Ph101: Solution 2 version 3

Jing Luan, E. Sterl Phinney

20 April 2009

1 Viscosity

a) Every term in the equation has the same dimension. For example, we can use

$$[\eta] = \frac{[\nabla p]}{[\nabla^2 \vec{u}]} \tag{1}$$

$$= \frac{[M][L]^{-2}[T]^{-2}}{[L]^{-1}[T]^{-1}} \tag{2}$$

$$= [M][L]^{-1}[T]^{-1}. \tag{3}$$

b) There are 5 variables: kT (T goes with k to make a dimension of energy), m, n, a and η. There are 3 independent dimensions (as is almost always the case). So there are two $\Pi's$. One Π is obvious: $\Pi_1 = na^3$. And suppose we have $\Pi_2 = \eta^x a^y m^z (kT)^w$ (i.e. Π_2 is independent of the n already in Π_1). Taking $x = 1$ and requiring Π_2 to be dimensionless, we find $y = 2$, $z = -1/2$ and $w = -1/2$.

$$\Pi_2 = \frac{\eta a^2}{\sqrt{mkT}}. \tag{4}$$

c) In general, $\Pi_2 = f(\Pi_1)$. η does not depend on n so $f(\Pi_1) = \text{const}$. So

$$\eta = \text{const} \times \sqrt{mkT}/a^2. \tag{5}$$

The exact result is (cf. Reif p. 545).

$$\eta = \frac{5}{64\sqrt{\pi}} \frac{\sqrt{mkT}}{a^2}. \tag{6}$$

Many inferior textbooks and scientific papers quote an estimate based on the mean free path, which gives a different (and incorrect!) numerical factor in front. Of course real molecules are not hard spheres, so their prefactor is neither the exact nor the approximate value!

d) $T_{room} \simeq 300$K can be read from the Purcell sheet. For, air mostly N_2, $m = 28m_p$. From class, we have a typical atom diameter of 3Å, or a radius $a \simeq 1.5\mathring{A}$. Since N_2 is a linear molecule, it will be about twice as long in one dimension, but we shan't concern ourselves with this complication. Then using equation 5 with const = 1, gives $\boxed{\eta \simeq 0.004 \text{dyn cm}^{-2} \text{ s}^{-1}}$. Unfortunately dimensionless numbers of order unity are sometimes more than an order of magnitude different from one, and this is one such case. If we use the "exact" hard-sphere factor from eq 6, then $\boxed{\eta \simeq 2 \times 10^{-4} \text{dyn cm}^{-2} \text{ s}^{-1}}$ which is 1/20 of our rough estimate, and happily but a bit accidentally the experimental value given on the Purcell sheet (we might not expect hard spheres or a spherical approximation for a linear molecule to work to better than a factor of two).

e) Charged particles interact by Coulumb scattering. Coulumb force can influence the motion of a particle when $e^2/r \sim kT$. Replacing a by r gives

$$\Pi_1 = n\frac{e^6}{(kT)^3} \tag{7}$$

$$\Pi_2 = \frac{\eta}{\sqrt{mkT}}\left(\frac{e^2}{kT}\right)^2. \tag{8}$$

f) Similarly we get

$$\eta = \sqrt{m}(kT)^{5/2}/e^4 \simeq 10^5 \text{dyn/cm}^2\text{/s}. \tag{9}$$

For the galaxy-sized sphere moving at 400km/s through the intergalactic medium,

$$Re = \frac{Rv}{\nu} = \frac{Rv\rho}{\eta} \tag{10}$$

$$= \frac{(2 \times 10^4 \times 3 \times 10^{18})(4 \times 10^7)(0.01 \times 1.6 \times 10^{-24})}{10^5} \tag{11}$$

$$\simeq 0.4. \tag{12}$$

Note that η is proportional to $\sqrt{m}$, so ion contribution to viscosity is 40 times larger than electron contribution. However the rough estimate neglects the dimensionless factor of 1/10 and a $1/\ln\Lambda$ (the famous Coulomb logarithm to scattering cross-sections, which arises because all impact parameters contribute equally until thermal or quantum mechanical effects limit them), so the actual value of η is about 1/200 of our estimate. So that the correct value (for zero magnetic field as assumed here!) is $\eta = 2\times10^3$g/cm/s. Apply this correct η to Re to give $Re \simeq 20$. For an application of this result, see Chris Reynolds et al 2005 MNRAS 357, 242. Note however that because the magnetic gyroradius is much smaller than the mean free path, others have argued that the actual (Osborne) Reynolds number is at least 10 orders of magnitude larger. . .!

g) e^2/kT is the radius at which the Coulomb energy is equal to the thermal energy. Π_1 is the Debye number: the mean number of electrons inside the volume where Coulomb energy exceeds the thermal kinetic energy. If $\Pi_1 \ll 1$, the electrons and ions are "free". If $\Pi_1 \gg 1$, the charged particles are "dressed" in shielding charges.

2 Gravitational radiation

a) We have variables $I = \mu a^2$ (just as for the electrical dipole, the mass quadrupole I should enter as a single variable), G, c, ω, P. Suppose the form

$$P = I^\alpha G^\beta c^\gamma \omega^\delta. \tag{13}$$

By analogy to electrical dipole radiation, we should have $\alpha = 2$. To make the right dimensions, we get

$$P = I^2 G c^{-5} \omega^6. \tag{14}$$

b) Two merging black holes radiating their mass-energy in a light crossing time produce a power $P = Mc^2/(GM/c^3) = c^5/G = 4\times10^{59}\text{erg s}^{-1} \gg 10^{10}\times10^{10}\times L_\odot = 4\times10^{53}\text{erg s}^{-1}$, i.e. about a million times the luminosity of all the stars in all the galaxies in the visible universe!

c) Because $M_\odot \gg M_\oplus$, we have

$$\mu = M_\odot M_\oplus/(M_\odot + M_\oplus) \simeq M_\oplus. \tag{15}$$

$$P_{gw} = G M_\oplus^2 a^4 \omega^6/c^5 \simeq 3.6\times10^8\text{erg s}^{-1}. \tag{16}$$

The exact formula for GW quadrupole radiation is $P_{gw} = \frac{32}{5} G\mu^2 a^4 \omega^6/c^5$, giving $P_{gw} \simeq 2\times10^9\text{erg s}^{-1} = 200W$. The recommended adult diet is

$2000\mathrm{Cal\ day}^{-1} = 2 \times 10^6 \mathrm{cal\ day}^{-1} = 10^9 \mathrm{erg\ s}^{-1}$, so your net infrared radiation is about the same at the earth is radiating in gravitational waves.

d) The time for the Earth to spiral into the Sun is

$$t = \frac{GM_\odot M_\oplus}{2aP_{gw}} \simeq 4 \times 10^{23}\mathrm{yr} \tag{17}$$

Since the sun will turn into a red giant and incinerate the earth in a mere 5×10^9yr, gravitational radiation is not important for the evolution of the solar system.

3 Blowing things up more thoroughly

a) R_{min} is reached when the total mass of air swept out by the expanding blast shock is comparable with the mass of the bomb's ejecta,

$$\rho_{air}\frac{4\pi}{3}R_{min}^3 = M_{bomb}, \tag{18}$$

which gives

$$\begin{aligned} R_{min} &= (M_{bomb}/4\rho_{air})^{1/3} && (19)\\ &= (5\mathrm{ton} \times 10^6 \mathrm{g\ ton}^{-1}/(4 \times 10^{-3}\mathrm{g\ cm}^{-3})^{1/3} && (20)\\ &= 10^3\mathrm{cm} = 10\mathrm{m}. && (21) \end{aligned}$$

When $R > R_{min}$, it is appropriate to neglect the mass of bomb.

b) In class we derived

$$R = (E/\rho_{air})^{1/5} t^{2/5}, \tag{22}$$

which gives

$$\dot{R} = (E/\rho_{air})^{1/5}\frac{2}{5}t^{-3/5}. \tag{23}$$

R_{max} is reached when the momentum flux at the shock is comparable with the air pressure,

$$\rho_{air}\dot{R}^2 \simeq p_{air}, \tag{24}$$

which gives $\boxed{R_{max} = (E/p_{air})^{1/3}(2/5)^{2/3} \simeq 0.5\mathrm{km}}$. This equation is also to oom, equivalent to $E = p_{air}R^3$ —i.e. the energy of the bomb is equal to the thermal energy of the pre-existing air —i.e. at larger radii, the bomb's energy isn't enough to increase the temperature by more than a fraction less than 1. When $R < R_{max}$, we can neglect the pressure of air.

c)Equation 24 can be written

$$\rho_{air}\dot{R}^2 = p_{air} \simeq \rho_{air}c_s^2, \tag{25}$$

where c_s is the sound speed of the air. Thus at R_{max}, the Mach number of expansion $M = \dot{R}/c_s \simeq 1$.

d) The wind speed of destructive hurricanes and tornados is $v_{des} \sim$ 100mph $\sim 4 \times 10^3$cm s^{-1} $= 0.13c_s$. After $R > R_{max}$, $\Delta v \propto 1/R$. So we have

$$v \sim \dot{R}(R_{max})\frac{R_{max}}{R} \sim c_s\frac{R_{max}}{R}. \tag{26}$$

Where we used part (c) in the last equality. The post-shock airspeed v falls to $v_{des} = 0.13c_s$ at $R \sim R_{max}/0.13 \sim$ 3km. At larger radii, the blast wave is not so destructive. More detailed analysis should also consider the sudden overpressure as the shock passes ($\Delta p \sim p_{atm}(R_{max}/R)$), and the reflection of the blast wave on the ground (the reflected shock catches up to and merges with the main blast, creating the extra-destructive Mach stem). Nevertheless, our estimate is roughly consistent with the result of detonating Little Boy (about 25kton, with 13kton in blast and the rest in radiation) 0.6km over Hiroshima: almost all buildings within 2km of ground zero were leveled, with considerable damage beyond.

Refer to the figure http://en.wikipedia.org/wiki/File:Hiroshima_aftermath.jpg to remind yourself why nuclear weapons are such a cost-effective way to sow terror.

4 Heat transport

We have 6 variables T_1, T_2, σ, κ, F and d. Thus there are $6 - 3 = 3$ dimensionless Π's. One dimensionless Π is obviously the variable we need to calculate: $\Pi_1 = T_2/T_1 < 1$. To find the other Π's, notice that there are four independent heat-fluxes relevant in this problem (all with dimensions of heat flux of course): F, $\kappa T_1/d$, σT_1^4, σT_2^4. The ratio of the last two just repeats Π_1^4. Two more dimensionless Π's can be formed from the ratios of the first and third ($\Pi_2 = \sigma T_1^4/F$) and the first and second, say. One of these just determines the characteristic temperature scale (i.e. $T_1^4 \sim (F/\sigma)\Pi_2$). Then the remaining one determines the distribution of temperature:

$$\Pi_3' = [F/(\kappa T_1/d)]^4 = \frac{F^4 d^4/\kappa^4}{(F/\sigma)\Pi_2} = \frac{F^3 \sigma d^4/\kappa^4}{\Pi_2}. \tag{27}$$

Or more conveniently

$$\Pi_3 = \Pi_3'/\Pi_2 = F^3 \sigma d^4/\kappa^4. \tag{28}$$

Generally, we have Π_1 as a function of Π_3

$$\Pi_1 = g(\Pi_3). \tag{29}$$

Case 1: when d goes to zero or κ to infinity ($\Pi_3 \ll 1$), thermal conduction is so efficient that both sides will be at the same temperature, each radiating half the flux F: $\Pi_1 = 1, \Pi_2 = 1/2$.

Case 2: When the κ goes to zero or d to infinity ($\Pi_3 \gg 1$), $T_2 \ll T_1$, so radiation from the back side is negligible and the front side radiates almost all of F: $\Pi_1 = 0, \Pi_2 = 1$.

From the previous analysis, we can see that the transition between the two behaviors should happen at $\Pi_3 \sim 1$. The solar flux is $F = 1.4 \times 10^6$erg cm^{-2}s^{-1}, the Stefan-Boltzmann constant is $\sigma = 5.7 \times 10^{-5}$erg cm^{-2}s^{-1}K^{-4}, and the Purcell sheet gives the heat conduction coefficient for Copper $\kappa_{Cu} =$ 1cals^{-1}cm^{-1}K^{-1} $= 4 \times 10^5$ergs^{-1}cm^{-1}K^{-1}, and for an insulator, $\kappa_{In} = 10^{-2}\kappa_{Cu}$. Thus

$$\begin{aligned} d_{Cu} &= \kappa_{Cu}/F^{3/4}\sigma^{1/4} && (30) \\ &= (4 \times 10^7)/(1.4 \times 10^6)^{3/4}/(5.7 \times 10^{-5})^{1/4} \simeq 10^4\text{cm} = 100\text{m}, && (31) \end{aligned}$$

and for the insulator

$$\begin{aligned} d_{In} &= \kappa_{In}/F^{3/4}\sigma^{1/4} && (32) \\ &= 10^{-2} d_{Cu} \simeq 1\text{m}. && (33) \end{aligned}$$

5 Interstellar exploration

You first need $\sqrt{2GM_\oplus/R_\oplus} = 11$km s^{-1} $= v_\oplus$ to escape from earth into solar orbit near the earth. How much extra velocity you need to escape from the solar system depends on the direction you choose to escape in. The smartest is in the direction of earth orbit around the sun: Then you need still to be moving relative to the earth at at least ($\sqrt{2}\times$(earth orbital velocity around sun) – earth orbital velocity around sun = $(\sqrt{2})30 = 13$km s$^{-1} \equiv v_2$), so that your total velocity exceeds $\sqrt{2GM_\odot/a} = 43$km s^{-1}, the escape velocity from the solar system. Thus if your mass is m,

your initial kinetic energy must be at least $(1/2)m(v_\oplus^2 + v_2^2)$, so that after subtracting your binding energy to earth $GM_\oplus m/R_\oplus = 1/2mv_\oplus^2$, you are still moving with speed v_2 in the direction of earth motion as you escape earth's pull.

If I take $m = 50-75$kg, I get an initial kinetic energy of $0.7-1\times10^{17}$erg. Eating the recommended 2000Cal/day $= 0.3\times10^{17}$erg y^{-1}, we find that it takes just a few years worth of food energy to escape the solar system.

Of course we have neglected the weight of rocket stages, life-support equipment and air-resistance, which combined with the rocket equation increase the required energy enormously ….