Ph101: Solution3 Version 2

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1 Natural Frequency of Millikan Library

Millikan has one end (ground) fixed, and one end free. The natural frequency of oscillation depends on what fraction of the building's mass is in vertical structural supports which resist bending in the given direction. For example, for oscillations in the E-W direction, the N-S walls, as well as the floors, ceilings, books and furniture are simply inertia, contributing nothing to the restoring force. If the walls were structural (e.g. solid steel plates), the natural frequency for oscillation in the x-direction would be

$$\omega^2 \sim 2\frac{Bl_x^3 l_y}{l_z^3 m}$$

where m is the mass of the library, l_y is the wall thickness, l_x is the extent of the wall along the oscillation direction, and the factor of 2 arises because there are two walls. Millikan has $l_{N-S} \approx$ 10m, $l_{E-W} \approx$15m and $l_z \approx$50m. If the walls were solid steel, ($\rho_w \approx 7\mathrm{gcm}^{-3}, \mathrm{B} \approx 10^{12}$) and as thick as the external pillars, $l_y \approx$ 50cm, then if the mass of the floors were comparable to that in a pair of walls, and the mass of books were again comparable, then we would have $m \approx 6\rho_w l_x l_y l_z$, so

$$\omega^2 \approx \frac{Bl_x^2}{l_z^4 3\rho_w}$$

and thus the natural period of oscillation would be

$$P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3\rho_w}{B}\frac{l_z}{l_x}} l_z,$$

which is about 0.7s for the N-S direction and about 0.5s for the E-W direction.

In fact, not all of the walls is structural (plaster, glass windows, granite facing, etc.); the structural members are braced pillars (visible outside the first floor), so the resonant periods are actually a bit longer. Notice that to the extent that all tall buildings have similar shapes (l_x/l_z), $P \propto l_z$, we have roughly derived the civil engineer's rule of thumb: oscillation period equals one second per 10 stories.

2 Cooking Vessels

The issue here is thermal expansion, combined with thermal diffusion. Consider Sterl's Pyrex pie pan, $\ell \sim 0.5$cm thick, a typical insulator with thermal diffusivity $\kappa \sim 3 \times 10^{-3}\text{cm}^2\text{s}^{-1}$. It will take a time $\sim \ell^2/\kappa \sim 10^2$s for heat to diffuse throughout the glass. If the temperature were raised/lowered very slowly, say a few degrees per minute, and the oven interior had no temperature gradients (not mine!), the glass would not break because the whole thing would expand/contract uniformly, and no internal stresses would develop (except small ones due to inhomogeneity in the glass). But when you put the dish in a preheated oven, the outer parts of the glass heat to oven temperature[1] before the heat has diffused into the middle. This means that the surface of the glass will expand relative to the inner parts, developing large horizontal stresses. If the dish survives this, and you then take the baked goods out of the oven, the surface will begin to cool and shrink, compressing the hot interior and itself being pulled into tension. If the resulting strains exceed the yield strains, the glass will break. Temperature gradients within the oven can also produce more persistent strains in the glass.

The coefficient of thermal expansion α relates the fractional change in length (strain) of any part of a substance to the change in temperature: $\boxed{\frac{\Delta L}{L} \equiv \epsilon = \alpha \Delta T}$. As shown in class this is due to the anharmonic part of the inter-atom potential, which generally is steeper at small separations and shallower at large separations. This causes the mean separation of atoms to increase as the thermal energy rises: $\alpha \Delta T \sim k\Delta T/E_b$. Using Boltzmann's constant $k = 9 \times 10^{-5}\text{eV K}^{-1}$, $E_b \sim 10$eV, we found in class

[1] Actually this gets a bit subtle: one needs to check that the radiative and conductive heat fluxes from the oven, the oven's heat capacity and the heating element output are in combination actually able to heat the dish in the 100 seconds diffusion time, which turns out to be just marginally true for realistic ovens.

$\alpha \sim 10^{-5}\mathrm{K}^{-1}$, typical of ordinary class. Pyrex and Vycor cleverly shrink their structure to partly cancel this, leading to their lower α's. Most materials have $\boxed{\text{breaking strains } \epsilon_{cr} \sim 10^{-3} - 10^{-2}}$ (cf. class discussion of Griffith cracks). Brittle materials like glass tend to be at the lower end of this range, say $\epsilon_{cr} \sim 10^{-3}$.

Heating from 20°C to 220°C (425°F baking temperature) produces $\Delta T \sim$ 200°C. So for ordinary glass $\boxed{\epsilon = \alpha \Delta T \sim 2 \times 10^{-3} = 2\epsilon_{cr}}$. So regular glass would likely break in the oven. For Pyrex, where $\alpha_{\rm pyrex} \simeq 0.3\alpha_{glass}$, the same temperature change would produce $\epsilon = 0.6\epsilon_{cr}$, so it will (just) survive, as long as you don't put the oven dish too close to the broiler flame in a gas oven, as a former TA learned the hard way. Vycor would be very safe, but very expensive.

A cheap way to observe this process is to put ice cubes into hot water, and watch and listen to them crack. More expensively, you can crack cast iron frying pans by putting them into a sinkful of cold water immediately after use. The British tradition of warming a fine china teapot by swirling warm water in it before pouring in the boiling water probably has its origin in sad experience, especially in the days before central heating.

3 Geothermal Sources

A couple of fun points before we start: this temperature gradient is a big deal for miners and oil well drillers (deep mines are unpleasantly hot -the Homestake Mine in South Dakota was 2.4km; gold mines in South Africa are $\sim$ 4km, and the deepest oil wells are $\sim$ 10km). Second, if you multiply this gradient by the radius of the earth, you get $\sim 10^5$K, whereas the actual central temperature is inferred to be around 6000–7000 K. You should have expected a change in the gradient because if you extend the gradient through the earth's crust ($\sim$ 40km thick) down the rigid mantle about 200 km, you get a temperature of 4000K. At this temperature, the rock has melted and the viscosity has become low enough that heat is efficiently carried by convection, not by conduction. Only a much smaller temperature gradient is required to drive the buoyant convection that carries the heat flux below 200km depth.

(a) The conductive energy flux from the interior of the earth is $F_{\oplus} = -kdT/dr$. Since rock is an insulator, the class derivation or the Purcell sheet gives a conductivity of about $k \sim 10^{-2}$cal s^{-1}cm^{-1}K^{-1}, [2] giving

$$\begin{aligned} F_\oplus &\simeq 10^{-2}\text{cal s}^{-1}\text{cm}^{-1}\text{K}^{-1} \times \frac{20\text{K}}{10^5\text{cm}} && (1) \\ &\simeq 80\text{erg cm}^{-2}\text{s}^{-1} \ . && (2) \end{aligned}$$

This is much less than mean irradiation from the sun now (see below). But if the sun were turned off, eventually the earth would cool until the surface of the earth radiated this $F_\oplus$ as a blackbody, $F_\oplus = \sigma T^4$, so

$$T = \left(\frac{F_\oplus}{\sigma}\right)^{1/4} = \left(\frac{80}{5.67 \times 10^{-5}}\right)^{1/4} \simeq 30\text{K} \ . \qquad (3)$$

At this temperature the atmosphere would have condensed, so there would be no greenhouse effect to worry about! If you don't happen to remember σ, but remember that the sun's overhead flux at earth $F_\odot = 1.4$kW m^{-2} = 1.4×10^6erg cm^{-2}s^{-1} keeps (with a bit of help from greenhouse effect) the earth at 300 K: $\pi R_\oplus^2 F_\odot = 4\pi R_\oplus^2 \sigma (300\text{K})^4$, then you can also write

$$\frac{T}{300\text{K}} \simeq \left(\frac{F_\oplus}{F_\odot/4}\right)^{1/4} , \qquad (4)$$

which gives the same answer.

(b) The total power from geothermal heat is

$$4\pi R_\oplus^2 F_\oplus \sim 4 \times 10^{20}\text{erg s}^{-1} = 4 \times 10^{13}\text{W} \qquad (5)$$

About 3/4 of this might be hard to access since it is under the ocean, but let's be optimistic about technology.

There are many ways to estimate the world energy use, ranging from using the annual rise in atmospheric carbon dioxide and the energy release per carbon atom from burning fossil fuels (the main source of world energy), to estimating that a significant fraction of energy use in rich countries is for cars and using mean annual miles driven and mpg's, to just looking it up

[2] $= 4$Ws^{-1}m^{-1}K^{-1}; actual rock measurements give $2 - 6$Ws^{-1}m^{-1}K^{-1}, depending or rock type and porosity, and as discussed in class, k decreases with temperature roughly as $k \propto T^{-1}$.

(e.g. the BP Statistical review of world energy, June 2006, gives 2006 world energy use as 10^{10}tons of oil equivalent, or 4×10^{20}J y^{-1} $\simeq 10^{13}$W, using 1ton oil equiv $= 4 \times 10^7$Btu and 1Btu $\simeq$ 1kJ $=$ 0.25kcal). The result is that if we could extract *all* the heat rising through the continents, we could just barely meet our current energy needs. Solar power seems much more promising!

4 Climbing

While mountain climbing, we don't want to haul more than $m = 10^4$g of ladder or rope up after us. Let's consider each in turn and determine the maximum length that can support the mountain climbers weight. We adopt a typical elastic modulus for steel of $B = 2 \times 10^{12}$erg cm^{-3}, and yield strain of $\epsilon_{cr} \sim 10^{-3}$ and a density for steel $\rho = 8$g cm^{-3}.

(a) Consider a laden climber of mass $M \approx 8 \times 10^4$g. The maximum mass a strut (our ladder with square cross section a^2, length L, and $\rho a^2 L = m$) can support on top of it without buckling was derived and demonstrated in class. It is reached when the gravitational energy released by bending (and thus lowering) the mass at the top exceeds the elastic energy stored in the strut by bending it.

$$M_{max} = \frac{Ba^4}{gL^2}\frac{\pi^2}{48} > M \tag{6}$$

So we see that,

$$\frac{Bm^2}{5\rho^2 L^4 g} > M \Rightarrow L < \left(\frac{Bm^2}{5\rho^2 g M}\right)^{1/4} \approx \left(\frac{10^{12}(10^4)^2}{5(8)^2(10^3)(8 \times 10^4)}\right)^{1/4} \text{cm} \approx 250\text{cm}. \tag{7}$$

We see that the maximum length of a ladder an average person could carry is approximately $\boxed{2.5 \text{ m}}$. This is consistent with everyday experience of actual ladders.

(b) For our steel rope not to break while we are hanging from it, the strain shall be less than $\epsilon_{cr} \sim 10^{-3}$. The strain is given by

$$\epsilon = \frac{\Delta L}{L} = \frac{(M+m)g}{B\pi r^2} < \epsilon_{cr} \tag{8}$$

from the stress-strain relationship. The mass of the circular rope is $m = \rho\pi r^2 L$, so we have,

$$L < \frac{m}{M+m}\frac{\epsilon_{cr} B}{g\rho} \simeq 0.1\frac{10^{-3}(2\times 10^{12})}{10^3 \times 8}\text{cm} \approx 2\times 10^4\text{cm} = 200\text{m}. \qquad (9)$$

So the maximum length of steel rope we could carry that would support our weight is $\boxed{\sim 200 \text{ m}}$ in length. It will be 1.4mm in radius. If we used fine piano wire ($\epsilon_{cr} = 10^{-2}$, but brittle) and didn't care about a margin of safety, we could maybe carry 2km of rope 0.4mm in diameter.

Climbers aren't stupid: they carry rope, not ladders. Actual climbing ropes are made of nylon, not steel ('stretchier', so doesn't jerk and cut you when you fall). Nylon has $B \sim 3\times 10^{10}$erg cm^{-3} and $\rho \sim 1$g cm^{-3}. It also has $\epsilon_{cr} \sim 10^{-2}$, so the figure of merit (see equation 9) $\epsilon_{cr}B/\rho$ is the same as for steel. Thus the maximum length of nylon rope that can support you is the same as for steel wire: ~ 200m. Because of the lower density, its radius will be about 4mm (8mm diameter). Actual nylon climbing ropes sold by REI are 7-10mm diameter, and 30-80m long.

5 Choo, choo, clackety clack

(a) The steel rail mass per unit length $\lambda = 64$kg m^{-1} $= 640$g cm^{-1}. Using the density of steel 8g cm^{-3} we come up with $a = \sqrt{\frac{\lambda}{\rho}} = \sqrt{\frac{640}{8}}\text{cm} = \boxed{9 \text{ cm}}$.

(b) If the temperature rises by ΔT the rail expands $\Delta L = L\alpha\Delta T$ where $\alpha \approx 10^{-5}$/K. Using the stress-strain relationship and an elastic modulus $B \sim 2\times 10^{12}$erg cm^{-3} for steel, we find,

$$F_c = a^2 B\alpha\Delta T = (9^2)(2\times 10^{12})(10^{-5}\times 20)\text{dyne} \approx 3\times 10^{10}\text{dyne} \qquad (10)$$

One ton weight is approximately a billion dynes. So the force of compression is $\boxed{30 \text{ tons}}$ in weight, independent of the length of the rail.

(c) When the temperature rises, the rail expands taking with it the wooden ties and the gravel between the wooden ties (so work is done). The mass per unit length of the gravel and wooden ties is about $\lambda' = (2\text{g cm}^{-3})(18\text{cm})(250\text{cm}) \approx 10^4\text{g cm}^{-1}$. The friction force per unit length $f \approx \lambda' g$, assuming a coefficient of sliding friction of around one. If the frictional force is to confine the rail against the thermal expansion forces of part (b), the length of the rail L must be at least such that $fL_{min} = \lambda' gL = F_c = 3 \times 10^{10}$dyn, which gives $L_{min} = 3 \times 10^3$cm $= 30$m. If $L < L_{min}$, the rail will push the gravel along allowing it to expand. If $L > L_{min}$, it can't.

(d) Turning the strut example from class on its side, the maximum compressive force a rail can support without buckling is $gM_{max} \sim g\frac{Ba^4}{5L^2g} = \frac{Ba^4}{5L^2}$. As long as $L < L_{min} = 32$m, the maximum compressive force is not F_c, but will be limited to the frictional force needed to move the gravel between the ties $\sim \lambda' Lg$, so

$$\lambda' Lg =< gM_{max} = \left(\frac{Ba^4}{5L^2g}\right) g \Rightarrow L < \left(\frac{Ba^4}{5\lambda' g}\right)^{1/3} \tag{11}$$

Plugging in the numbers, we find that $\boxed{L < 6\text{m}}$. Longer rails, including the actual 20m sections, would buckle in summer if laid in cool weather.

(e) If the rails are laid when the temperature is at the summer maximum T_{max} (if air the temperature is not that high, then the rails are flame heated before being laid), then the rail will always be thermally contracting -i.e. be in tension, not compression, and will never buckle. By contrast, if the rails were anchored 10 or 20 K below T_{max} then as we saw in parts b-d, the compressive force resisting thermal expansion can exceed the buckling force, with catastrophic results for passing trains.

For 20 m unwelded rail segments, the force of thermal contraction exceeds the resisting force of the gravel (part d), so $T = T_{max} - 20$K, the rail will contract by about $\Delta L = (2000\text{cm})(10^{-5})(20K) = 0.4$cm. This is the width of the typical clickety clack gap.

Continuously welded rails are safe, since for $\Delta T < \epsilon_{cr}/\alpha \sim 100$K, the compressive strain is less than the breaking strain of steel. There are very few inhabited parts of the world where temperature changes of 100 K occur (though remember that in the sun, rail temperature can be 20 K above air temperature)! Continuous welding is commonly employed especially on high speed rail lines in Europe.

(f) As explained in part (e) above, factor (1) is meant the track was under compressive stress about sufficient to cause buckling. Factor (2) meant that there was not as much gravel as usual on the sides to help prevent the rail from buckling by providing resistance to sideways changes in the rails' position. Factors (3) and (4) were the final straw: the centrifugal force of the heavier train going around the curve applied a large sideways force to the rails, pushing what must have been a rail on the verge of bucking beyond its limits.