Ph101: Solution 5 version 2

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1 Piano Physics

(a) The speed of transverse waves in the wire is $V_t = \sqrt{T/\mu}$, where T is the tension and μ the mass per unit length in the wire (you can derive this from dimensional analysis if you'd forgotten it). Since $T = A\sigma$ and $\mu = A\rho$ where A is the cross-sectional area of the wire, $\rho \sim 8\text{g cm}^{-3}$ its density, and $\sigma =$ its stress, we also have $V_t = (\sigma/\rho)^{1/2}$. Since we are told that piano strings are stretched to the yield point of steel, $\sigma \simeq \epsilon_Y B$, where $B \sim 2\times 10^{12}\text{erg cm}^{-3}$ is the elastic modulus and $\epsilon_Y \sim 10^{-2}$ is the yield strain (cheap steel has $\epsilon_Y \sim 10^{-3}$, but for reasons that will become apparent, piano wire is made from cold-drawn tempered high-carbon spring steel and is an order of magnitude better). Thus

$$V_t = \left(\frac{\sigma}{\rho}\right)^{1/2} = \left(\frac{B\epsilon_Y}{\rho}\right)^{1/2} = \left(\frac{2\times 10^{12}\times 10^{-2}}{8}\right)^{1/2} = \boxed{5\times 10^4\text{cm s}^{-1}}\,. \tag{1}$$

This is comparable to the speed of sound in air, 340m/s.

One may briefly wonder why V_t is not the same as the speed of sound in steel. This would occur if $\epsilon_Y \sim 1$. Compressional waves have a restoring force set by the interatomic forces, which are huge—so compressional waves move quickly. Transverse waves have a restoring force set by the tension. But you can't get the tension in real macroscopic materials anywhere near to the interatomic force (i.e., double all interatomic spacings) —they flow or fracture long before that (recall discussion of Griffith crack lengths). So $V_t \sim \sqrt{\epsilon_Y} \sim 0.1$ times the sound speed.

(b) A piano string is fixed at both ends, so the lowest frequency f (the fundamental mode) fits half a wavelength into the string length: $L = \lambda/2$.

Since $\lambda = V_t/f$, we have

$$L = \lambda/2 = \frac{V_t}{2f} = \frac{5 \times 10^4 \text{cm s}^{-1}}{2 \times 262 \text{s}^{-1}} \simeq 100 \left(\frac{\epsilon_Y}{10^{-2}}\right)^{1/2} \text{cm} \tag{2}$$

In actuality the wires are not pulled exactly to their breaking strain (remember they get pounded by the hammers when playing!), but a bit below. The actual length of the middle C strings in Sterl's grand piano is 60 cm.

(c) The string is not exactly horizontal but is inclined by some angle θ at its ends with respect to the horizontal plane. The vertical component of the tension T in the wire must balance gravity:

$$\begin{aligned} 2T\theta &\simeq \mu L g & (3) \\ Lg/\theta \simeq 2(T/\mu) &= 2V_t^2 = 2(L2f)^2 \qquad \text{(from part (b))} & (4) \\ \theta\left(\frac{L}{2}\right) &\simeq \frac{g}{16f^2} = \frac{980}{16 \times 440^2} \simeq \boxed{3 \times 10^{-4}\text{cm}} & (5) \\ & & (6) \end{aligned}$$

Notice that the sag distance $g/16f^2$ depends only on the frequency to which the string is tuned, and is independent of its length, thickness or tension!

(d) A long vibrating string is a dipole in two dimensions. Air on the leading edge of the string is being compressed (pushed out of the way), and on the trailing edge is a rarefaction into which air is flowing, so the motion is like that of one long "source" or air followed by one long "sink" - i.e. a mass dipole, with no net mass flux. Net momentum, however, is being transported: the string upon each swing in one direction imparts a fraction of its own instantaneous momentum to both waves. Momentum flux implies dipolar radiation.

The thin wire on its own doesn't push around much air because its cross-section is so small. The strings on their own would be inaudible as a musical instrument. But when clamped to the wooden soundboard, the wire imparts its vibrational energy to the board, whose large surface area pushes much more air, so the power emitted is much larger. The string sets the frequency. The sound-board driven at that frequency pushes the air to radiate the sound.

(e) First a little warning: pianos are rarely played in pastures. In a room, the sound radiated from the piano reflects off the walls, and standing waves are set up. The mean square pressure your ear or dB meter feels (the loudness or decibel level) will depend on the sound absorption coefficient of the room surfaces. Concrete absorbs only $\sim 2\%$ of the incident acoustic power, while a carpeted room with acoustic tiles, open windows and a lot of upholstered furniture might absorb $\sim 30\%$. Adopting 10% as a typical value in a living room or chamber music venue, the circulating power will be about 10 times larger (and so the decibel level 10 larger) than if the piano were played in a pasture with no reflected waves.[1]

From experience, we know that a piano player playing a loud piece can compete with a vacuum cleaner (70–90dB depending on model). If we adopt a peak SPL of 90dB in a room, corresponding to 80dB in a pasture (i.e., in free space —see previous paragraph) at a distance of 2m, and using the relation between decibel SPL and unidirectional power given in class SPL $=$ 80dB $= 10\log_{10}(F/10^{-12}\text{W m}^{-2})$ we estimate the acoustic power radiated by the piano to be

$$P = 4\pi(2\text{m})^2 \times 10^8 \times 10^{-12}\text{W m}^{-2} \simeq 0.005\text{W} \tag{7}$$

A vicious Beethoven attack on the keys might consist of dropping a pair of $m = 2$ kg arms $h = 20$cm onto the keys, giving an energy $2mgh = 8 \times 10^7$erg $=$ 8J. But it is hard to play fast that way —perhaps just one such stroke per second. A minimum effort might be raising and dropping by 1cm four 15g fingers four times each second, requiring energy 6×10^4erg s^{-1}. Playing *fortissimo* seems like more effort than just dropping the fingers, so we'll split the geometric difference between the two estimates and adopt 2×10^6erg s^{-1} $=$ 0.2W as the mechanical power input. Thus the ratio of acoustic power to mechanical power is $\boxed{\sim 0.005\text{W}/0.2\text{W} = 0.02}$.

Given the uncertainty in the decibel level and the mechanical input, we shouldn't expect this to be good to better than a factor of 10. But actual measurements (using a weight to pull the piano keys down) indicate acoustic efficiencies of about 2% in the lower octaves, dropping to about 0.2% in the upper octaves.

2 Waves on Titan

(a) Roughly speaking, a substance boils at the temperature at which the thermal energy kT equals a fixed fraction of the atomic binding energy, E_b.

[1] In an all-concrete environment, it would be 50 times larger = 17dB louder — unpleasant for the piano, but desirable for flautists busking in the tunnels of the Paris metro. No one could hear them in a park, but they are gloriously loud in a subway tunnel.

Surface tension arises from the fact that the molecules on the surface of a liquid have fewer neighbors to bond with than the molecules in the depths of the liquid. From dimensional analysis, the surface tension can be written:

$$\gamma \approx \frac{E_b}{a^2} \tag{8}$$

with a the intermolecular distance, which we take to be the same in all liquids. Putting the argument together, we have:

$$\frac{\gamma_{CH4}}{\gamma_{H2O}} = \frac{Eb_{CH4}}{Eb_{H2O}} = \frac{112K}{373K} \approx \frac{1}{3} \tag{9}$$

(b) The dispersion relation for waves in deep water is:

$$\omega^2 = g_T k + \gamma k^3/\rho \tag{10}$$

Setting the two terms on the right equal gives the transition from ripples to gravity waves.

$$g_T k_{crit} = \gamma k_{crit}^3/\rho \tag{11}$$

$$k_{crit} = (\rho g_T/\gamma)^{1/2} \approx 2\text{cm}^{-1} \tag{12}$$

$$\lambda_{crit} = 2\pi/k_{crit} = 3\text{cm} \tag{13}$$

And the phase velocity at this wavelength is:

$$V_{ph} = \frac{w}{k_{crit}} = \frac{(2g_T k_{crit})^{1/2}}{k_{crit}} \approx 12\text{cm s}^{-1} \tag{14}$$

(c) The dispersion relation and phase velocity for shallow water waves is:

$$\omega^2 = g_T k^2 h \tag{15}$$

$$V_{ph} = \frac{\omega}{k} = \sqrt{gh} \tag{16}$$

Hence, if we made observations of dispersionless waves, we would know that we had reached a depth, h, less than the depth given by plugging in the wind velocity into V_{ph} in equation (18) above.

(d) A resonance occurs when two natural frequencies form a ratio of small integers. The angular frequency of Titan's orbit (the tidal forcing) is given by:

$$\omega_T = \frac{2\pi}{P} = \frac{2\pi}{15\text{day}} \sim 2 \times 10^{-5}\text{s}^{-1} \tag{17}$$

Typically, when we think of water waves (i.e. waves at the beach, ripples) the frequencies we experience are much faster. The only way we could imagine waves with frequencies this low would be to consider very long wavelength waves on Titan. However, the wavelength of waves on Titan is restricted by the size of the lakes (100 m depth, 100 km extent). For deep waves, we have:

$$\omega_d = \sqrt{g_T k} = \sqrt{134\text{cm s}^{-2} \times \frac{2\pi}{10^7\text{cm}}} = 10^{-2}\text{s}^{-1} \tag{18}$$

which is much higher than the tidal frequency. For shallow waves, we have:

$$\omega_s = \sqrt{g_T h k^2} = \sqrt{(134\text{cm s}^{-2})(10^4\text{cm})(\frac{2\pi}{10^7\text{cm}})^2} \sim 7 \times 10^{-4}\text{s}^{-1} \tag{19}$$

which is still faster than the frequency of tidal forcing. One way in which the wave frequency can be lowered is by considering shallower lakes, i.e. lakes only a few meters thick. Where such conditions are met, it's possible to have what might truly be called tidal waves, i.e. waves excited by tidal forces.

3 Water Bug Propulsion and Navigation

A brief digression on the meaning of non-wettable: When a solid meets a liquid interface, there are three surface tensions: solid/air, solid/liquid and liquid/air. Equilibrium of forces at the 3-way interface line (see figure 1a) requires

$$\gamma_{la} \cos\theta_E = \gamma_{sa} - \gamma_{sl} \tag{20}$$

If $\gamma_{sa} > \gamma_{sl} + \gamma_{la}$, then θ_E is zero and the liquid spreads to an atomically thin layer to minimize its surface energy. This is known as "total wetting". If the inequality is the other way round, there is partial wetting. *wettable* means the angle of contact θ_E of the surface of the drop with the surface is nearer 0 than $\pi/2$; *non-wettable means the angle of contact is nearer* π*.* The consequences are shown in figure 1.

(a) To jump on water, the marine water strider *Halobates* (see: http://www.jochemnet.de/fiu/Halobates.jpg) must use its non-wettable legs to push off of the water. By dimensional analysis or the discussion in the caption of figure 1, the surface tension force is given by: $F = \gamma L$ where L is the length of contact with the water. Note that unlike walking on land, the force is *not* proportional to the surface area of contact. We can solve for the

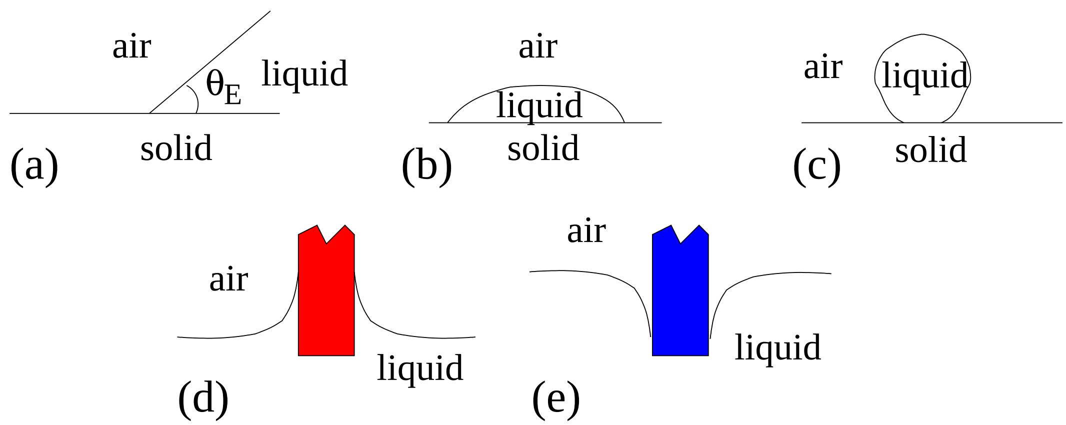

Figure 1: (a) Equilibrium between the three surface tensions of the three interfaces at the contact point sets the angle of contact θ_E. (b) A water drop spreads on a partly wettable surface like plastic or dirty glass, $\theta_E < \pi/2$. (c) A water drop balls up on a non-wettable surface like wax, $\theta_E > \pi/2$. (d) a wettable bug leg (red) is pulled *into* the water (watch bees and flies struggling, usually unsuccessfully, to get out of a pool), while (e) a non-wettable bug leg (blue) is pushed out of the water. In cases (d) and (e) the force is approximately γ_{la} times the circumference of the leg.

length of contact by equating the work done to the energy needed to reach a height of several centimeters:

$$\gamma L d = mgh \tag{21}$$

where d is a few millimeters, i.e. the length of the bug's legs, which is roughly the distance over which the bug can exert force on the water by bending and straightening its "knees" (this is not a biology course, so we regrettably have to skip the jargon). Solving for L, we have:

$$L = \frac{mgh}{d\gamma} = \frac{0.01\text{g} \times 10^3\text{cm s}^{-2} \times 3\text{cm}}{50\text{erg cm}^{-2} \times 0.2} = 3\text{cm} \ . \tag{22}$$

(b) The *Stenus* squirts behind it a fluid which lowers the surface tension dramatically. So we can simplify the problem to one where normal surface tension force on the front drags the *Stenus* forward, while there is no surface tension force on the behind. So the net surface tension force is $F = \gamma L$ where

L is the size of the *Stenus* $\sim$ 5mm. The drag force exerted on the submerged part of the bug by the water is $\sim \frac{1}{2} C_D \rho_{water} v^2 L^2$. An upper limit is $C_D \sim 1$, which gives

$$\gamma L = \frac{1}{2} C_D \rho_w v^2 L^2 \tag{23}$$

$$v = \sqrt{\frac{2\gamma}{C_D \rho_w L}} = \sqrt{\frac{2 \times 50\text{erg cm}^{-2}}{1 \times 1\text{g cm}^{-3} \times 0.5\text{cm}}} = \boxed{14\text{cm s}^{-1}(1/C_D)^{1/2}} \tag{24}$$

If we'd taken $C_D \sim 0.1$ then $v \sim 44\text{cm s}^{-1}$ which is greater than the minimal phase velocity for water waves ($\sim 20\text{cm s}^{-1}$). Wave drag sets in when $v > v_{ph}$ and carries away even more momentum and energy from the poor little bug. So he's probably limited to somewhere between 14 and 20cm s^{-1}.

(c) A molecular thickness (~ 5Å) of this material is enough to decrease the surface tension at the back. The mass of material that must be excreted per unit time is:

$$f = \rho a L v \simeq 1\text{g cm}^{-3} \times (5 \times 10^{-8}) \times 0.5\text{cm} \times (14\text{cm s}^{-1}) = 3 \times 10^{-7}\text{g s}^{-1}\ , \tag{25}$$

where we have assumed that the surface-tension reducing material (like soap) has about the same water-like density as the rest of the insect. The thrust produced by this mechanism of propulsion is: $F = \gamma L = 25$dyn. Now, if the insect instead tried to use the jet propulsion by ejecting the same amount of material with an overpressure of order the atmospheric pressure, the ejection velocity would be of order the sound speed in air, c_s, which is $\sim 3 \times 10^4\text{cm s}^{-1}$. The thrust of such a jet would be:

$$F_{jet} = \dot{m} c_s = 3 \times 10^{-7}\text{g s}^{-1} \times 3 \times 10^4\text{cm s}^{-1} = 9 \times 10^{-3}\text{dyn}\ , \tag{26}$$

which is 0.0004 of the force the bug produces by surface tension reduction. Once again, evolution has crafted some pretty crafty creatures.

(d) The whirligig beetles can't move faster than the phase velocity of the ripples they make, otherwise the waves can't feed them information in time. The abdomen is smooth, so the smallest wavelength it can produce is roughly its size, i.e. $\lambda_{min} \sim 1$cm and $k_{max} = 2\pi/\lambda_{min} \sim 6\text{cm}^{-1}$. So for ripples we have from the disperson relation for capillary waves $\omega^2 = \gamma k^3/\rho$

$$V_{max} = \frac{d\omega}{dk} = \frac{3}{2}\sqrt{\frac{\gamma k_{max}}{\rho}} \sim 25\text{cm s}^{-1} \tag{27}$$

which is comparable to the top speeds that these little creatures have been observed to move. In principle, they could move faster briefly but only if it's worthwhile to do so blindly (i.e., if being chased by a predator).

4 Exciting Waves

(a) *Why is it difficult to walk carrying a bowl of soup without spilling any?* Recall from class that waves in shallow water of depth h have $\omega/k = \nu\lambda = \sqrt{gh}$. A bowl of diameter l and soup depth $h << l$ (shallow wave) has a fundamental sloshing mode of wavelength $\sim 2l$ and frequency $\nu \sim \sqrt{gh}/(2l) \sim$ 1Hz (This formula just happens to be exact for rectangular soup bowls of length l, but is approximate for more usual round ones, for which the factor 2 should be 1.7 if the depth is uniform, or $\pi/2$ for a parabolic bottom) For a typical bowl of soup, $L = 20$ cm, $h = 2$ cm, and the frequency of the resonant sloshing mode is $\boxed{\nu \sim 1}$Hz. Walking at normal speeds provides footfall impulses at about 1 Hz which resonantly drive the sloshing mode.

(b) *Estimate the energy per unit area stored in ocean waves. Express your answer in terms of seconds of solar insolation.* Averaged over a wavelength, a wave with a crest height ξ is created by lifting a mass per unit area $\sim \rho\xi$ from the trough and putting it on the crest, increasing the potential energy per unit mass by $\sim g\xi$. Thus, the energy density is $\mathcal{E} \sim \rho g \xi^2 \sim 10^7$ erg cm^{-2} for a wave of crest height $\xi \sim 1$ m. The sun overhead radiates a flux 1.4×10^6erg cm^{-2}s^{-1}, so the waves have about $\boxed{\text{7 s of insolation.}}$

5 Ship Wakes

(a) *Prove that surface gravity waves generated by a ship in steady motion are confined to a wedge with opening angle $2\theta = 2\sin^{-1}(1/3)$.*

As the ship moves along in deep water at velocity $\mathbf{v}$, its hull and churning propellers create waves with a wide range of directions and wavelengths. In a stationary "beachfront" reference frame with coordinates $(\mathbf{x}, t)$ the waves produced by the ship can be described as a superposition of plane waves of the form

$$e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \quad . \tag{28}$$

We won't concern ourselves with the little capillary (surface-tension) ripples of wavelength less than 2 cm, but just with the big buoyancy waves, whose dispersion relation, derived in class, is $\omega^2 = gk$. Such waves have a phase velocity $v_p = \omega/k = \sqrt{g/k}$ and a group velocity $v_g = d\omega/dk = v_p/2$.

To solve the problem, we need just two key ideas: 1) in the rest frame of the steadily moving ship, the wake must be time-independent, and 2) the packets of large-amplitude waves move outwards at the *group* velocity, not the phase velocity. The crests of the individual plane waves contributing to the packet move through it from inside to outside, at twice the speed of the packet itself (see figure 2).

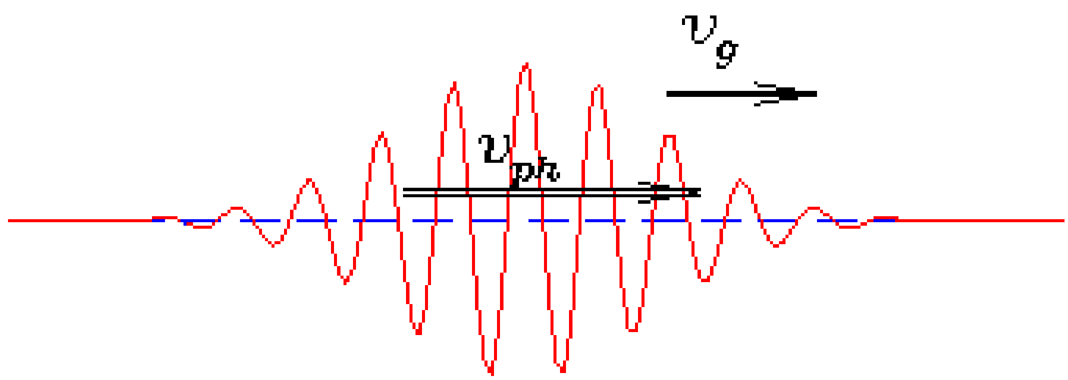

Figure 2: Wave packet of deep water waves. The pattern of large-amplitude waves moves at the group velocity v_g. Individual plane wavecrests move *through* the packet at the phase velocity v_p, which is twice the group velocity.

To implement idea (1), we transform equation 28 to coordinates $(\mathbf{x}', t)$ in which the boat is at rest. Since $\mathbf{x}' = \mathbf{x} - \mathbf{v}t$ we get

$$e^{i((\omega - \mathbf{k}\cdot\mathbf{v})t - \mathbf{k}\cdot\mathbf{x}')} \ . \tag{29}$$

From this, we see that the only waves that can contribute to a pattern which is time-independent in the ship's frame have $\omega - \mathbf{k}\cdot\mathbf{v} = 0$, i.e. $\omega = kv\cos\theta$, where θ is the angle between the direction of the wave-vector and the ship's direction of motion. Notice that such waves have $v_p = \omega/k = v\cos\theta$ and, since from the dispersion relation $\omega = \sqrt{gk}$, the waves propagating in direction θ that contribute to the steady-state wake have wavelength

$$\lambda = 2\pi v^2 \cos^2\theta / g \ . \tag{30}$$

Figure 3 shows a ship which at $t = 0$ was at point P: $x = 0, y = 0$, and at some later time t is at point S: $x = vt, y = 0$. The waves it generated at $t = 0$ move out from point P at the group velocity. Recalling that the only waves contributing to the steady-state wake have $v_p = v\cos\theta$ and thus $v_g = (1/2)v\cos\theta$ in direction θ (magenta line in figure 3), we see that at time t the position of the large-amplitude group of waves that were launched at $x = 0, t = 0$ is

$$x = v_g t\cos\theta = \frac{1}{2}vt\cos^2\theta = vt(\frac{1}{4} + \frac{1}{4}\cos 2\theta) \qquad (31)$$

$$y = v_g t\sin\theta = \frac{1}{2}vt\cos\theta\sin\theta = vt\frac{1}{4}\sin 2\theta \qquad (32)$$

As θ varies from 0 to 2π, this describes a circle (shown in blue in figure 3) of radius $vt/4$, centered at $x = vt/4$, i.e., a distance $(3/4)vt$ behind the current ship position. The envelope of these circles radiated at all times before the present is a cone of half-angle $\alpha = \sin^{-1}(1/3) = 19.5°$ as seen from the ship.

The region inside the envelope of circles is known as the "Kelvin wedge" after William Thompson, Lord Kelvin (same guy both the cross-section and temperature scale are named after. He discovered the cross-section before he was given the title). If you tilt the magenta line of figure 3 until it touches the tangent point T of the group-velocity circle, you will observe that the waves contributing to the wake are *not* moving perpendicular to it, but at $\theta_T = (90 - \alpha)/2 = 35°$. Their wavelength is $\lambda = 2\pi v^2\cos^2\theta_T/g = (2/3)2\pi v^2/g$, i.e. is (2/3) of the wavelength, given in part (b), of the forward propagating waves in the middle of the wake.

Note that the result depends on the deep-water dispersion relation. If the ship is propagating in a channel whose depth is not large compared to the wavelength, the wake is wider —see figure 4.

(b)*How does the wavelength of forward propagating waves depend upon the speed of the ship?*

Waves propagating forward with the ship ($\theta = 0$) have, from equation 30, wavelength $2\pi/k = 2\pi v^2/g$. Or simply note that their phase velocity must be equal to the speed of the ship, so from the dispersion relation $k = g/v^2$.

Ship speeds are usually given in knots, which are nautical miles per hour. A nautical mile is the distance which subtends a minute of arc of longitude on the earth, i.e. $R_\oplus(\pi/180)/60 = 1.8$km. So 10knots = 5m s^{-1}, and $\lambda = 17(v/10\text{knots})^2$m. When $1/k$ is of order the length l of the ship, the ship will be tilted up by its own bow wave (as familiar from "cigarette boats" in power racing). With a tilted hull, the engine has both to keep the ship from "falling back" into the wave, and overcome an increasing ram pressure drag,

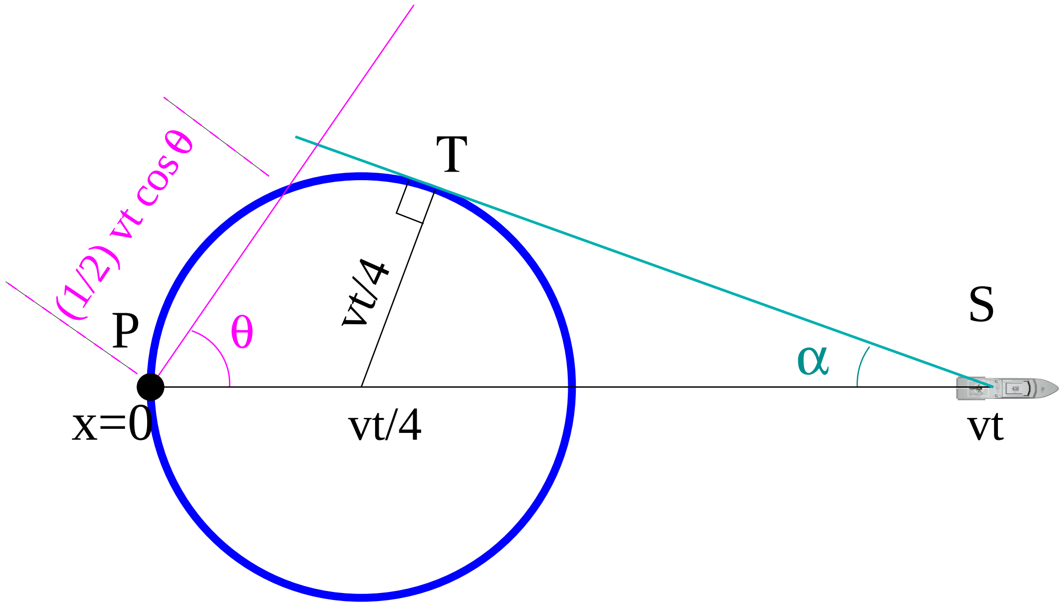

Figure 3: A ship was at point P ($x = 0$) at time $t = 0$, and has moved to the right to point S ($x = vt$) at time t. The waves contributing to its steady-state wake have phase velocity $v_p = v \cos\theta$ and group velocity $v_g = v_p/2 = (1/2)v\cos\theta$ (magenta). At time t, the locus of the large-amplitude waves, whose pattern moves out at the group velocity, is a circle (blue) of radius $vt/4$ centered at $x = vt/4$. The envelope of such circles for all t subtends angle α with $\alpha = \sin^{-1}[(vt/4)/(3vt/4)] = \sin^{-1}(1/3) = 19.5°$.

so the engine power required to maintain speed rises rapidly as the Froude number $Fr = v^2/lg$ approaches unity.

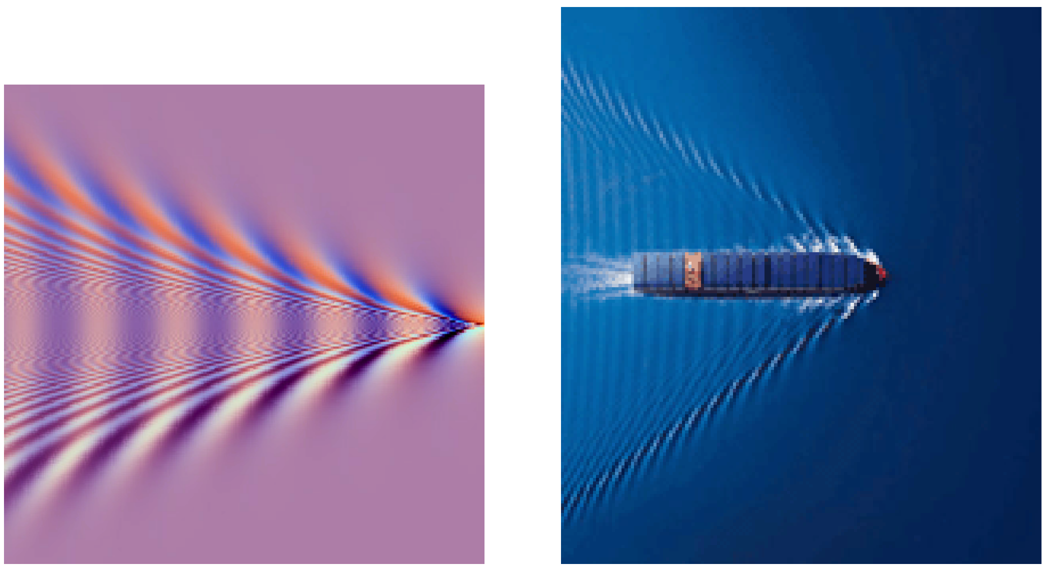

Figure 4: **Left**: Deep water wake (mathematical model from Sir Michael Berry). **Right** Photo of a container ship moving at $Fr \ll 1$ in *shallow* water.