Energy-dissipation functionals in fracture mechanics

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Systems with evolving microstructure

$$\partial\Psi(\dot{u}) + DE(t,u) = 0, \quad \begin{cases} \Psi \equiv \text{dissipation potential} \\ E \equiv \text{energy} \end{cases}$$

- Systems of interest are time-dependent, dissipative, irreversible and hysteretic
- The behavior of the system is governed by both energy and kinetics
- However: Energies of interest often lack differentiability and lower-semicontinuity.
- Meaning of ‘solutions’ in the presence of evolving fine microstructure?
- One approach: Time-discretized variational problems

Classical rate variational problems

- Rate potential: $G : [0, T] \times Y \times Y \to \mathbb{R}$, e. g.,

$$G(t, u, v) \equiv \Psi(v) + DE(t, u)(v)$$

- Potential energy:

$$E(t, u) = \int_\Omega [W(u, \nabla u) - f(t) \cdot u]\, dx$$

- Dissipation potential:

$$\Psi(v) = \int_\Omega \psi(v, \nabla v)\, dx$$

- IBV problem: For $t \in [0, T]$,

$$v(t) \in \operatorname{argmin} G(t, u(t), \cdot)$$

$$\dot{u}(t) = v(t),\ u(0) = u_0$$

Rate problems − Time discretization

- Incremental functional: $u \in Y$,

$$F(u_{n+1}; u_n) = \inf_{\text{paths}} \int_{t_n}^{t_{n+1}} G(t, u(t), \dot{u}(t))\, dt$$

- Example: $G(t, u, v) \equiv \Psi(v) + DE(t, u)\, v$,

$$F(u_{n+1}; u_n) = \Delta t \Psi\left(\frac{u_{n+1} - u_n}{\Delta t}\right) + E(u_{n+1}) - E(u_n)$$

- Incremental problem: For $t_0, \ldots, t_n, t_{n+1}, \ldots$, $u(t_0) = u_0$,

$$\inf_{u_{n+1} \in Y} F(u_{n+1}; u_n)$$

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Rate problems − Time discretization

- IBVP reduced to a sequence of minimization problems to be solved sequentially:

$$\inf_{u_1 \in Y} F(u_1; u_0) \to u_1$$

$$\inf_{u_2 \in Y} F(u_2; u_1) \to u_2$$

$$\dots$$

- But incremental problems may lack attainment!
- Initial conditions for next minimum problem may be ill-defined → scheme breaks down
- Instead: Devise a single minimum principle for entire trajectories

Energy-dissipation functionals

- Energy-dissipation functional: $\lambda_n > 0$, $u \in Y^N$,

$$F(u) \equiv \sum_{n=0}^{N-1} \lambda_{n+1} F(u_{n+1}; u_n)$$

- But *causality* demands: $\lambda_1 \gg \lambda_2 \gg \ldots$
- Introduce function $\lambda_\epsilon(t) > 0$, $\epsilon \geq 0$, s. t.

$$\lim_{\epsilon \longrightarrow 0} \frac{\lambda_\epsilon(t_{n+1})}{\lambda_\epsilon(t_n)} = 0$$

- Sequence of functionals:

$$F_\epsilon(u) \equiv \sum_{n=0}^{N-1} \lambda_\epsilon(t_{n+1}) F(u_{n+1}; u_n)$$

Energy-dissipation functionals

- Suppose: $G(t,u,v) \equiv \Psi(v) + DE(t,u)(v)$
- Sequence of energy-dissipation functionals:

$$F_\epsilon(u) = \sum_{n=0}^{N-1} \lambda_\epsilon(t_{n+1}) \left\{ \Psi\left(\frac{\triangle u}{\triangle t}\right) + \frac{\triangle E}{\triangle t} \right\} \triangle t$$

- Formally, as $\Delta t \to 0$, $u : [0,T] \to Y$,

$$F_\epsilon(u) = \int_0^T \lambda_\epsilon(t)[\,\Psi(\dot{u}) + \dot{E}\,]\,dt$$

- Minimum principle: $\mathbb{Y} = \{u : [0,T] \to Y\}$,

$$\inf_{u \in \mathbb{Y}} F_\epsilon(u)$$

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Energy-dissipation functionals

- Particular choice: $\lambda_\epsilon(t) = \mathrm{e}^{-t/\epsilon}$,

$$F_\epsilon(u) = \int_0^T \mathrm{e}^{-t/\epsilon}[\,\Psi(\dot{u}) + \dot{E}\,]\,dt$$

- Alternatively, integrating by parts,

$$F_\epsilon(u) = \int_0^T \mathrm{e}^{-t/\epsilon}[\Psi(\dot{u}) + \frac{1}{\epsilon}E(t,u)]\,dt + [\mathrm{e}^{-t/\epsilon}E]_0^T$$

- Euler-Lagrange equations:

$$-\epsilon D^2\Psi(\dot{u})\ddot{u} + D\Psi(\dot{u}) + DE(t,u) = 0$$

- Evolutionary problem is (formally) recovered in the limit of $\epsilon \to 0$.

Energy-dissipation functionals

- Energy-dissipation functionals represent **elliptic regularizations** of the evolutionary problem
- The system is endowed with a small amount of '**foresight**' over small time intervals of size ε

$$F_\epsilon(u) = \int_0^T \underbrace{\mathrm{e}^{-t/\epsilon}}_{\text{“Arrow of time”}}\underbrace{[\Psi(\dot u)}_{\text{Dissipation}} + \underbrace{\frac{1}{\epsilon}E(t,u)}_{\text{Energy}}]\,dt + [\mathrm{e}^{-t/\epsilon}E]_0^T$$

- Minimizers do not define a gradient flow: $E = E(u)$,

$$\epsilon D\Psi(\dot u)(\ddot u) + D\Psi(\dot u)(\dot u) + \dot E = 0$$

Energy-dissipation functionals

- Minimization of energy-dissipation functionals characterizes **entire trajectories** of the system
- 'Solutions', understood as **minimizers**, make sense even when the energy is not differentiable
- 'Solutions', understood as **minimizing sequences**, make sense even when the energy lacks lower semicontinuity
- *Existence* of solutions at fixed ε?
- *Causal* limit $\varepsilon \to 0$?
- Limiting behavior of trajectories of *sequences* of energy-dissipation functionals?

Causal limit – Rate-independent case

- Banach spaces Z, X compactly embedded.
- Sequences of functionals Ψ_k (rate indep.), E_k.
- Energy-dissipation functionals:

$$F_{\epsilon,k}(u) = \int_0^T \mathrm{e}^{-t/\epsilon}[\Psi_k(\dot{u}) + \frac{1}{\epsilon} E_k(t,u)]\, dt + [\mathrm{e}^{-t/\epsilon} E_k]_0^T$$

- Coercivity: $\Psi(v) \geq c\|v\|_Z.$
  $E(t,u) \geq c\|u\|_X^\alpha - C$
- Power control: $|\partial_t E(t,u)| \leq c_1^E (E(t,u) + c_0^E)$
- IC stability: $E(0,u_0) \leq E(0,\tilde{u}) + \Psi(\tilde{u} - u_0)$

$\Big\}(A)$

Causal limit – Rate-independent case

- Weak continuous convergence of Ψ_k:

$$v_k \rightharpoonup v \Rightarrow \Psi_k(v_k) \to \Psi(v)$$

- Weak Γ-convergence of E_k:

$$E = \Gamma - \lim_{k\to\infty} E_k, \text{ weakly in } X$$

- Weak continuity of external power:

$$\left.\begin{array}{l} u_k \rightharpoonup u \\ E_k(t,u_k) \to E(t,u) \end{array}\right\} \Rightarrow \partial_t E_k(t,u_k) \to \partial_t E(t,u)$$

(B)

Causal limit – Rate-independent case

Theorem (A. Mielke & MO) *Let assumptions (A) and (B) hold uniformly in* k*. Then:*

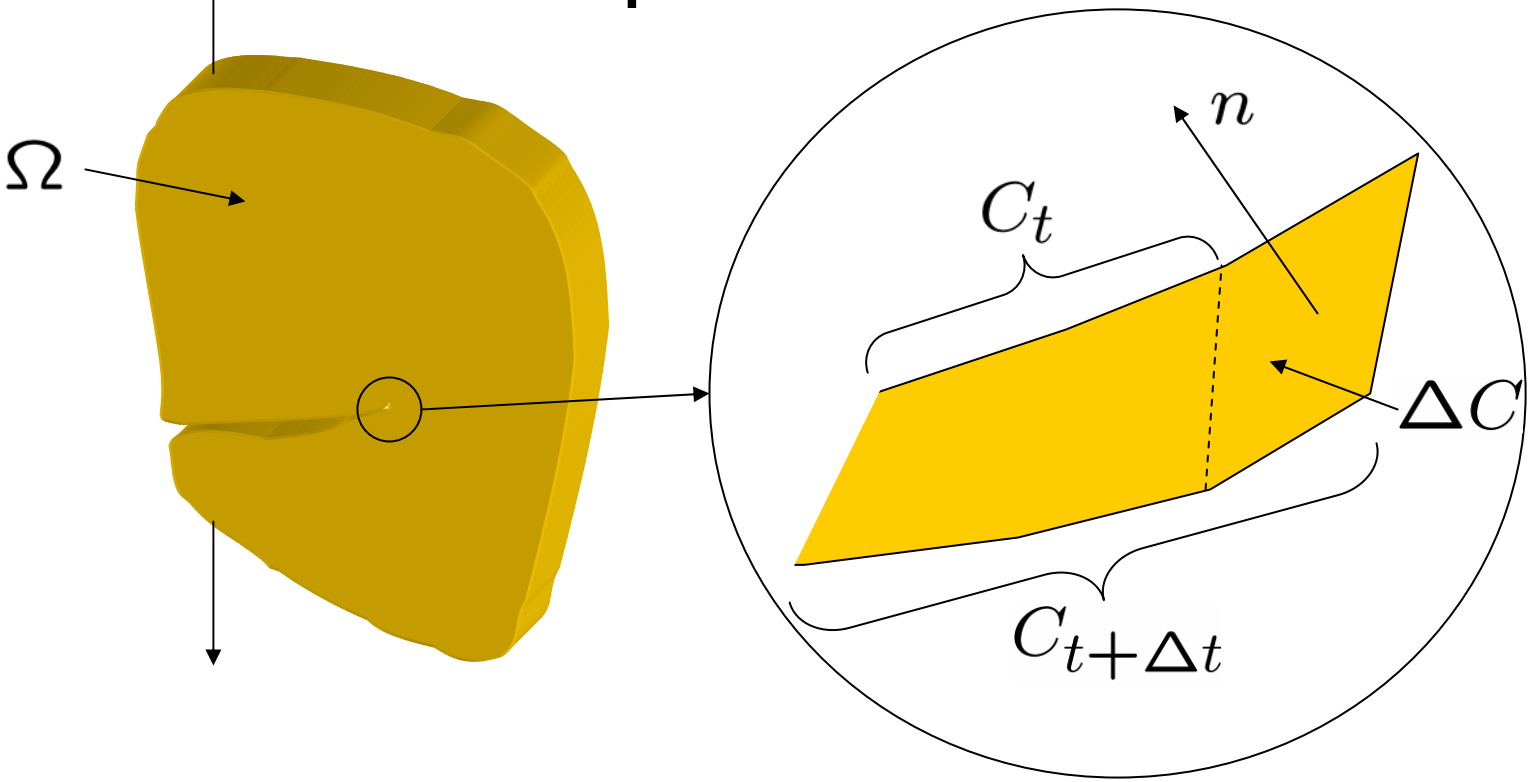
i) Any family $u_{\epsilon,k}$ *of minimizers of* $F_{\epsilon,k}$ *is weakly precompact in* $\mathbb{Y} \equiv \mathsf{L}^\infty([0,T],X) \cap \mathsf{BV}([0,T],Z)$.

ii) Any limit point $u \in \mathbb{Y}$ *of* $u_{\epsilon,k}$ *is a solution of the* energetic formulation

$$\forall\, v \in X\colon E(t,u(t)) \le E(t,v) + \Psi(v-u(t)),$$

$$E(t,u(t)) + \int_0^t \Psi(du) = E(0,u(0)) + \int_0^t \partial_s E(s,u(s))ds$$

The rate problem of LEFM



- Assume regularity, smoothness . . .
- Load increment, crack extension:

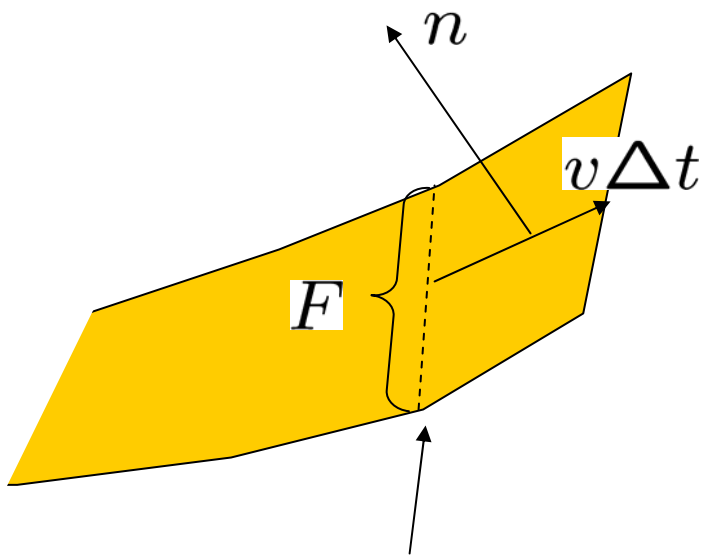
$$-\Delta E = \int_{\Delta C} [DW(\nabla u_t)\, n] \cdot [\![u_{t+\Delta t}]\!]\, d\mathcal{H}^2 + h.o.t.$$

The rate problem of LEFM

- Energy-release rate:

$$G = \lim_{\Delta t \to 0} -\frac{\Delta E}{\Delta t}$$

$$= \int_F f(n)\, v\, d\mathcal{H}^1$$



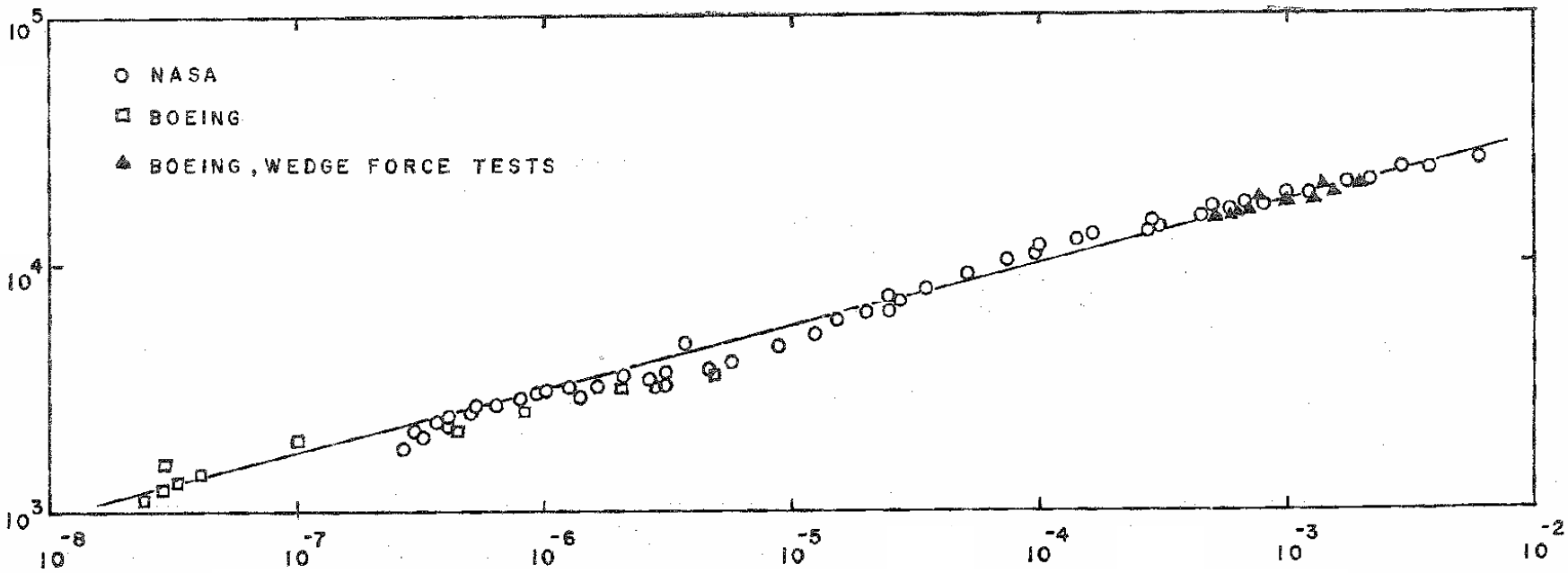
- Driving force:

$$f(n) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [DW(\nabla u_t)\, n] \cdot [\![u_{t+\Delta t}]\!]$$

- Crack-tip equation of motion: $f = \partial\psi(v)$
- Dissipation: $\Psi(v) = \int_F \psi(v)\, d\mathcal{H}^1$

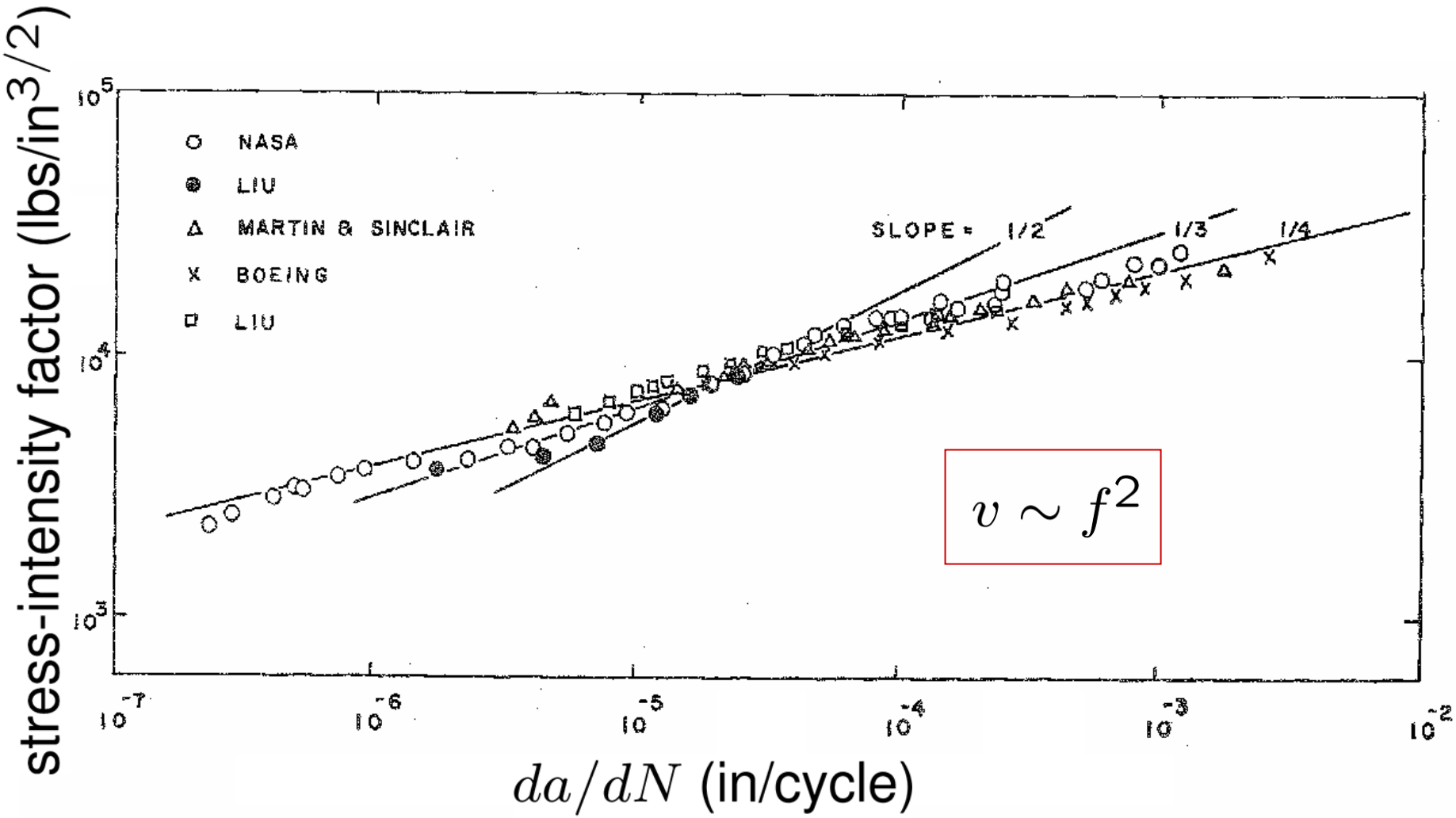
The rate problem of LEFM

Crack-tip equation of motion for fatigue crack growth



Crack-growth data for 2024-T3 aluminum alloy
(P. Paris and F. Erdogan, ASME Trans (1963)

The rate problem of LEFM

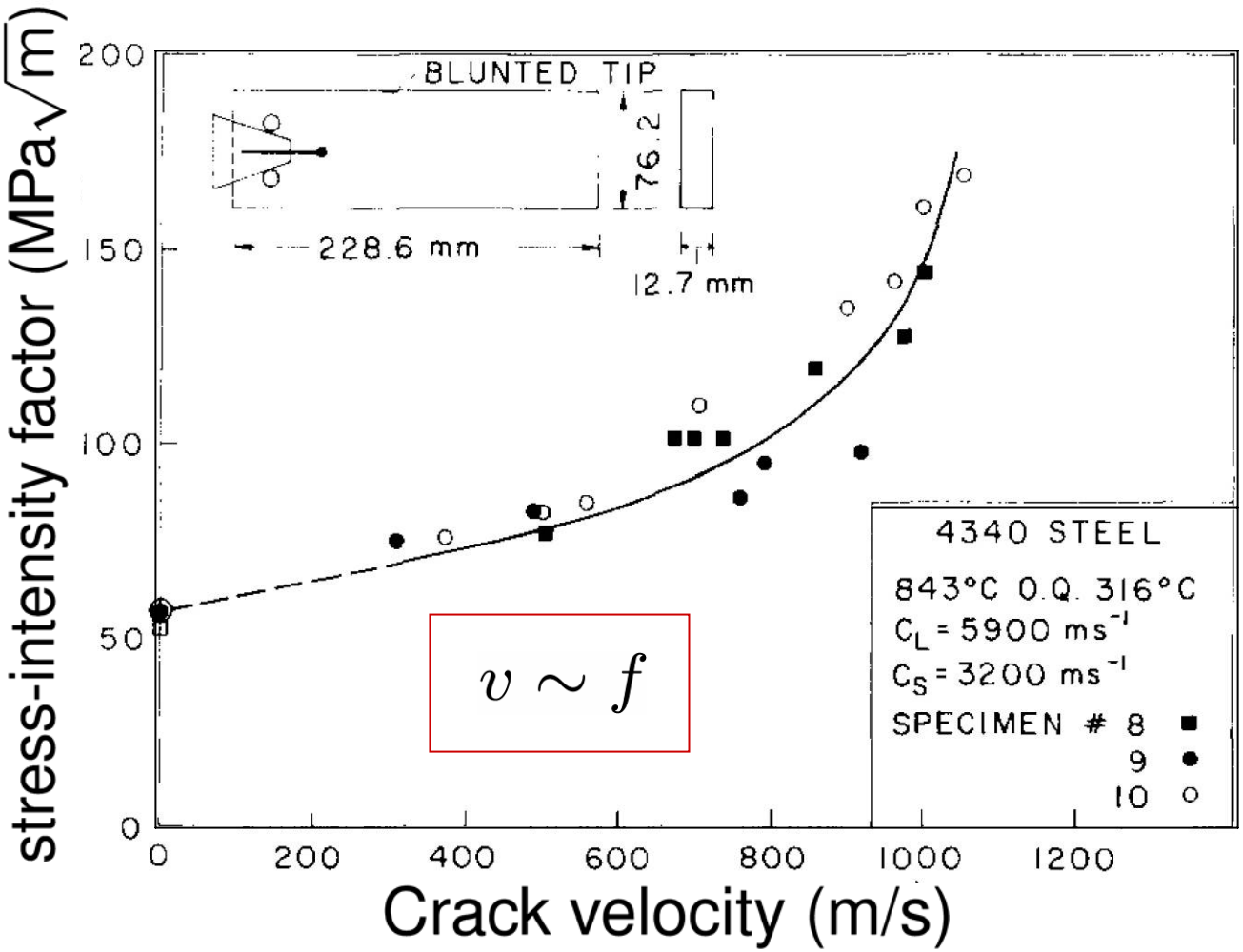


$$v \sim f^2$$

Crack-growth data for 2024-T3 aluminum alloy
(P. Paris and F. Erdogan, ASME Trans (1963)

The rate problem of LEFM

Dynamic crack-tip equation of motion



Rosakis, Duffy and Freund, *JMPS* (1984)

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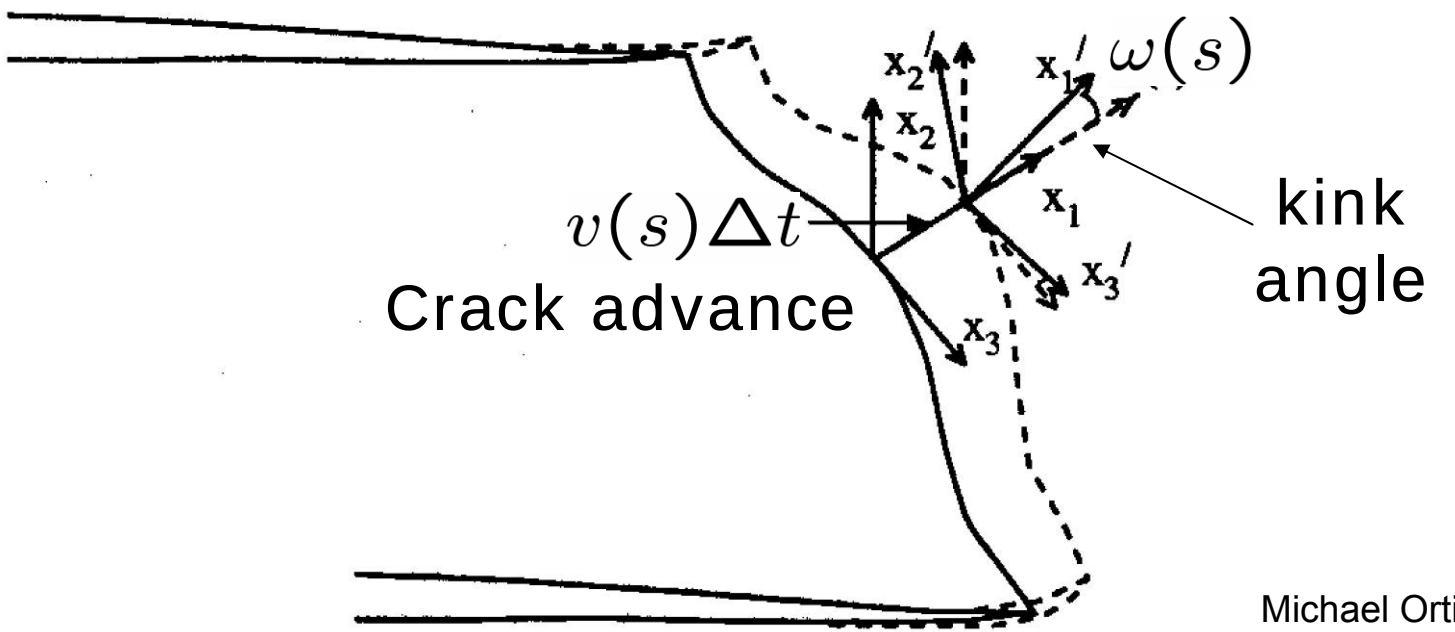
The rate problem of LEFM

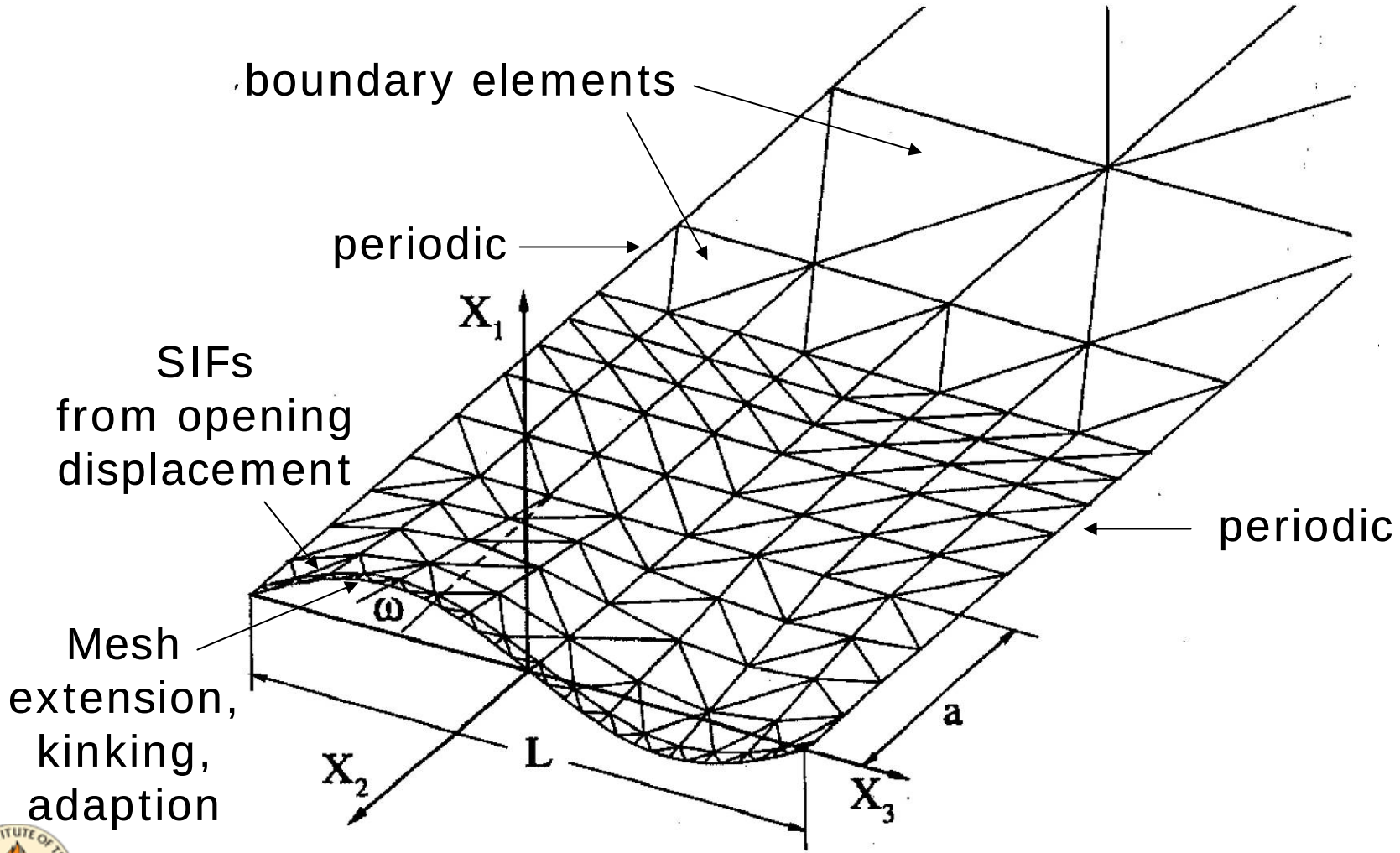
- Rate problem: $\inf_{v,n} \int_F [\psi(v) - f(n)\, v]\, d\mathcal{H}^1$

$$\Rightarrow \left\{ \begin{array}{l} \partial\psi(v) = f(n) \\ \partial\psi^*(f(n)) = 0 \end{array} \right\} \longrightarrow (v, n)$$

(maximum driving force)



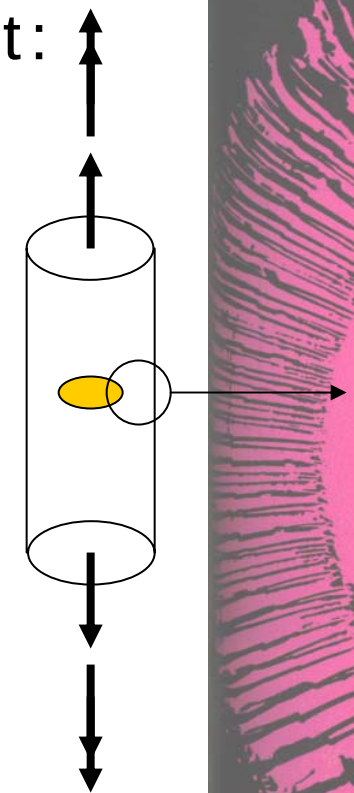
LEFM rate problem – Numerical analysis

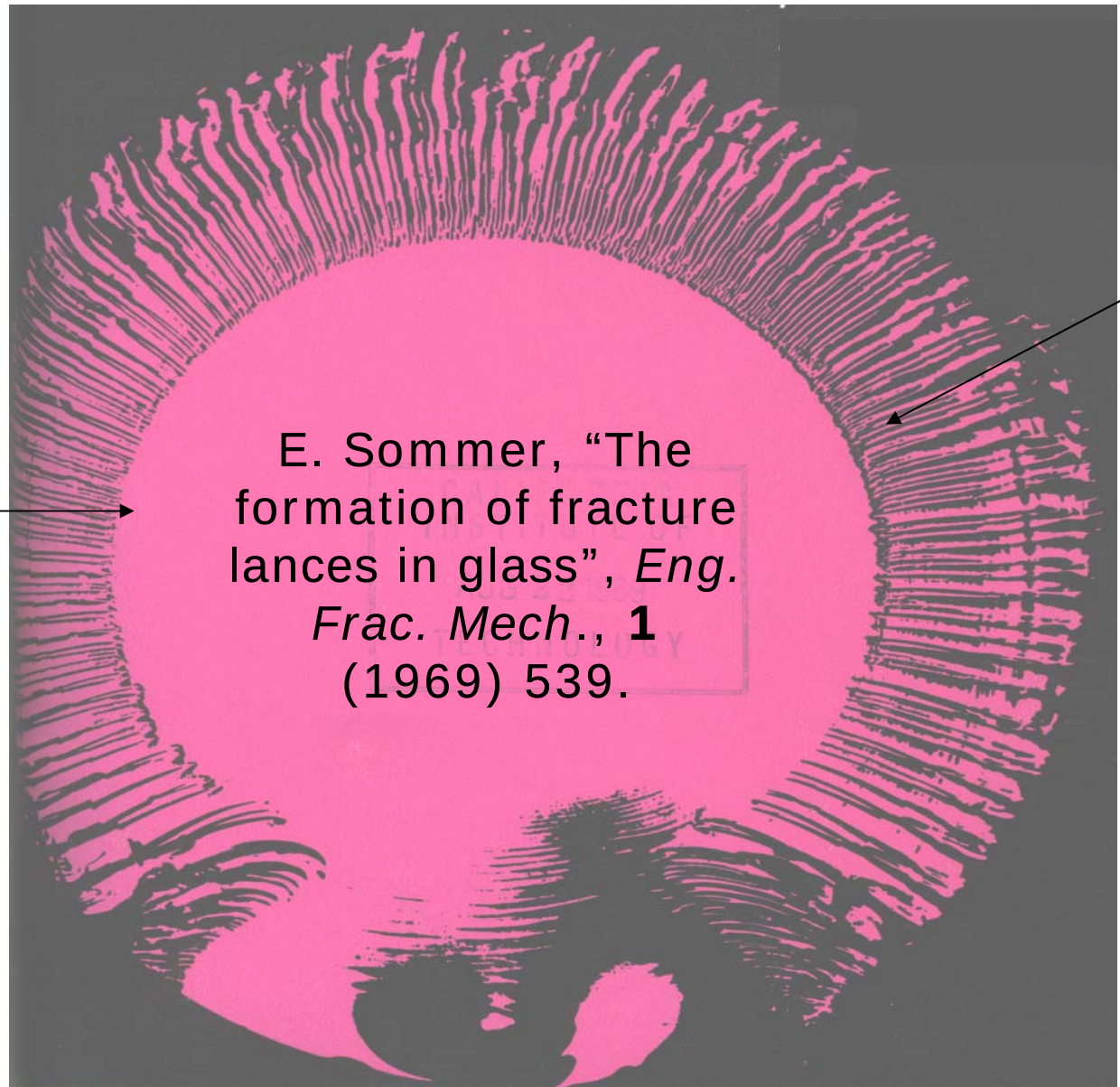


Xu, Bower and Ortiz, *IJSS* (1994)

LEFM rate problem – Numerical analysis

tension-torsion test:





E. Sommer, “The formation of fracture lances in glass”, *Eng. Frac. Mech*., **1** (1969) 539.

lances

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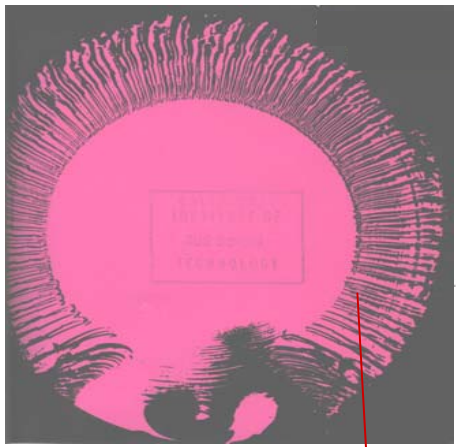
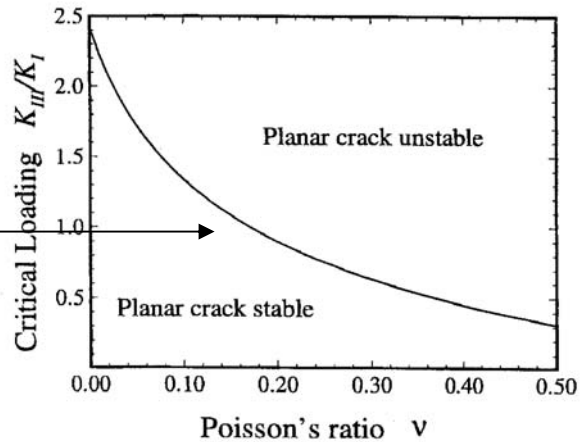
LEFM rate problem – Numerical analysis





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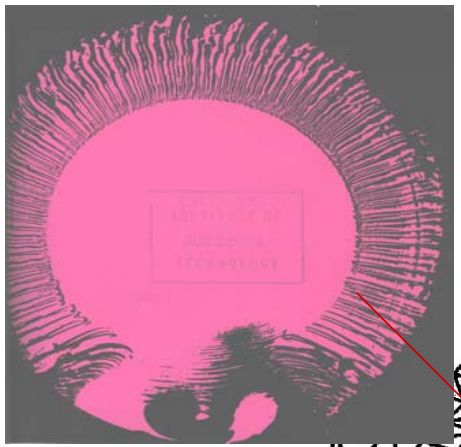

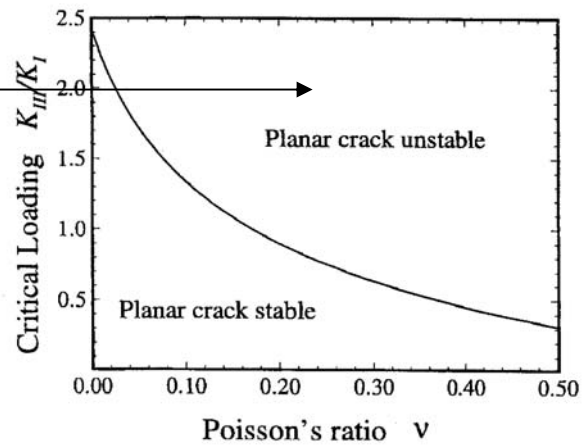

Xu, Bower and Ortiz,
IJSS (1994)

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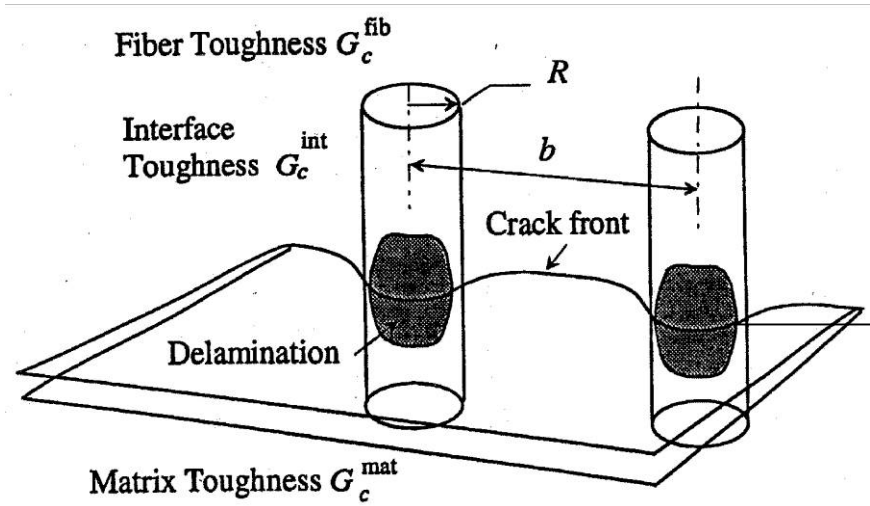
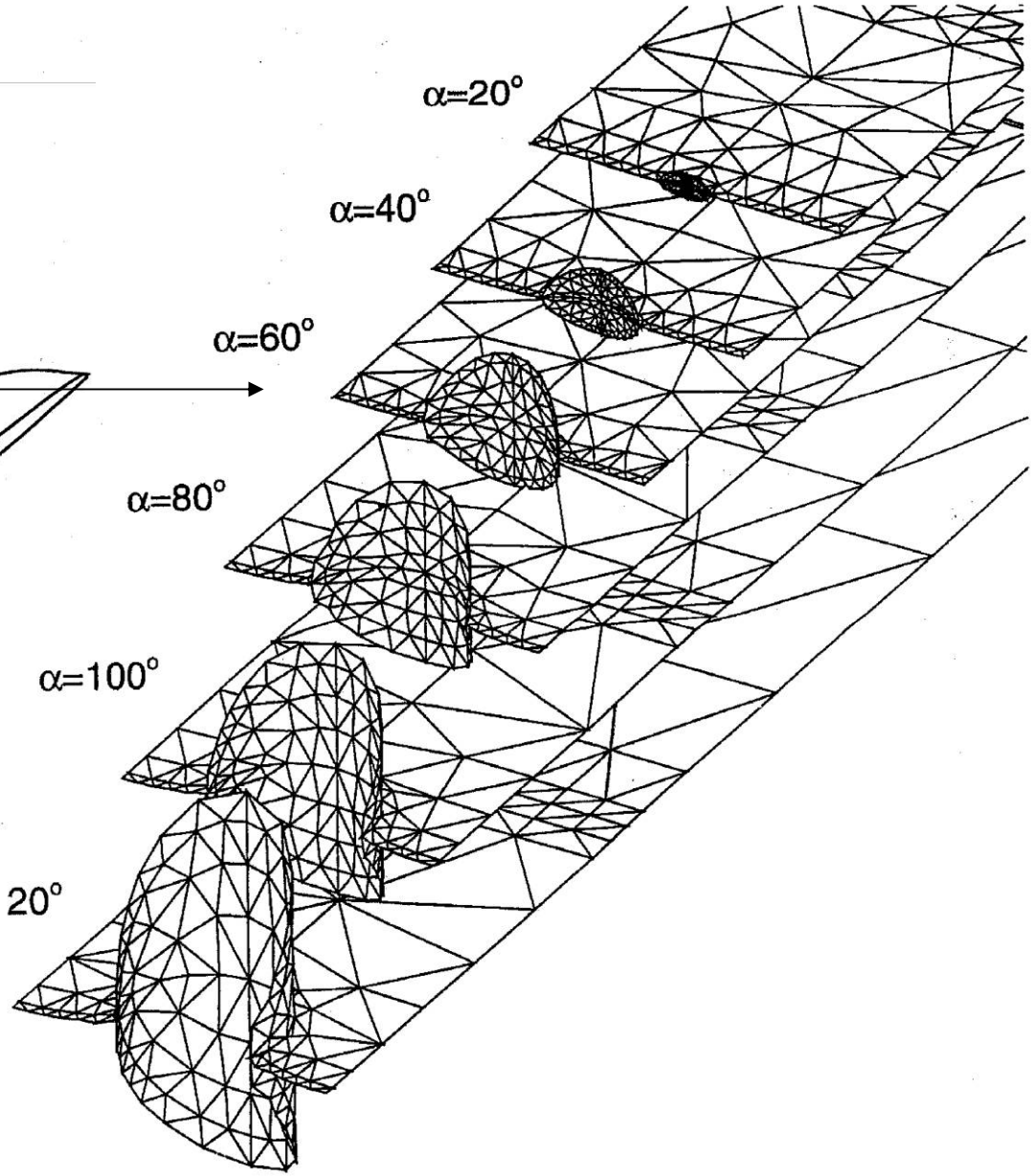
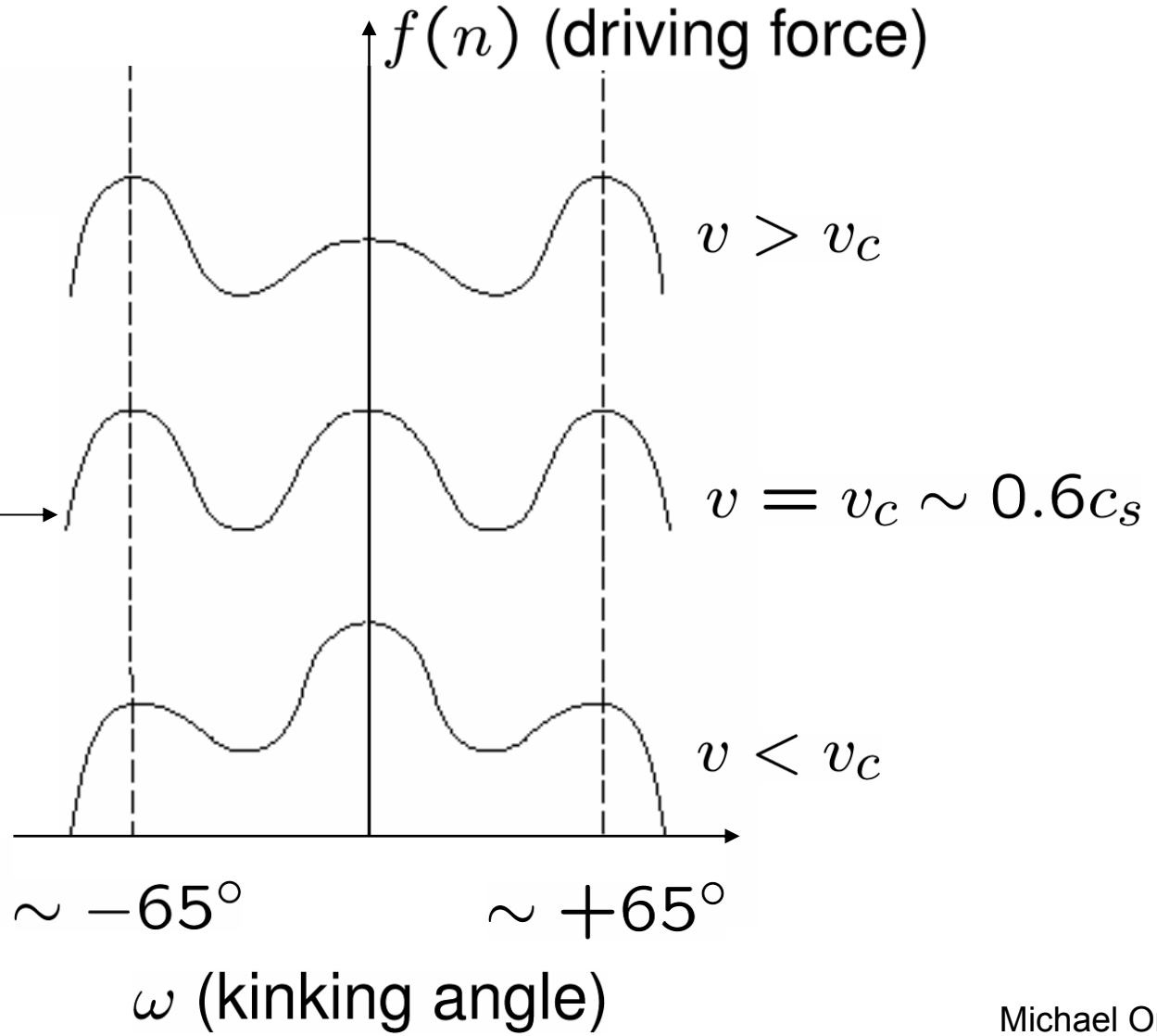




Fiber debonding in composites

Xu, Bower and Ortiz, *JMPS* (1998)

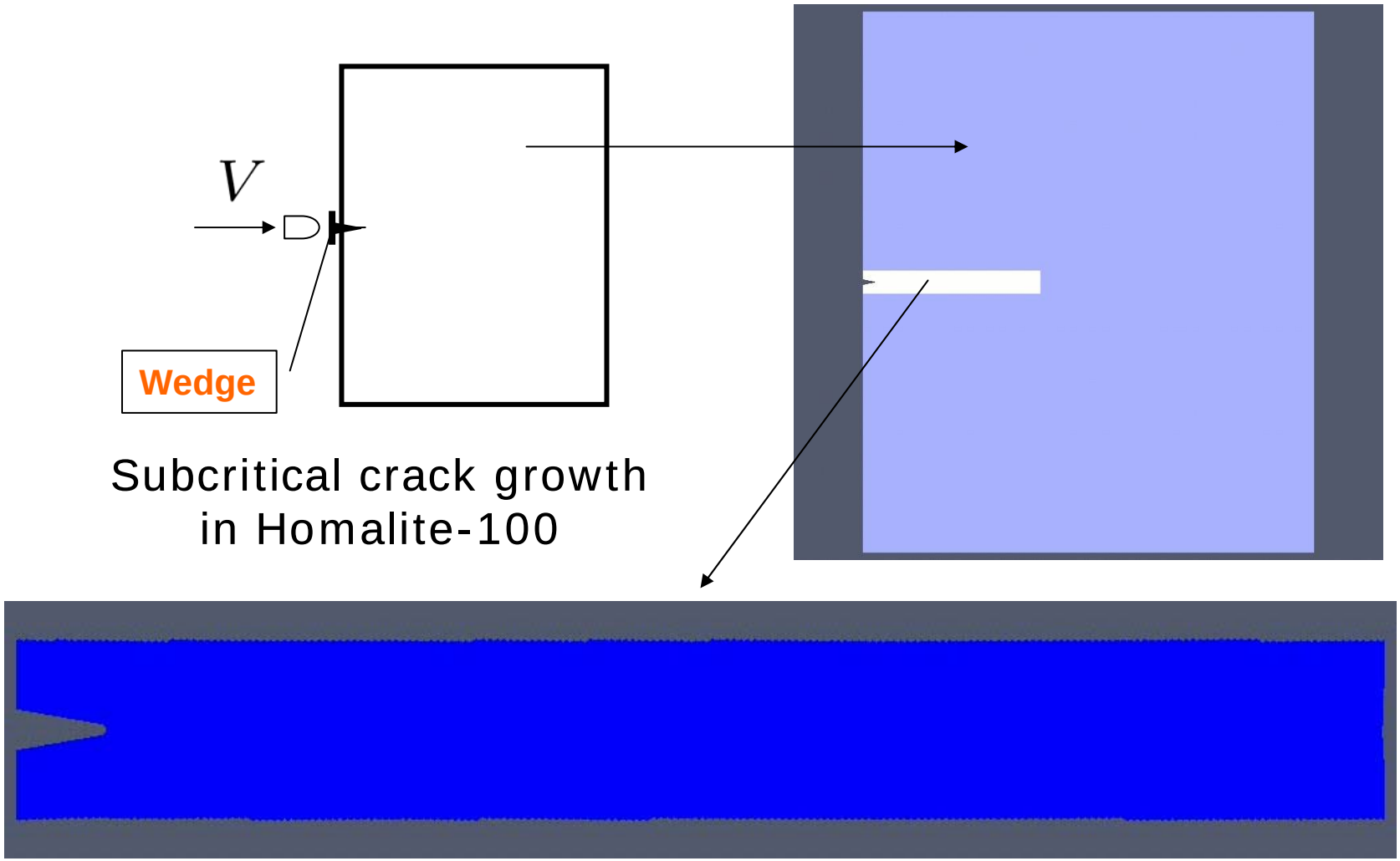
LEFM rate problem — Dynamics



branching instability

E.H. Yoffe, *Phil. Mag.* (1951)

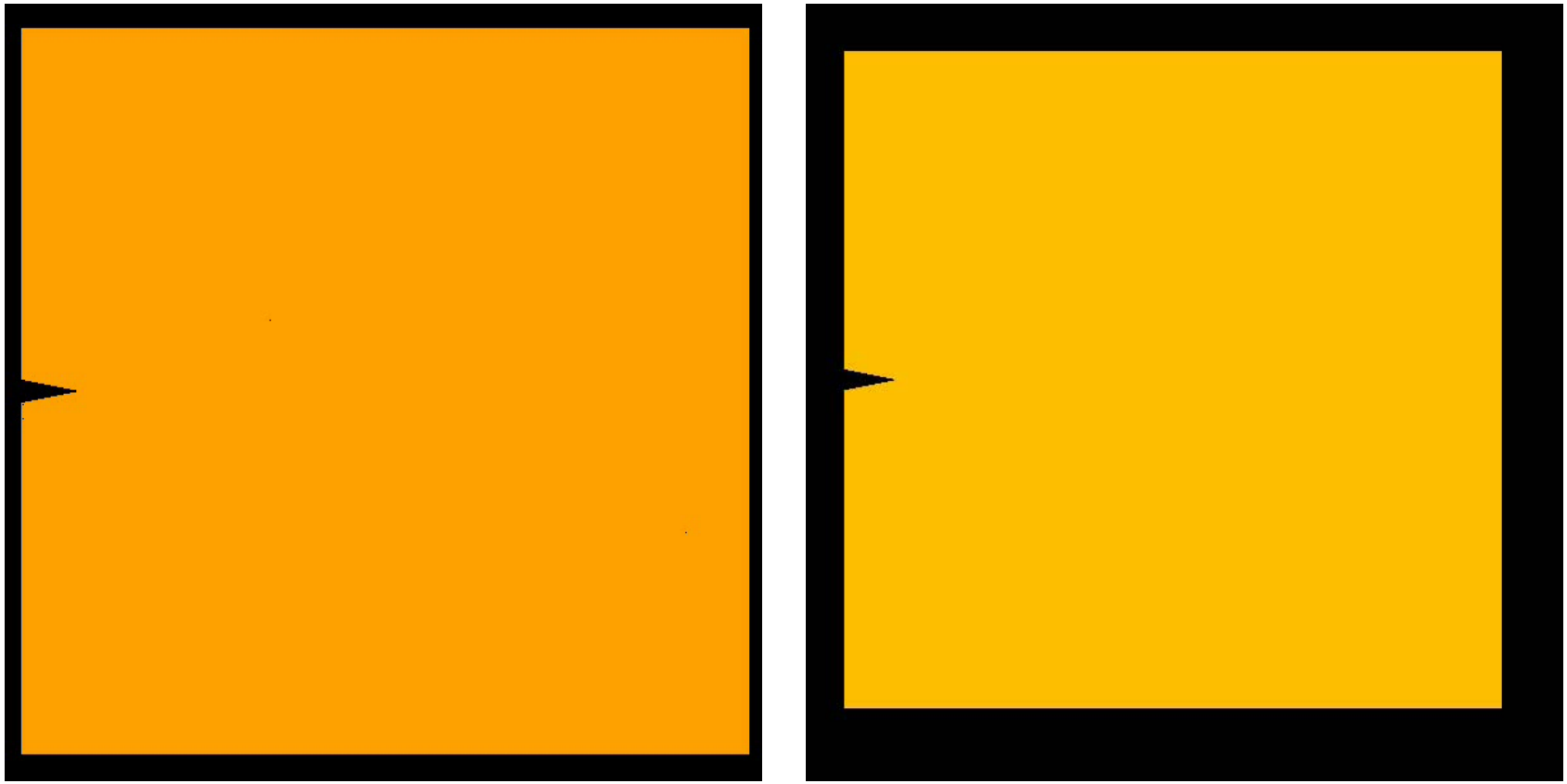
LEFM rate problem — Dynamics



Subcritical crack growth
in Homalite-100

I. Arias et al., *CMAME* (2007)

LEFM rate problem — Dynamics



Supercritical crack growth in Homalite-100
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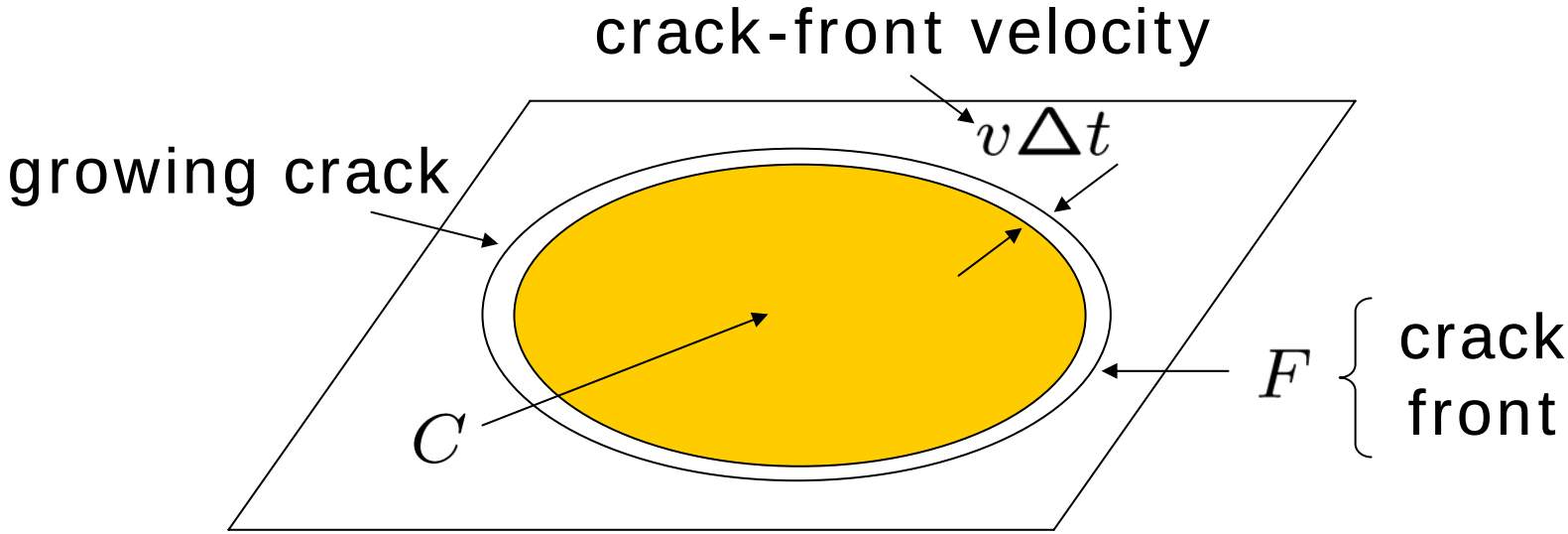
LEFM energy-dissipation functionals



- Crack set: C, $0 < \mathcal{H}^{n-1}(C) < +\infty$.
- Crack-front measure: $\forall \varphi \in C_0^1([0,T]), \forall f \in C_0(\Omega)$,

$$\int_0^T \dot{\varphi} \int_{C(t)} f d\mathcal{H}^{n-1} dt = -\int_0^T \varphi \int_\Omega f d\mu_t(x) dt$$

- Crack front, velocity: $d\mu_t = v_t d\mathcal{H}^{n-2} \lfloor F(t)$.

LEFM energy-dissipation functionals

- Path: $p := (u, C)$.
- Energy-dissipation functional:

$$F_\epsilon(p) := \int_0^T \mathrm{e}^{-t/\epsilon} \left\{ \Psi(v) + \frac{1}{\epsilon} E(u) \right\} dt$$

- Energy: $E(u) = \int_\Omega W(\nabla u)\, dx$
- Dissipation: $\Psi(v) = \int_{F(t)} (\alpha + v^p)\, d\mathcal{H}^{n-1}$

nucleation energy

rate-dependent crack-tip equation of motion

LEFM energy-dissipation functionals

- Assume $\Omega \subset \mathbb{R}^2$.
- Minimize $F_\epsilon(p)$ over the class $\mathcal{P}$ of paths:

  i) $u(\cdot,t) \in SBV(\Omega), \forall t \in [0,T]$.

  ii) $u(\cdot,t) = g \in L^\infty(\Omega)$ on $\partial\Omega \setminus C(t), \forall t \in [0,T]$.

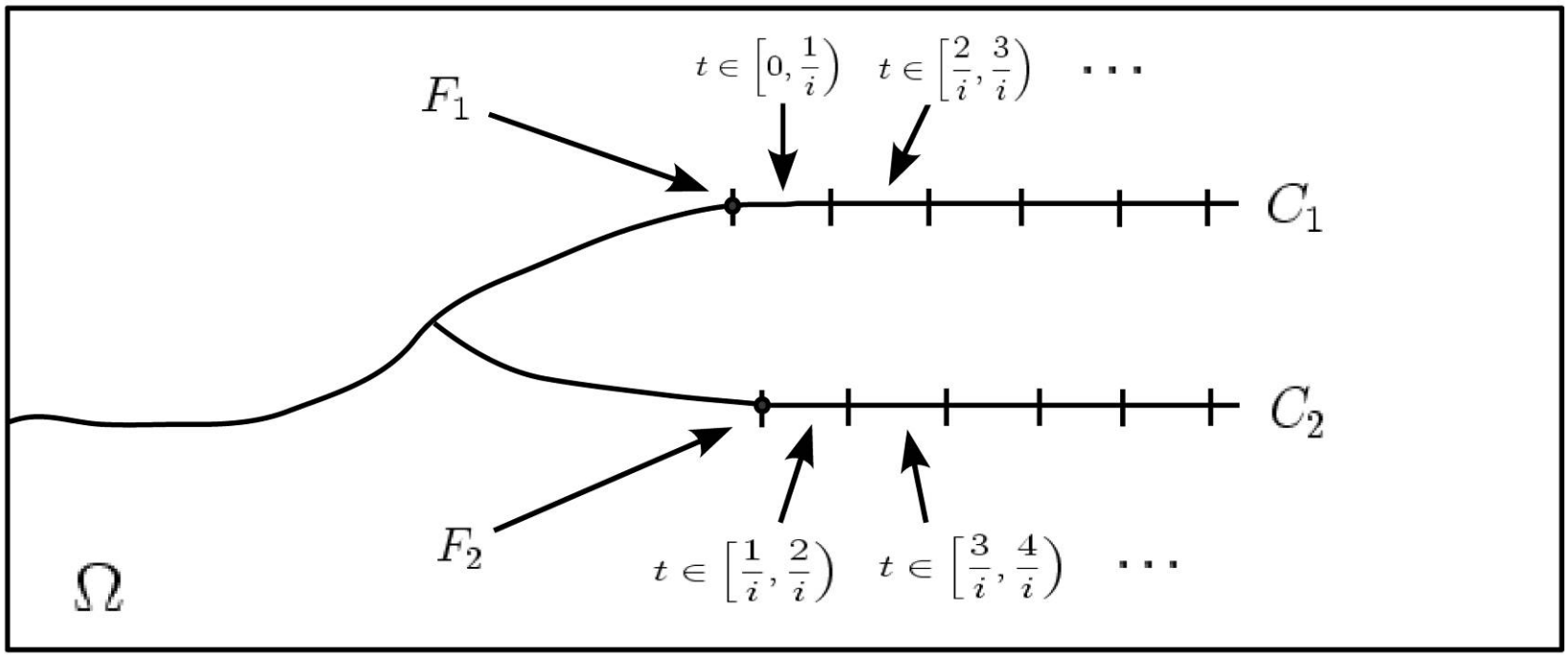
  iii) $C : [0,T] \to \left\{K : K \subset \bar{\Omega}, \mathcal{H}^1(K) < \infty\right\}$ s. t.:

  iii.a) C nondecreasing: $\forall \tau < t, C(\tau) \overset{\mathcal{H}^1}{\subset} C(t)$.

  iii.b) $\forall \tau < t, S(u(\tau)) \overset{\mathcal{H}^1}{\subset} C(t)$.

  iii.c) $\exists$ CFM $d\mu_t = v_t d\mathcal{H}^0 \lfloor F(t)$,
  $F : [0,T] \to 2^\Omega$, $v(t) : F(t) \to \mathbb{R}$.

LEFM energy-dissipation functionals



- Failure of l.s.c.: For $v \leq \left(\frac{\alpha}{1-2^{1-p}}\right)^{1/p}$,

$$\alpha + v^p = \liminf_{i\to\infty} \Psi(p_i) \leq \Psi(p) = 2\alpha + (v/2)^p$$

LEFM energy-dissipation functionals

Theorem (C. Larsen, MO, C.L. Richardson) *The lower semicontinuous envelop of* F_ϵ *in* $\mathcal{P}$ *is:*

$$sc^- F_\epsilon(p) =$$

$$\int_0^T e^{-t/\epsilon} \left\{ \frac{1}{\epsilon} \int_\Omega W(\nabla u) dx + \gamma \int_{F(t)} v d\mathcal{H}^0 \right\} dt$$

where: $\gamma = p \left(\frac{\alpha}{p-1} \right)^{\frac{p-1}{p}}$

- Relaxed energy-dissipation functional is **rate-independent**!

LEFM energy-dissipation functionals

Sketch of proof:

1. Consider an arbitrary $p \in \mathcal{P}$. The plan is to construct a sequence $p_i \longrightarrow p$ such that p_i builds the crack set of p with minimal dissipation.
2. Break up $[0, T]$ into increments of size $\frac{1}{i}$.
3. For each increment, use rectifiability to finely cover the crack set with balls in which the crack set is close to a segment.
4. Use the Besicovitch covering theorem to get a countable disjoint subcover that covers almost all of the crack set.

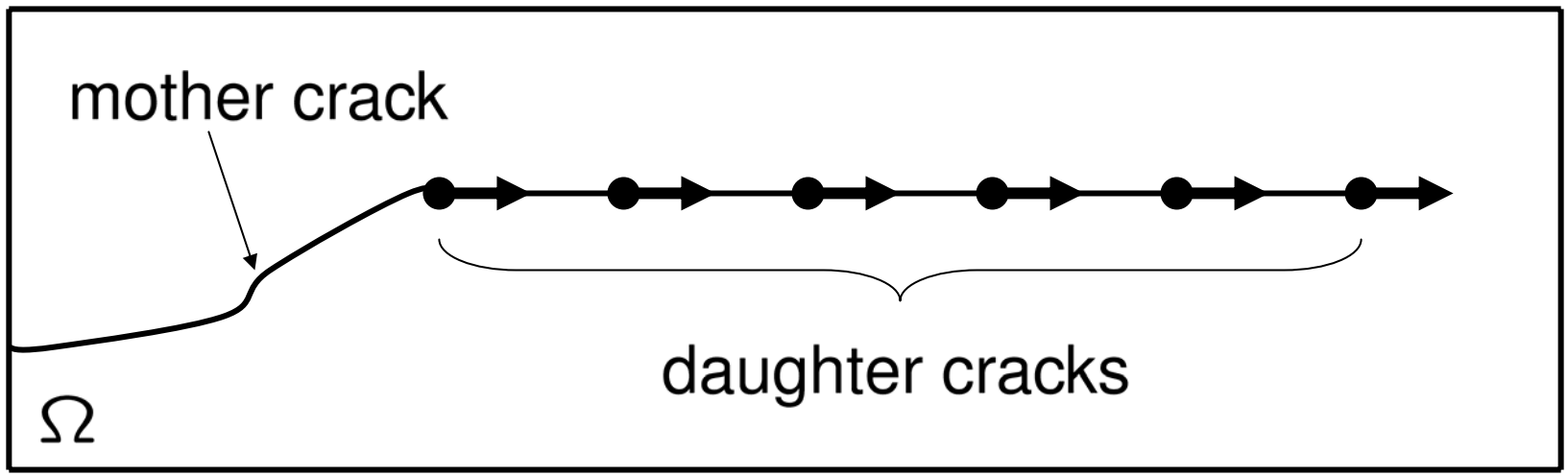
LEFM energy-dissipation functionals

Sketch of proof (continued):

5. Calculate the minimal dissipation required to build such a hyperplane, and construct the configuration of fronts that achieves the minimum.
6. Define the crack set for p_i locally in space by this construction, on each time increment.

LEFM energy-dissipation functionals

Sketch of proof: Mother-daughter mechanism:



- Twin daughters (optimal by Jensen's inequality):

$$\Psi = n\alpha + n\left(\frac{v}{n}\right)^p \to \min \Rightarrow$$

$$n_{\min} = \left(\frac{p-1}{\alpha}\right)^{(1/p)} v, \quad \Psi_{\min} = p\left(\frac{\alpha}{p-1}\right)^{(1-1/p)} v$$

Concluding remarks

- Energy-dissipation functionals provide a useful tool for understanding microstructure evolution within the framework of the calculus of variations.
- The application to fracture requires a careful analysis of the crack front. The dissipation attendant to crack growth is concentrated on the crack front
- Calculus of variations tools should be most effective for understanding complex fracture processes and defining effective problems, including:
  - *Crack branching, fragmentation phenomena*
  - *Crack propagation through heterogeneous media*
- Calculus of variations tools could be useful for understanding the convergence of numerical methods for fracture mechanics