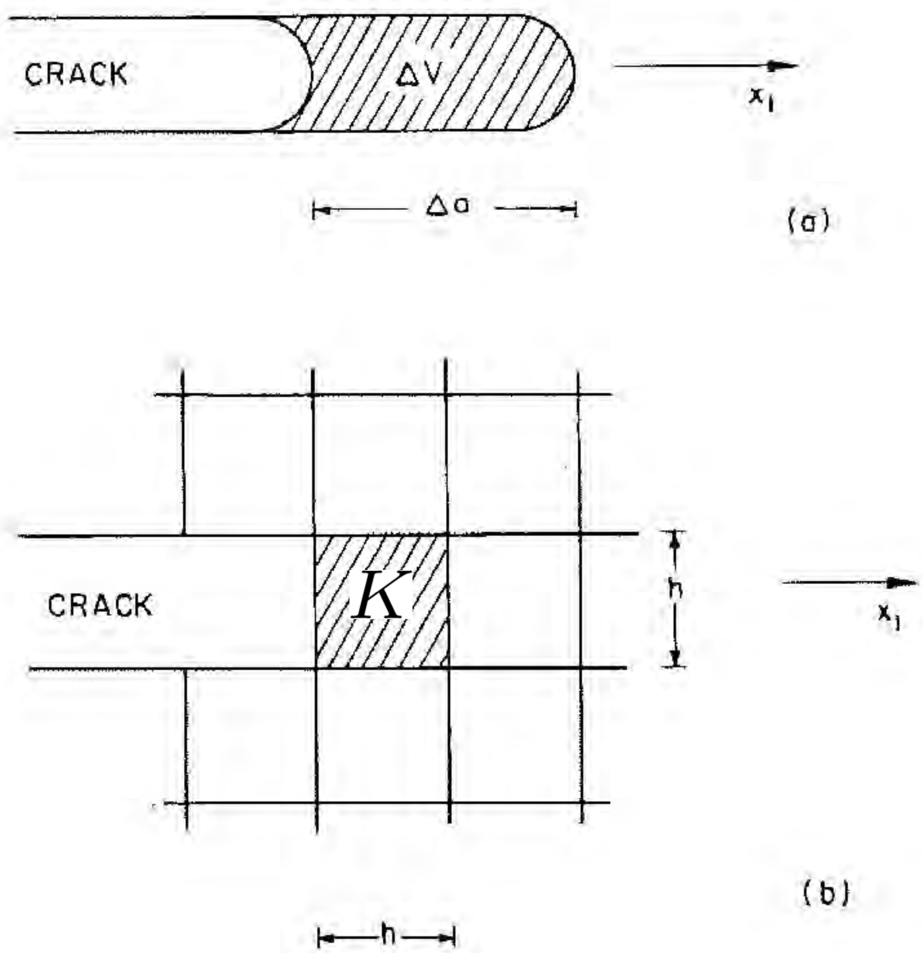
$$E_\epsilon(u,\beta) = \int_\Omega W(\nabla u - \beta)\,dx + \frac{\gamma}{2\epsilon}|\{\beta \neq 0\}_\epsilon|$$

- Quasistatic problem: $E_\epsilon(u,\beta) \to \inf!$
- Γ-limit in $L^1(\Omega) \times \mathcal{M}(\Omega)$: $E(u,\beta) =$

$$\begin{cases} \int_{\Omega\setminus J_u} W(\nabla u)\,dx + \gamma \mathcal{H}^{n-1}(J_u), & \text{if } u \in SBV(\Omega), \\ & \text{and } \|u\|_{L^\infty(\Omega)} < K, \\ & \text{and } \beta = D^s u, \\ +\infty, & \text{otherwise.} \end{cases}$$

Schmidt, B., *et al.*, *SIAM Multi. Model.*, **7** *(*2009) 1237

Eigendeformations and fracture



- Energy-release rate:

$$G \sim \frac{1}{h}\int_K W(\nabla u)\,dx$$

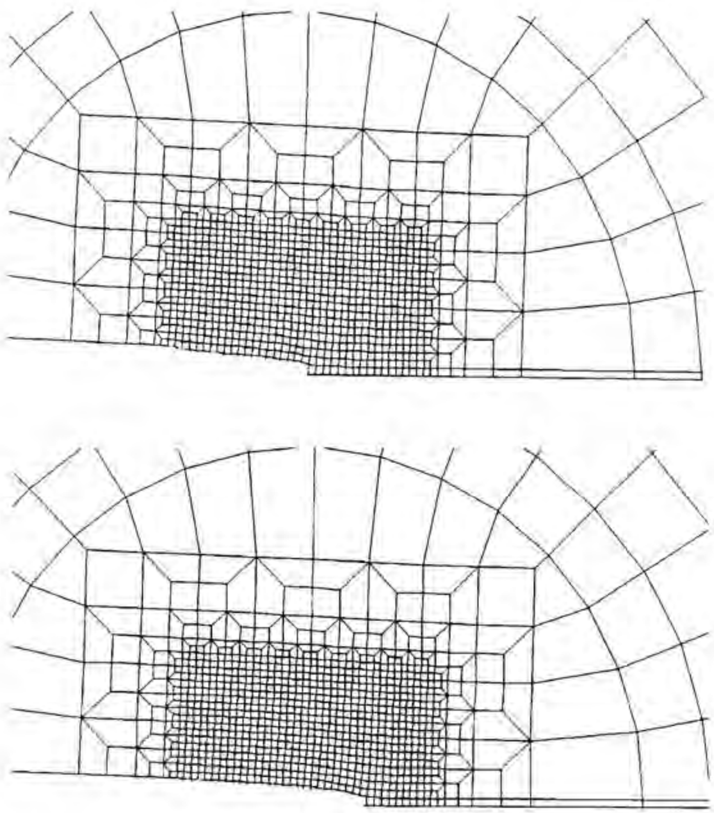
- Erosion criterion:

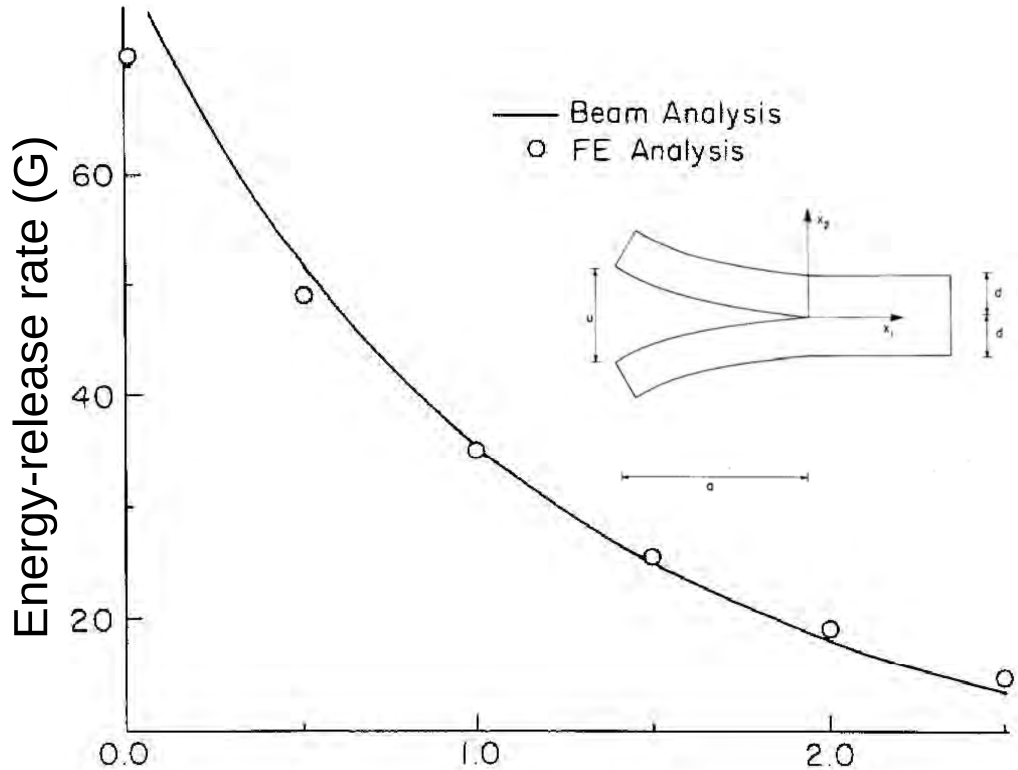
$$G \geq G_c$$

- Implementation:
  i) Order elements by G
  ii) Pop top element

M0 and Giannakopoulos, A.E., *Int. J. Fracture,* **44** (1990) 233

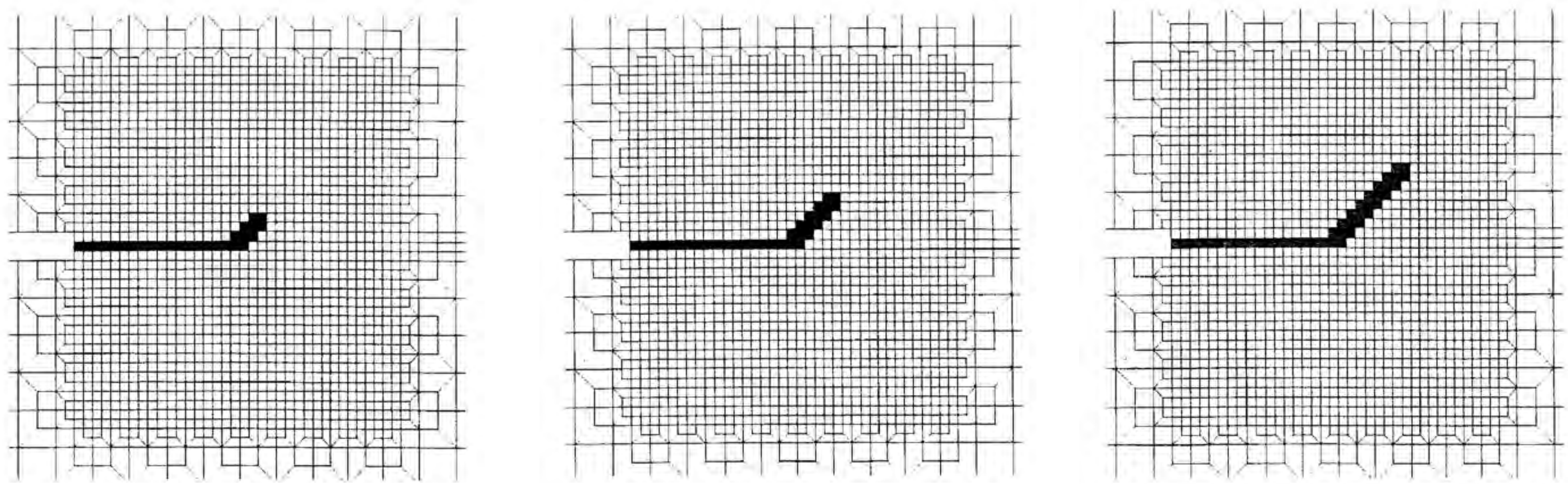
Eigendeformations and fracture





M0 and Giannakopoulos, A.E., *Int. J. Fracture,* **44** (1990) 233

Eigendeformations and fracture



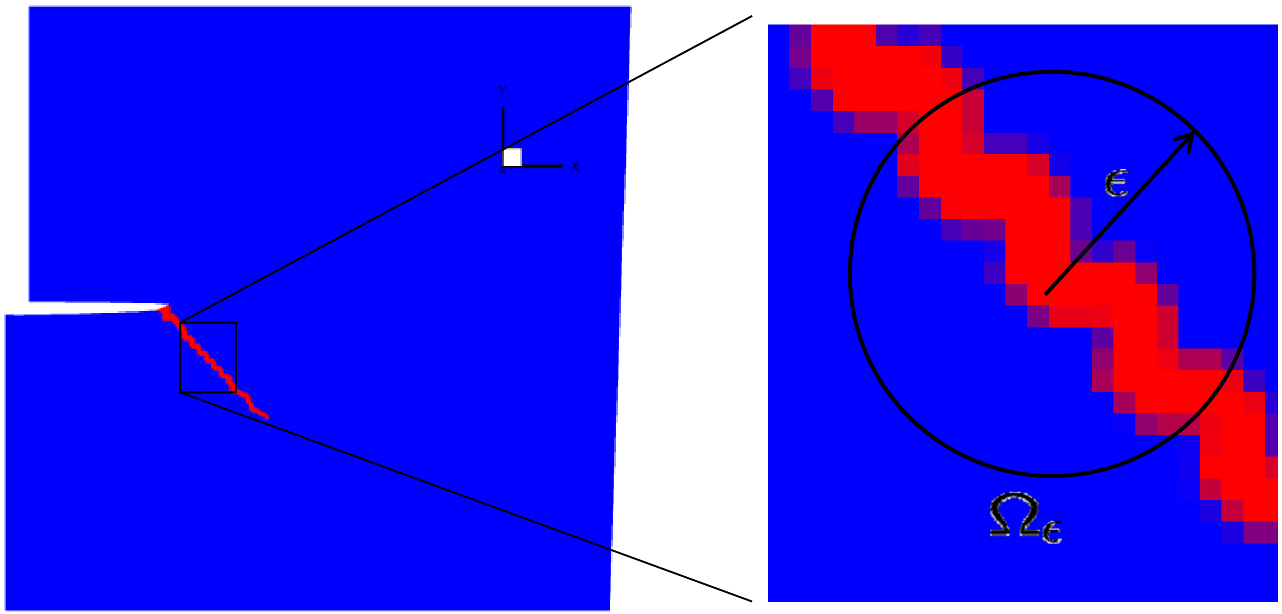
Crack growth in mixed mode

M. Ortiz and A.E. Giannakopoulos,
Int. J. Fracture, **44** (1990) 233-258.

- Fracture energy over-estimated as $h \rightarrow 0$*!*
- Non-convergence for general paths, meshes!

M0 and Giannakopoulos, A.E., *Int. J. Fracture,* **44** (1990) 233

Eigendeformations and fracture



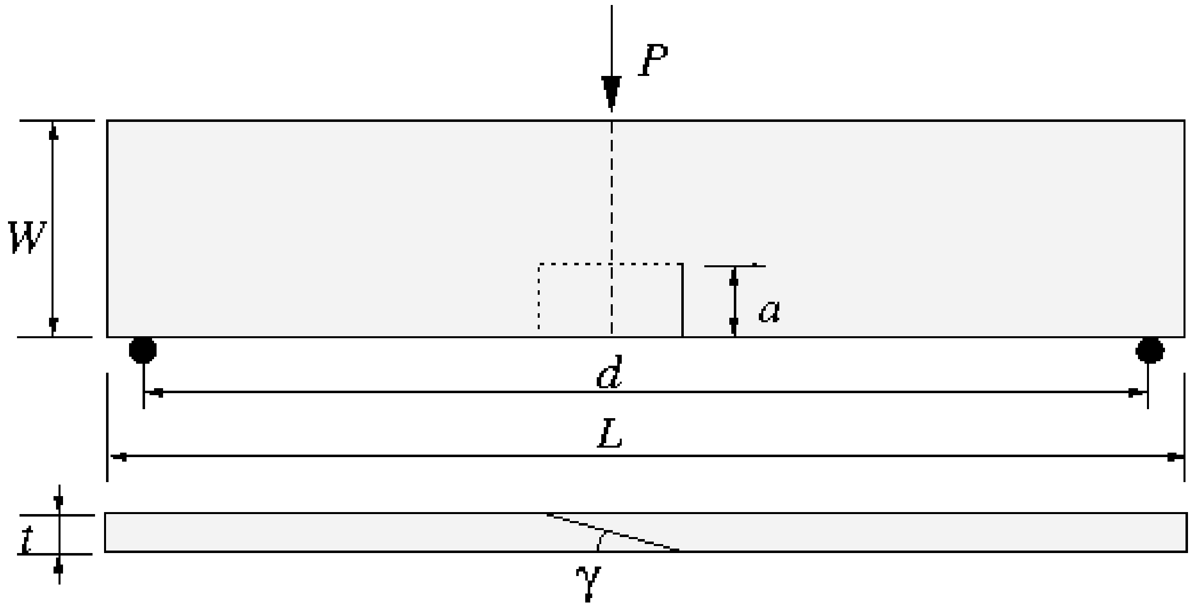
ε-neighborhood construction

- Energy-release rate and element-erosion criterion:

$$G_\epsilon \sim \frac{1}{A_\epsilon} \int_{\Omega_\epsilon} W(\nabla u)\, dx \geq G_c$$

Proof of convergence: Schmidt, B., Fraternali, F. and Ortiz, M., *SIAM J. Multiscale Model. Simul.*, **7**(3) (2009) 1237-1366.

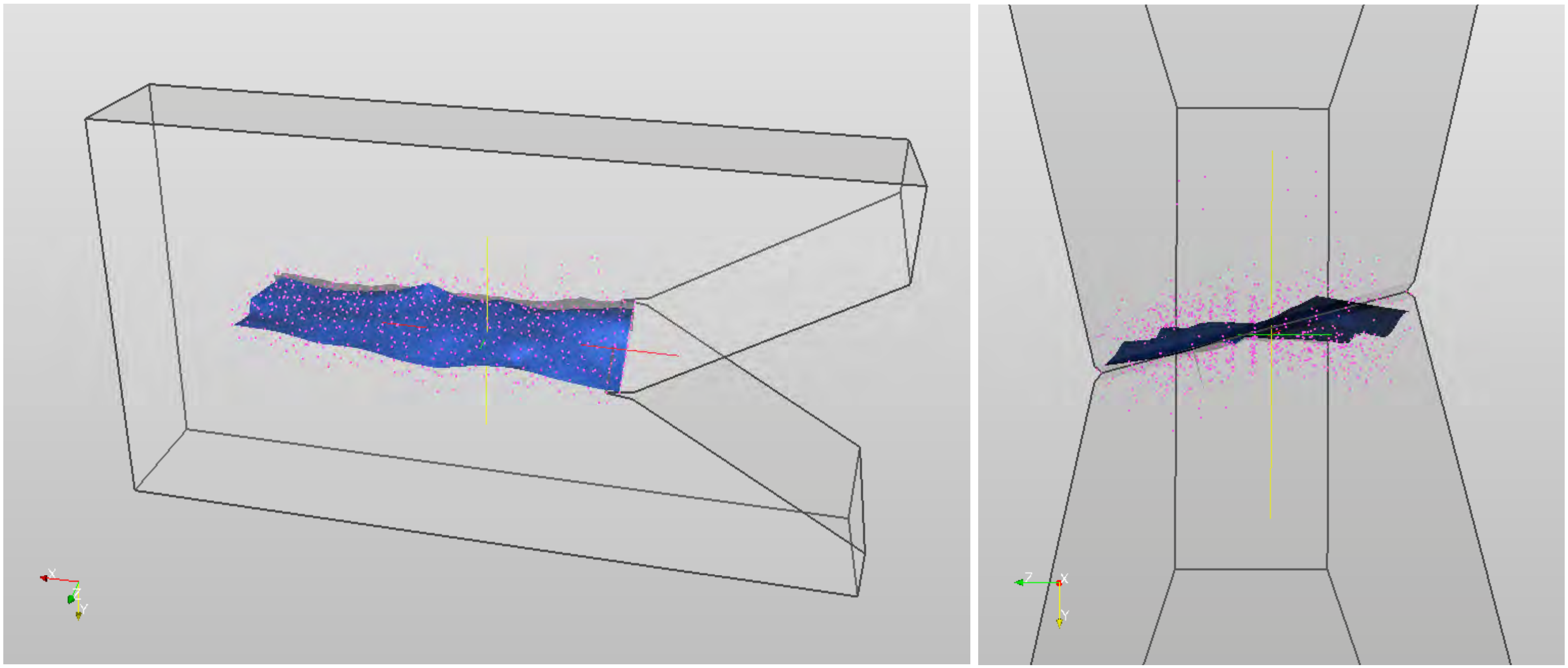
Verification: Mode I-III 3-point bending



- Mixed-mode 3-point bending tests, PMMA plates (260x60x10 mm, a = 20 mm)
- Inclination g of notch: 75°, 60°, 45°
- E = 2800 MPa, n = 0.38, G_c = 0.54 N/mm

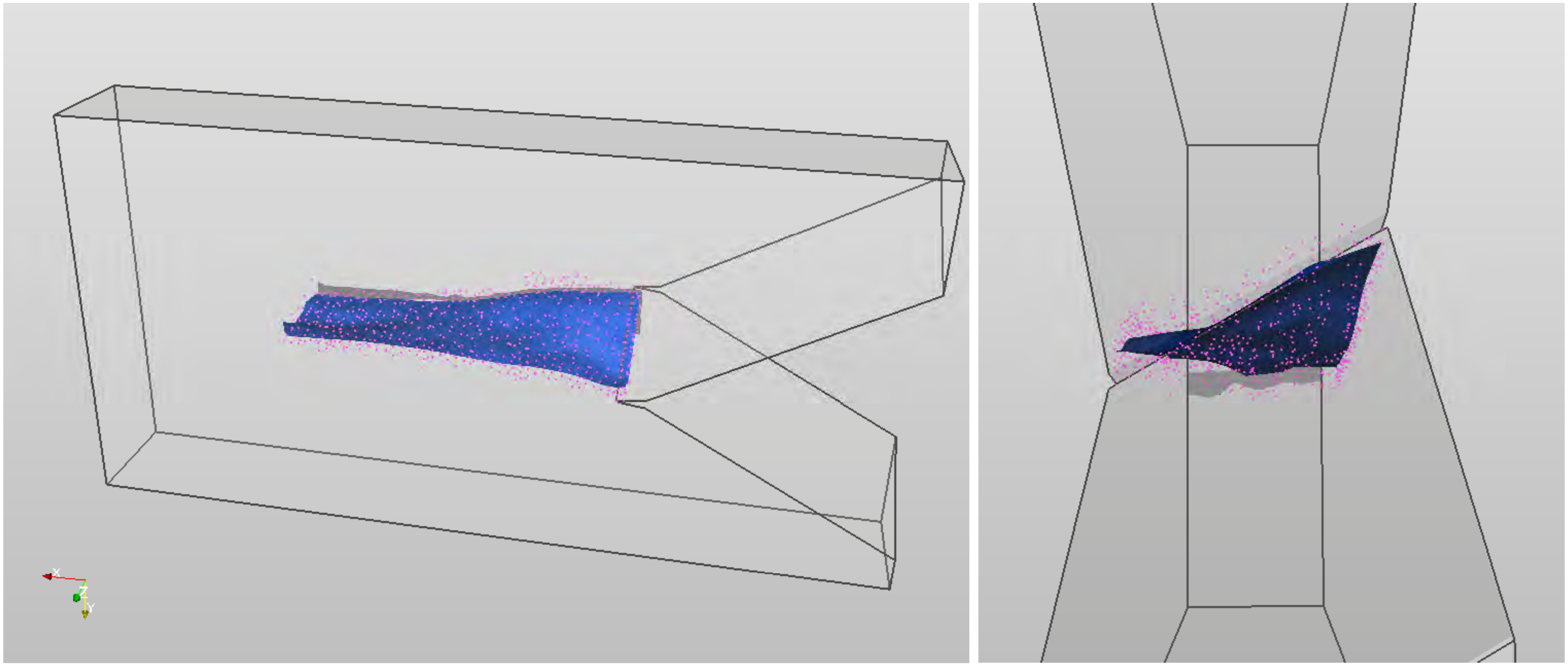
[Lazarus *et al*., 2008]

Predominant Mode I (γ = 75°)



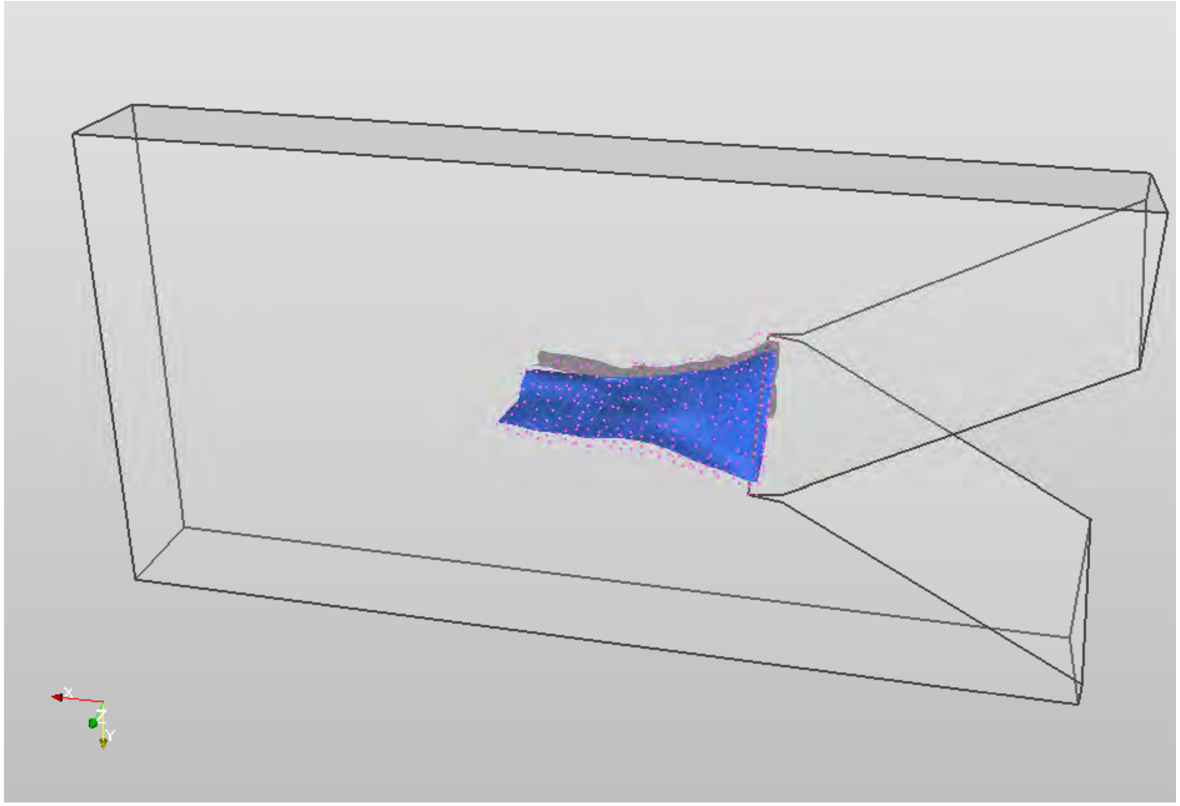
Pandolfi, A. and MO (in preparation)

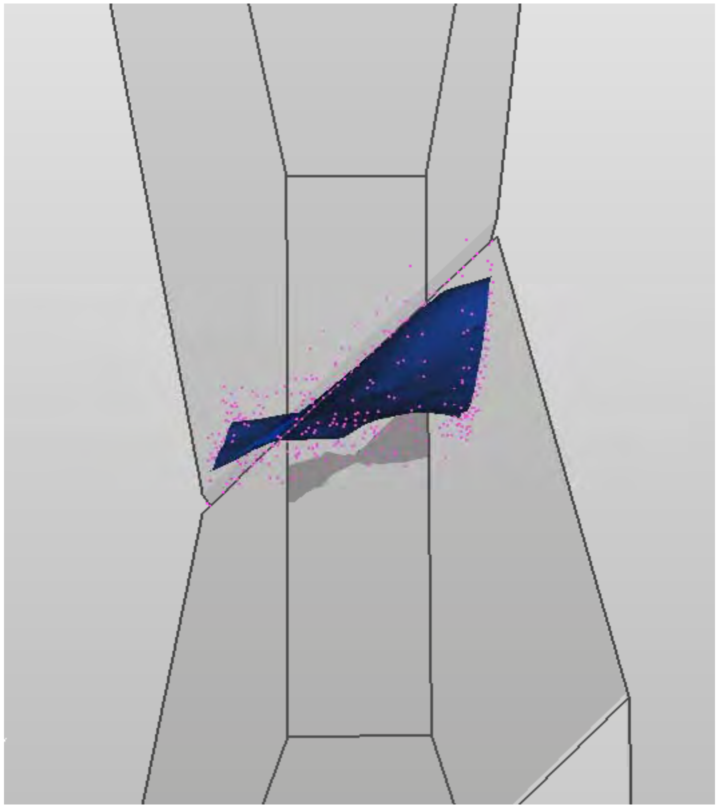
Mixed Mode I-III ($\gamma = 60°$)



Pandolfi, A. and MO (in preparation)

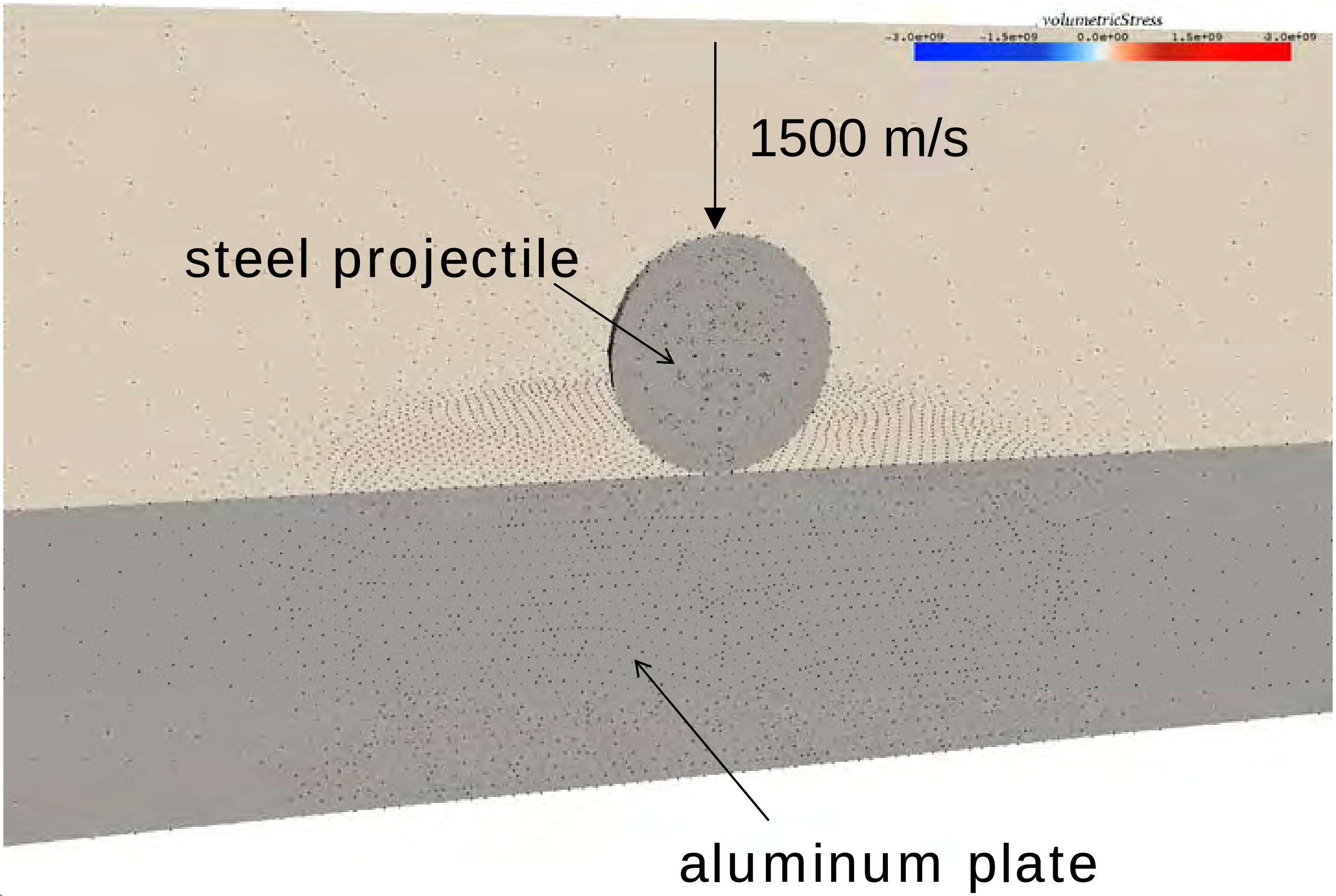
Predominant Mode III ($\gamma = 45°$)



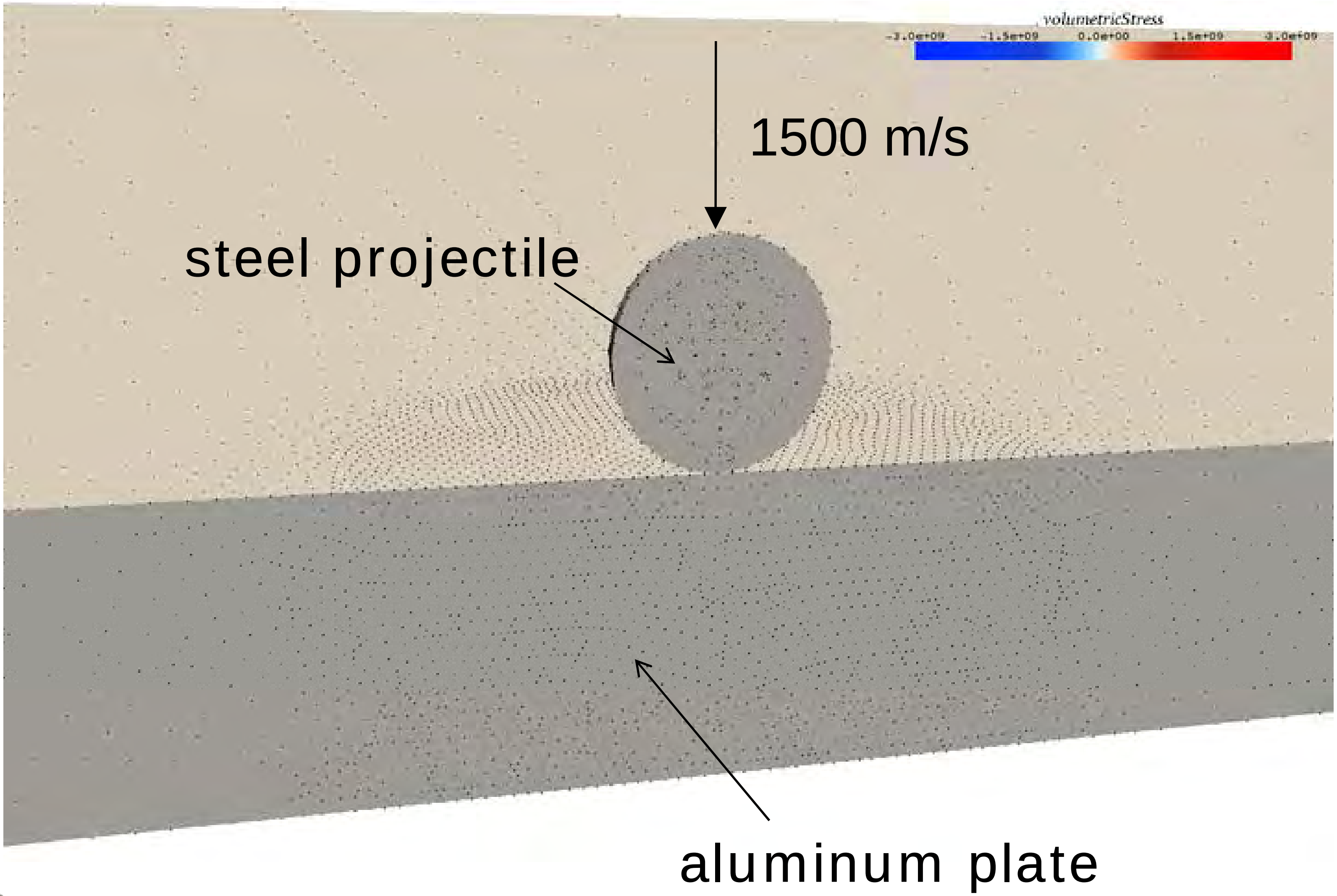


Pandolfi, A. and MO (in preparation)

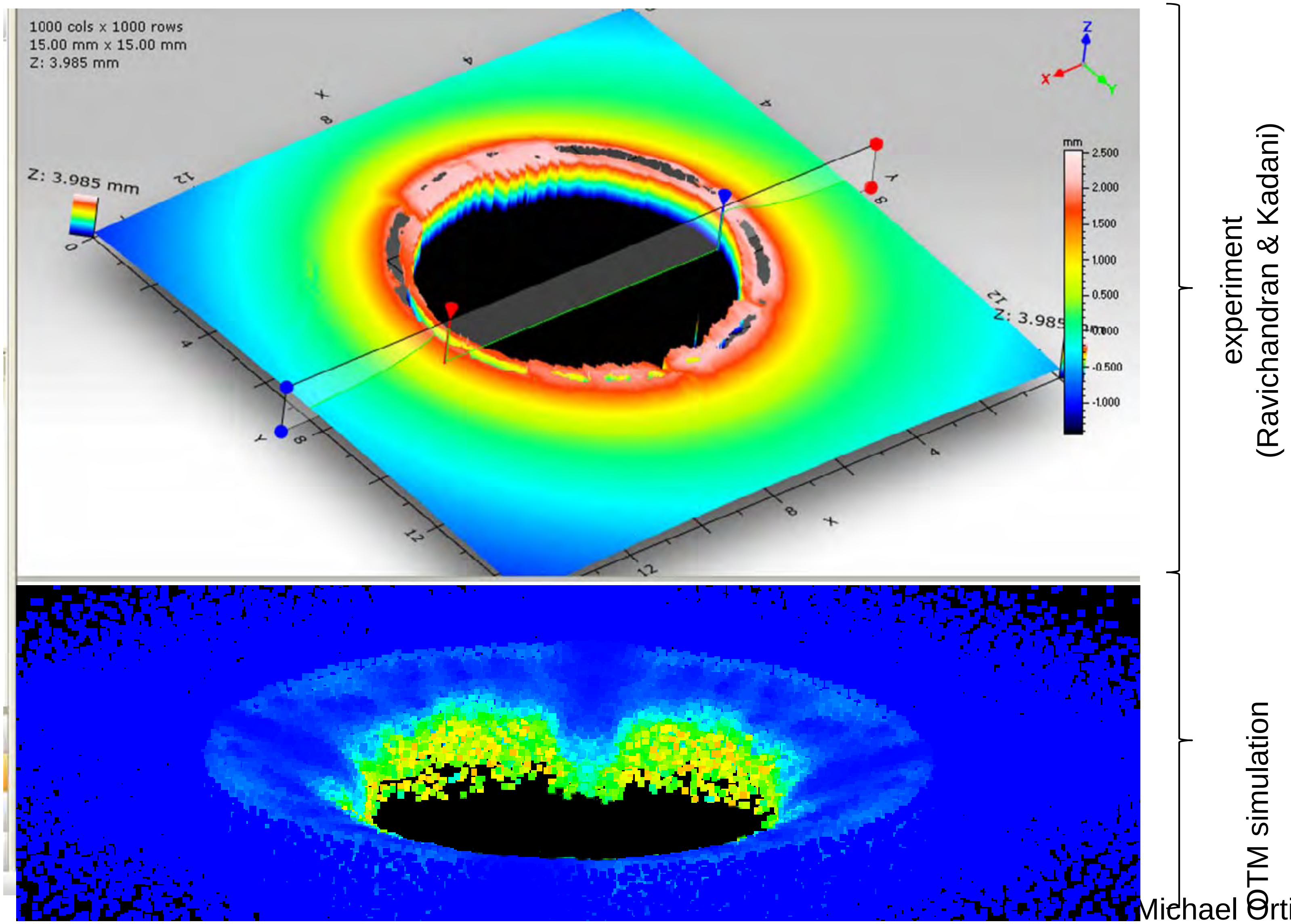
OTM – Back to terminal ballistics



OTM – Back to terminal ballistics



OTM – 440C Steel/Al6061-T6 – 800 m/s



Li, B. *et al.*, Int. J. Impact Eng. (submitted)

Open problems

1. Existence theory for dynamic fracture including crack-path prediction
2. Effective models (in the sense of weak convergence) for compressive comminution including frictional sliding
3. Effective models (in the sense of weak convergence) for dynamic fragmentation in the limit of fine (large number of) fragments