Ph101: Solution 4 Version 2

Sterl Phinney

May 20, 2009

1 Cost of Raiding the Fridge

When you open a refrigerator door, the cold dense air inside falls out onto the floor, and warm room air flows in to replace it. Then the fridge has to cool that warm air. Adopt a hulking refrigerator of interior height $L = 1.6$m and cross-sectional area $A = 0.5\text{m}^2$ (this is ~ 24 cubic feet, about the largest sold). We didn't specify whether we were talking about the freezer or fridge compartments, so let's adopt a mean temperature of 0C = 273K for the fridge air, and 24 C for the room air.

The cold air is in pressure equilibrium with the warm air, so is denser by $\Delta\rho/\rho = -\Delta T/T = -24/297 = -0.08$. The cold air initially accelerates according to $D\mathbf{v}/Dt = -\mathbf{g}\Delta\rho$, and reaches a terminal velocity when the ram pressure force on the blob of cold air balances the (anti)buoyancy force $ALg\Delta\rho = (1/2)\rho v^2 A$, or $v \sim \sqrt{g\Delta\rho L/\rho}$. This is exactly the same as the speed attained in falling a distance L calculated from the initial acceleration equation (with no ram pressure drag). Thus the time it takes the cold air to fall out of the fridge is

$$t_{\text{fall}} = \frac{L}{v} = \sqrt{\frac{L}{2g\Delta\rho/\rho}} = 1\text{s} \ . \tag{1}$$

So unless you are incredibly fast, the cold air will fall out, and the refrigerator will have to cool down the warm air you let in.

The specific heat capacity of air at constant pressure is $c_p = \frac{7}{2}k_B/m_{N_2}$. The difference between the temperatures of the room (297K) and the desired air temperature inside the refrigerator (273K) is ΔT24K. so the energy the fridge has to remove from the warm air is

$$\Delta E = c_p\rho_{air}LA\Delta T \simeq 10^7(10^{-3})(0.8\times10^6)(24) = 2\times10^{11}\text{erg} = 2\times10^4\text{J} \ . \tag{2}$$

How much electrical energy do we have to supply the refrigerator compressor to remove this energy? To change the interior energy by 1 erg, fortunately the refrigerator does not need to use 1 erg of electricity —it takes less than an erg of pdV work to compressionally

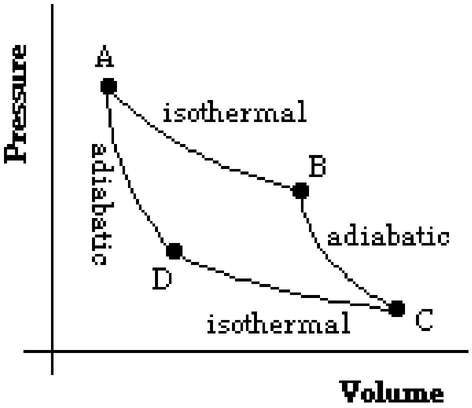

Figure 1: Carnot cycle of a refrigerator's working fluid. It is heated by adiabatic compression on DA. On the top isothermal curve AB the working fluid exchanges heat with the outside air. It is then cooled by adiabatic compression on BC, and on the bottom isothermal curve CD the cooled fluid absorbs heat from the air inside the refrigerator.

heat a working fluid from below the fridge temperature to more than the room temperature, and then heat transfer will magically do the rest. The best the fridge can do is have the Carnot coefficient of performance (COP_C), which implies that

$$\mathrm{COP}_C = \frac{\Delta E \text{removed from fridge}}{\text{pdV work done}} < \frac{T_c}{T_h - T_c} = \frac{273}{297-273} \sim 11 \tag{3}$$

In this cycle recall that a perfectly efficient and adiabatic compressor heats the fridge's working fluid (starting at just below the fridge temperature) to just a bit above room temperature and sends it frictionlessly and infinitely slowly through cooling fins where it infinitely slowly loses the heat to the room air, before being adiabatically expanded back to just below fridge temperature and sent through the fridge to accept heat from the interior of the fridge. And round and round again (see figure 1). In reality, to get the refrigerator to cool on a finite timescale, the working fluid in the cooling fins is about 10 C hotter than the air, and in the fridge about 10 C cooler than the desired fridge temperature, so a practical COP_C is ~ 6 rather than 11. Since the motors aren't perfectly efficient and the insulation not perfect, we guesstimate

$$\frac{\Delta E \text{ removed from fridge}}{\text{Electrical work}} \sim \frac{1}{2}(\text{practical } \mathrm{COP}_C) = \frac{1}{2}\left(\frac{273-10}{297+10-(273-10)}\right) = 3\ . \tag{4}$$

Thus we need an electrical energy of only $\sim \Delta E/3 \sim 0.6 \times 10^4\text{J} \sim 0.002\text{kWh}$. Electricity typically costs 10 cents/kWh, so the cost of opening the refrigerator is about $\boxed{0.02 \text{ cents}}$.

Dad was exaggerating the cost by a factor of 1000! Dads do that sometimes. Always with your best interests at heart. You know: "If you don't eat your broccoli, you'll shrivel up and turn into a toad." *Harmless little white lies.*

We can check our estimate: Current (2009) Energy-Star rated top-mount freezer fridges in 24 cubic foot sizes use about 400kWh per year, or about 1kWh per day -i.e. 10 cents of electricity per day. If the fridge gets opened 10 times for each meal, plus another 10 times per day for snacks, that is 40 openings per day, which by our estimate adds up to about a penny per day, or 10% of the cost of operation, so our estimate is certainly good to an order of magnitude. Someone with a Kill-A-Watt meter could check this by comparing power use while on vacation versus while using the fridge.

2 Wind Power

(a) We can use Buckingham's Pi theorem. Taking the relevant variables to be the power P, the wind speed v, the air density ρ and propeller diameter D (or radius R), and neglecting the viscosity ν as a variable here (the Reynolds number for the turbine of part (b) in a 5m/s wind is $Re = Dv/\nu = 2.5 \times 10^7$). Then the only dimensionless variable is $P/\rho D^2 v^3$ or some function thereof, giving $P \sim \rho D^2 v^3$.

Alternatively, we note that the kinetic energy of wind passing through the circle cut by spinning blades of radius R per second is $\boxed{\frac{1}{2}\rho\pi R^2 v^3.}$ We guess that efficient rotors can extract a fraction of order unity of this incident kinetic energy. Indeed modern rotors extract up to about 0.45 of this value (Betz's law gives the maximum possible extractable fraction of power as 0.59, so they are doing pretty well!).

(b) $R = 50\text{m}$, $v = 5$ m/s, $\rho = 1\text{kg/m}^3$, so power $P \sim 0.45\frac{1}{2}\rho v^3 \pi R^2 \sim 2.5 \times 10^5\text{W}$ or 250kW.[1] 1 year is 365×24 hours, so the energy produced is about $2 \times 10^6\text{kWh}$. Your instructor's electricity bill lists "DWR/SCE generation charges" as 0.06 per kWh (the bill is actually about double that, but the rest is claimed to be for "distribution" and "customer service", and the producer of electricity presumably just gets paid for the "generation charges"). So the wind generator could get paid \$0.06 per kWh, or $\boxed{\$\ 10^5 \text{ per year}}$ if the wind speed were steady at 5m/s for the year.

(c) Note that $P \propto v^3$. For 15% of the time the speed is twice as high, resulting in power output that is eight times as large. Similarly, 1% of the time the speed is four times as high,

---

[1] In the wind-power industry, turbines are specified by their "rated power", which is usually the electric generator output in a wind speed of 13m s^{-1} (29 mph). So this turbine would have a "rated power" of 4MW times the gearbox and generator efficiencies. The turbines are generally designed to shut down if the wind speed exceeds 25m s^{-1}.

and the power is sixty-four times as large. Thus the realistic total annual energy output should be

$$0.84 + 0.15 \times 8 + 0.01 \times 64 = 2.68 \tag{5}$$

times the value estimated in part (b).

The calm 'normal' 5m/s days account for only about $0.84/2.68 = 31\%$ of the power produced. The rest is produced on stormy days, so the actual energy produced is about 5×10^6kWh, with value \$ 3×10^6, *if* the stormy-day power could actually be sold. This isn't a problem if wind generation is only a tiny proportion of the total power going into the grid. But if wind turbines become a large fraction, storage for later resale of that majority of the power produced on only a few stormy periods becomes vital (the demand, and hence price of power is a lot higher on hot sunny afternoons than on cold stormy nights).

(d) The length of the eastern seabord of the USA is approximately $L = 3000$km. The total mean US electricity consumption is approximately 0.45TW (peak summer demand is 0.9TW), or $1.5 \times 10^{19} J$ each year. The equation of continuity means that the streamlines of the wind must expand behind the turbines where the wind is slowed. To avoid complicated interactions, and allow for the wind direction to vary without turbines shadowing each other, we should probably space the turbines by at least twice their diameter (industry standard is 3 times the diameter).

If we neglect the serious problem of matching capacity to demand (i.e. assume we solve the storage problem somehow), the total mean power produced is the product of the number of turbines we can fit along the seabord $L/(3 \times 2R)$, and the total mean power produced per turbine 2.68×0.25MW = 0.67MW.

$$E_{tot} = \frac{L}{3 \times 2R} E_{tur} = \frac{3 \times 10^6\text{m}}{300\text{m}} \times 2.68 \times 0.25\text{MW} = 7 \times 10^3\text{MW} \tag{6}$$

where we have used the improved estimate of the mean energy produced per turbine from part (c). We see that the fraction of the total US energy consumption we could generate using this single line of turbines is $\boxed{\sim 1.4\%}$.

(e) The largest turbulent eddies in the atmosphere have scale equal to the atmospheric scale height $H \sim 10$km (you can see these in the shapes of cumulus clouds). The turbulent cascade carries the energy in these eddies (driven directly by solar heating and indirectly by condensation of solar-evaporated water) to smaller scale eddies.

A single rotor of diameter $D = 2R$ will extract the wind energy from eddies in a region of size $D \ll H$. So there is no point in putting another turbine a distance $\ll D$ behind it, since it would then be in the slowed flow. But if we put a second turbine a distance of more than a few times D behind the first, it will be in independent eddies of scale D, and since the first rotor extracted only a small fraction of the energy of the atmosphere-scale wind and the large H-scale eddies, these will be nearly as fast as those the first rotor feels. Experiments show that about 50% of the power is lost if the turbines are separated by $4D$, but only 15% is lost if they are separated by $8D$ [2] We can repeat this process until the ranks of turbines cover a linear extent $\sim H$, extracting most of the energy from the largest H-scale eddy and the mean flow. The number of ranks is thus $\sim H/(5D) \sim 20$.

But remember that solar and condensation heating replenish the largest eddies on their turnover time. As we discussed in class ~ 0.3 of the mean solar energy flux $F_{\odot}/4$ contributes to weather, so the input is $F_w \sim 0.3F_{\odot}/4 \sim 10^5 \text{erg cm}^{-2}\text{s}^{-1}$, and the characteristic wind speeds were determined by $F_w = \rho v^3 = H\rho v^2(v/H)$. Thus the solar flux replenishes the 'typical' winds on their characteristic eddy scale H (high speed storm winds build up over longer scales). So in fact we could continue our ranks of turbines at spacing $\sim 5D$ indefinitely, though with the side effect of slowing storm winds to breezes.

In summary: we can cover any area A with $\sim A/(3D \times 5D)$ turbines, each extracting a power $\sim 0.2\rho v^3 D^2$, i.e. a $\boxed{\text{power per unit area of wind farm of} \sim 0.01\rho v^3}$, independent of D. Since $\rho v^3 \sim F_w$, we see that the wind farm extracts about 0.003 of the mean solar flux on its area. This requires installing turbines on much more land than we'd need if we tiled the same area with solar cells (efficiency about 0.1), but at current prices somewhat cheaper for given power output. And it has the enormous advantage that most of the land or water area is unshaded and free for other uses like farming or fishing. Tiling with the $D = 100$m turbine of parts (a) and (b) yields $\boxed{4\text{MW km}^{-2}}$, so to produce the 0.45TW US electrical power we'd need a farm covering at least 10^5km^2, and more realistically probably 10 times more to meet peak demand at times of unfavorable wind speeds. Texas or a few midwestern states would do nicely, as T. Boone Pickens has noticed. The total cost, at current prices of turbines and transmission lines: a few billion dollars per GW, is 'only' \$ 10^{12}, an Iraq war or recent bank bailouts and seems a much more productive long-term contribution to the economy. But we're just physicists, not politicians.

3 Freeway Physics

(a) Let v be the average velocity of the cars, d be the distance between the front of a car the rear of the car ahead, and $L \sim 5$m the length of a typical car. The flux of cars in N lanes will then be $f = Nv/(d + L)$. Driving schools teach the "two second rule" (that is, leave t_2= two seconds of space between your car and the next). If everyone did this, $d = vt_2$ and therefore,

$$f = Nv/(vt_2 + L) \ . \tag{7}$$

For $v \gg L/t_2 \sim 2.5\text{m s}^{-1} = 9\text{km h}^{-1} = 6\text{mph}$, this gives a velocity-independent flux

---

[2] Masters, Gilbert *Renewable and Efficient Electric Power Systems*, Wiley-IEEE Press, 2004, p. 352 Fig 6.28

of $0.5N$cars/s or $2000N$ cars/hour. So $\boxed{f = 1.5\text{cars/s or } 6000 \text{ cars/hour}}$ is a good estimate for the maximum unidirectional flux on a 3-lane freeway. Notice the important feature that because of the L term, the flux is very low at low (parking lot) speeds, and increases slowly with v even at freeway speeds. This means that a small perturbation ("spectator slowing", braking for merging traffic, etc.) when a freeway is carrying near-maximum flux at 65mph can lead to the incoming flux exceeding the flux that can be carried at *any* lower speed —i.e., a growing region of stopped traffic must appear to absorb the difference in flux. This, of course is known as a traffic jam. [3]

For a more complete and realistic answer, it is interesting to consider the origin and limitations of the two-second rule. When drivers see a car slowing ahead of them, they respond after a reaction time τ by decelerating their own car with acceleration a. [4] If everyone decelerated at exactly the same rate after the same reaction time, then the safe following distance would be $v\tau$. However, if there is a distribution in the ability to decelerate, or a dead stop is required, the safe stopping distance becomes quadratic in v: A car moving at speed v can be brought to a stop after a distance

$$d_s = v\tau + v^2/(2a) \ . \tag{8}$$

If everyone kept $d = d_s$, the maximum flux per lane $v/(d_s + L)$ would initially rise with v to a maximum of $v_m/(v_m\tau + 2L) = 0.6\text{s}^{-1} = 2000\text{cars h}^{-1}$ at $v = v_m = \sqrt{2La} = 10\text{m s}^{-1} = 35\text{km/h} = 22\text{mph}$, and then drop as $1/v$ at higher speeds (at $65\text{mph} = 30\text{m s}^{-1}$, $d_s = 100\text{m}$ and the flux per lane would be 0.3s^{-1}, just half the value at $v_m = 22\text{mph}$).

The two second rule is a compromise: it gives a margin of safety over the reaction-time ("0.75 second rule") to allow for some differences in deceleration and inattention. But since at freeway speeds it gives $d < d_s$, it does not allow for drivers to stop in the event of a crash or obstacle in front. [5]

(b) A typical tire goes bald in $L = 5 \times 10^4$ miles. During this time, the four tires each of width w, radius R and tread thickness t lose a volume of rubber $4V = 4 \times 2\pi Rwt$ $4V \sim 4 \times 2\pi \times 30\text{cm} \times 10\text{cm} \times 1\text{cm} \sim 7 \times 10^3\text{cm}^3$. The density of tire rubber is approximately

---

[3] The late Prof. Gerald B. Whitham of Caltech carried out in the 1950-1970s pioneering studies of the equations governing the 'hydrodynamics' of traffic flow which were influential in the design of roadways and traffic controls worldwide.

[4] The California DMV publishes a chart of safe stopping distances which is quadratic in v. Fitting it gives $\tau = 0.75$s and $a = 0.6g$, about what we'd expect from the coefficient of friction for rubber on dry pavement. We use these numbers for the numerical values in the problem.

[5] Your instructor was once driving at 60mph in the center lane of the 101 freeway, when the car in front suddenly swerved into the left lane, revealing a large sofa sitting squarely across his lane. Your instructor, following the 2 second rule, was luckily also able to swerve, but the driver behind him was blocked by cars in neighboring lanes, plowed into the sofa, and was in turn run into by several cars behind her, in a classic pileup.

1.5 g cm^{-3}, giving a mass loss of 0.2g/mile. Given a day-night average flux of cars in 6 lanes of $0.25 \times 6 \times 2000 = 3000$ cars per hour, about $\boxed{\text{50tons}}$ of rubber is shed on the 10mile Pasadena stretch of 210 freeway each year!

(c) From (b), a single tire must on average deposit a layer of thickness $V/Lt \sim 2 \times 10^{-8}$cm $= 2$Å, i.e. about $\boxed{\text{one atom thick}}$. The variance is likely large: much thicker layers are left during heavy acceleration, braking or cornering ("laying rubber") and on the edges of potholes.

4 House Lights

(a) The current I running through the light bulb is roughly $I = P/V = 100\text{W}/120\text{V} = 0.8$ A.[6]

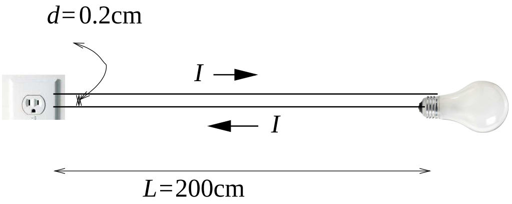

Figure 2: Geometry of the House Lights problem.

The power dissipated in the wires is related to the current I running through the wires, the total length of the wires $L = 2 \times 200 = 400$cm and the radius of the wires r. The current I is related to current density j by $I = j(\pi r^2)$ and by Ohm's law $j = \sigma E$ to the electric field driving the current. The power dissipated per unit volume in the wires (of conductivity σ and resistivity $\rho = 1/\sigma$; the Purcell sheet gives the resistivity of copper $\rho = 2\times10^{-6}$Ohm-cm) is $j \cdot E = \sigma E^2 = j^2/\sigma = j^2\rho$, so the total power dissipated in the two wires is

$$\begin{aligned} P &= j^2\rho(\pi r^2 L) = \left(\frac{I}{\pi r^2}\right)^2 \rho\left(\pi r^2 L\right) = I^2\frac{\rho L}{\pi r^2} \equiv I^2 R && (9)\\ &= \frac{(0.8\text{Amp})^2 2\times10^{-6}\text{Ohm-cm}\times 2\times 200\text{cm}}{\pi(0.1\text{cm})^2} \simeq 0.02\text{Amp}^2\text{Ohm} && (10)\\ &\simeq \boxed{0.02\text{Watt}} && (11)\end{aligned}$$

---

[6] This assumes that the power dissipated in the wires is negligible compared with the 100 W power dissipated in the bulb, which we will see is correct. If it weren't there would be an economic incentive to fix the problem!

This is much less than the power dissipated in the bulb, justifying our initial assumption. The total resistance R of the 4 m length of wire is 0.02 Ohm.

(b) The current I is related to the electron drift velocity v by $I/e = \pi r^2 n_e v$ The current of $I = 0.8$Amp flowing in the wire is 0.8 Coulomb per second, or $I/e = 0.8/1.6 \times 10^{-19} = 5 \times 10^{18}$ electrons per second. The mean density of electrons is about $n_e = 2n_s$, where n_s is the number density of copper atoms. Copper has $\rho \sim 9\text{g cm}^{-3}$ and $A = 65$, so $n_e \simeq 2\rho/(Am_p) = 2 \times 10^{23}\text{cm}^{-3}$. This gives a drift velocity

$$v = \frac{(I/e)}{\pi r^2 n_e} = \frac{5 \times 10^{18}\text{s}^{-1}}{\pi(0.1\text{cm})^2(2 \times 10^{23}\text{cm}^{-3})} \boxed{\simeq 10^{-3}\text{cm s}^{-1}}\,. \tag{12}$$

(c) To find the magnetic field in the two wires, we superpose the (almost exactly opposite) fields from each separate *single* wire. To find the single wire fields at $R = 100$cm, we use Ampère's law, $\oint B = 4\pi I/c$, so $B = 2I/(cR)$ in Gaussian units. Since we have fields from two wires with oppositely directed currents separated by $d = 0.2$cm, the net field is smaller by d/R.

$$\begin{aligned} B = (2I/c)(R^{-1} - (R+d)^{-1}) &\simeq (2I/cR)(d/R) && (13)\\ &= \frac{2 \times 0.8\text{Amp} \times 3 \times 10^9\text{esu s}^{-1}}{3 \times 10^{10}\text{cm s}^{-1} \times 100\text{cm}} \left(\frac{0.2\text{cm}}{100\text{cm}}\right) && (14)\\ &= \boxed{3 \times 10^{-6}\text{G} = 3 \times 10^{-10}\text{T}}\,. && (15)\end{aligned}$$

(d) In AC, the current reverses direction at $\nu = 60$Hz. Thus the electrons do not drift steadily in direction, but the drift oscillates back and forth, covering a distance of only $v/(2\pi\nu) \simeq 3 \times 10^{-6}\text{cm} = 300$Å.

One might worry about the skin depth, but at 60 Hz, this is 0.8 cm, much larger than the 0.1 cm radius of the wires, so the current density is uniform across the wires, and skin effects can be neglected.

5 City Heat Islands

(a) If there are 10^7 people generating $10\text{kW} = 10^{11}\text{erg s}^{-1}$ each in an area of $20\text{km} \times 20\text{km} = 4 \times 10^{12}\text{cm}^2$, then the extra heat flux is

$$\Delta F = \frac{(10^7) \times (10^{11}\text{erg s}^{-1})}{4 \times 10^{12}\text{cm}^2} = 2.5 \times 10^5\text{erg cm}^{-2}\text{s}^{-1};. \tag{16}$$

This is $\boxed{\text{equal to the mean solar flux on earth,}}$ so this is like having two suns in the sky over the city!

If there is no conduction or convection, the city must heat until the extra flux can be *radiated*. The Stefan-Boltzmann law $F = \sigma T^4$ gives the difference ΔT between the city and the surroundings needed to radiate the extra flux

$$\Delta F = 4\sigma T^3 \Delta T \tag{17}$$

Assuming a temperature of about 290K for the surrounding countryside, we find,

$$\Delta T = \frac{2.5 \times 10^5}{4(6 \times 10^{-5}(290)^3} \mathrm{K} \approx \boxed{40\mathrm{K.}} \tag{18}$$

Cities would be unbearable if this were the case!

(b) Convection will carry the heat up, and advection by the wind will carry it off to the countryside. If the heat is mixed up to an altitude h, the city has width L and the wind velocity is v, then the mass of warmed air (density ρ, heat capacity $c_p = \frac{7}{2} k_B / m_{\mathrm{N}_2} = 10^7 \mathrm{erg\ g^{-1} K^{-1}}$ at constant pressure) per unit time carried away from the city is $\rho L h v$, and the power it carries off is

$$P = c_p(\rho L h v) \Delta T \ . \tag{19}$$

We saw in class that the solar flux induces convection over the whole atmospheric scale height (troposphere) at speed given by $\rho v^3 = F$, $v \sim 6\mathrm{m\ s^{-1}}$. Since the city heat island produces an additional flux about equal to the mean solar flux, we might expect a slightly ($2^{1/3}$) higher buoyant convection speed, about $v \sim 7.5\mathrm{m\ s^{-1}}$. This is also a reasonable value for the horizontal wind speed (at altitude). Since the city is 20 km across, it takes this wind about 45 minutes to drive the air across the city, in which time the convection will have carried it on a total path also of about 20 km -i.e. up and down the whole atmospheric scale height. This it is reasonable to assume that the heat is well mixed over h equal to the full atmospheric scale height, $h \sim 10\mathrm{km}$. The air temperature increase over the city for the winds to carry off the city's extra power $P = \Delta F L^2$ is then given by equation 19,

$$\Delta T = \frac{\Delta F L^2}{c_p \rho L h v} = \frac{\Delta F}{c_p \rho v} \frac{L}{h} = \frac{2.5 \times 10^5}{(10^7)(10^{-3})(750)} \left(\frac{20\mathrm{km}}{10\mathrm{km}} \right) = \boxed{0.1\mathrm{K}} \ . \tag{20}$$

Convection is essential to making cities liveable! The real problem is those hot city nights. At night, the ground, and the air near it, cools more rapidly by radiation than the air at high altitudes, creating a $\boxed{\text{temperature inversion.}}$ This shuts down buoyant convection to the high-altitude air. If we recalculate equation 20 with $h = 0.3\mathrm{km}$, then $\Delta T = 3\mathrm{K}$. Actual measurements in cities reveal daytime temperature differences of only a few tenths Kelvin compared to the surrounding countryside, but night-time temperatures several Kelvin hotter.

Of course there are other contributors to the heat island effect than energy use: cities often have less vegetation (evaporative cooling), more black asphalt roads and roofs (lower albedo, absorbs heat during the day) than their surroundings, but since the anthropogenic flux is comparable to the mean solar flux, they probably don't dominate.