Ph101: Solution 1 version 3

Jing Luan, E. Sterl Phinney

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1 Crowded Space

a) Suppose that we have N satellites in randomly oriented low earth orbits[1]

The cross-section of each satellite is $A = 10\text{m}^2$. Consider two satellites of circular cross-section, radius a, area $A = \pi a^2$. They will collide if their centers

[1] some students, thinking all satellites were in equatorial circular orbits about a spherical earth, wondered how satellites' orbits would ever cross, and why anyone would consider random orbits a good approximation. There are several reasons: first, the earth is not spherical, so no satellite orbit can be circular (at 800km height, typical minimum eccentricities correspond to $\sim$ 10km differences between perigee and apogee) and inclined orbits cannot be planar: all satellite orbits inclined to the equator precess because of the earth's rotation-induced oblateness, and sweep out a shell around the earth. Second, many satellites either purposely (to make close passes over regions of interest, for scientific reasons, or to avoid having delicate electronics fried in the South Atlantic Anomaly) or accidentally (failed rocket stages) have quite high orbital eccentricity). Third, spy satellites, and global communications satellites like those of the Russian Cosmos series and the US Iridium constellation are placed in polar (or highly inclined) orbits so they will pass over all (inhabited) parts of the earth. Remote sensing satellites and weather satellites are also placed in near-polar sun-synchronous orbits whose precession rate just matches the earth orbit, so the satellites always pass over a given point on earth at the same time of day. The poles are a dangerous crowding region for all such satellites (indeed Cosmos 2251 and Iridium 33 collided near the north pole over northern Siberia; they were on near-polar orbits with nearly orthogonal orbit planes). Other satellites, like most astronomy satellites, manned missions and such, are placed in more modestly inclined orbits that pass over their launch country. To reduce the need for rocket fuel at launch, they usually orbit in the same direction as earth rotation (a free 0.5km s^{-1} contribution towards orbital speed if you orbit prograde; to orbit retrograde you have to boost an extra 1km s^{-1}). The orbits of these two groups of satellites (polar and near-equatorial) intersect at relative velocities of order $\sqrt{2}$ times the orbital velocity, i.e. 11km s^{-1}; this is also the maximum relative velocity of satellites on polar orbits. Satellites in the lower inclination prograde group have much lower relative velocities.

pass within a distance $2a$ of each other. Thus the collision cross-section is $\pi(2a)^2 = 4A$. The surface area of the earth is $4\pi R_\oplus^2 = 4\pi \times (6 \times 10^6)^2 \simeq 4 \times 10^{14}\mathrm{m}^2$. The range of heights in the so-called low earth orbit region is $h = 2000\mathrm{km} - 300\mathrm{km} = 1700\mathrm{km} = 1.7 \times 10^6\mathrm{m}$, so the N satellites fill a region of volume $V = 4\pi R_\oplus^2 h$. The number density of satellites is thus

$$n = \frac{N}{V}. \tag{1}$$

If the satellites have characteristic relative velocity v, the number of collisions in time t is thus

$$N_{coll} = n^2(4A)vtV = N^2(4A)vt/V \tag{2}$$

The desired N which gives $N_{coll} = 1$ in t is thus

$$\boxed{N = \sqrt{V/(4Avt)}} \tag{3}$$

For the characteristic relative velocity of the satellites we take their orbital velocity $v = \sqrt{GM/R_\oplus} = 7\mathrm{km\ s^{-1}}$ ($\sqrt{2}$ times larger for orbits intersecting at right angles, a modest fraction of $7\mathrm{km\ s^{-1}}$ for orbits intersecting at small angles).

Before giving the numerical answer, we give another derivation of the result in equation 3, which may be more understandable for those not used to cross-section and reaction rate calculations:

Each satellite travels a distance vt during time t. The collision volume it sweeps out through is then $vt4A$ (i.e. if the center of another satellite is in this volume, a collision will have occured). The number of satellites in this volume is $nvt4A$, which is the number of collisions a single has during time t. So the collision rate for one satellite is $nv4A$. We have N satellites in total, so the number of collisions during time $t = 10\mathrm{yr} \simeq 3 \times 10^8\mathrm{s}$ is $Nnv4At = N\frac{N}{4\pi R_\oplus^2 h}v4At$. And we require this number to be equal to one, which gives us the same result as equation 3:

$$N = \sqrt{\frac{4\pi R_\oplus^2 h}{v4At}} \simeq \sqrt{\frac{5 \times 10^{14}\mathrm{m}^2 \times 1.7 \times 10^6\mathrm{m}}{7000\mathrm{m/s} \times 4 \times 10\mathrm{m}^2 \times 3 \times 10^8\mathrm{s}}} \simeq \boxed{3000}. \tag{4}$$

b) With ~ 2000 payloads (plus several times more upper rocket stages and lost parts), the collision between Iridium 33 and Cosmos 2251 was just about on schedule. This suggests that all the ‘collision avoidance’ maneuvers satellites undertake are not helping much.

2 Water use

This will vary from person to person of course. In cases like this, you need to state clearly the assumptions and logic behind your estimates.

a) I have a hotpot whose occupation is about 1.5 liters. I drink about 2 hotpots of water per day, so $\boxed{3\text{ l/day}}$ for drinking;

b)California low-water use toilets use 1.6gals = 6l while the old kind use 5gals = 20l. I use the toilet around 8 times per day, so $\sim 50 - 150$l/day for flushing;

The volume of a washing machine is about $(30\text{cm})^3 = 27$ l, and I use it about once per week, so $27/7 \simeq 4$l/day for washing; As for cleaning, I have a bath every day. I measure the water flux: it takes 44.5 seconds for the tap to fill a container of $7.5 \times 17 \times 30\text{cm}^3$, so the flux is 0.086 l/s. Each bath takes me around 10 min, so the water for bathing is 50 l/day. Compared with bathing, cleaning hands, face, floors, cars, hamster cages can be neglected. Dish washing uses a few sink-fulls, not a bathtub-full, so probably is not a big perturbation either.

Adding these together, we get $\boxed{100\text{ to }200\text{l/day}}$ for household drinking and sanitation.

However, these estimates (by a grad student) neglect the main household use in suburban southern california: watering the yard. Your instructor is ashamed to report that his family of four uses an annual average of 4400 cubic feet per month, or per person 1,000l/day. Since the summer use is 6 times higher than the winter use (when sprinklers are off), it is clear that the bulk of this is going to the 0.3 acre yard. About 1/3 of this goes to 11 species of fruit, so would be counted in (c), but the other 2/3, about $\boxed{550\text{l/day}}$ is for watering inedible grass, flowers and shade trees.

c) About 30% of the earth's surface is covered by land, and about 10% of that is used for farming (if you have driven across the US, you spent more than 10% of your driving time admiring midwestern wheat fields —remember the US is a net exporter of food!). Average rainfall is 1m/yr (we will calculate this from the solar flux later in the course). The population of earth is about 7 billion. So water for farmed food is

$$4\pi R_\oplus^2 \times 0.3 \times 0.1\text{m}/365\text{days}/7 \times 10^9 \simeq 6\text{m}^3\text{day}^{-1} = \boxed{6000\text{l/day}}. \quad (5)$$

Adding the additional 20% of (somewhat drier than average) land used for pasture could roughly double this. Food production is by far the largest human demand on the water supply.[2]

Other water uses not required to be considered in this problem, but interesting to compare, are industrial: evaporative cooling in powerplants, water use in mining, paper and plastic manufacture, growing cotton for your clothes, etc. These are small compared to farming.

3 Hollywood vs Physics

The radius of one balloon is $r = 6 \times 2.5\text{cm} \simeq 15\text{cm}$, and its volume $V_b = 1.4 \times 10^4\text{cm}^3$, while the radius of the balloon cluster is R. The total number of balloons is N. We have

$$R = rN^{1/3}. \tag{6}$$

The densities of Helium gas, air, rubber and string are denoted by ρ_{He}, ρ_{air}, ρ_r and ρ_s. Air is mainly N_2 and has density one thousandth of that of water. The density of Helium gas at STP is $M_{He}/M_{air} \times \rho_{air} = 4/28\rho_{air}$. The balloon skin is made from rubber whose density is roughly the density of water. The string is made from cotton or nylon, whose density is also roughly the density of water.

a) Firstly, neglect the weight of balloons and string. The size of a modest house is for example $L = 700\text{cm}$. The house is wooden (attempting to lift a brick or stone house with balloons would be unwise because mortar has very low tensile and shear strength!). Most of the mass is in the beams and wall/floor/roofing, about 10cm thick and $t = 5\text{cm}$ thick respectively. We presume they are made of pine wood, $\rho_{wood} \simeq 0.5\text{g cm}^{-3}$. Thinking of the house as a cube divided into two floors with 4 rooms on each floor, the area of walls, floors and roof is about $9L^2$. So we estimate the mass of the house as $m_{house} \simeq 9L^2 t\rho_{wood} = 1.1 \times 10^7\text{g}$, and its mean density as

$$\bar{\rho}_{house} = m_{house}/L^3 = 9(t/L)\rho_{wood} = 0.03\text{g cm}^{-3} \tag{7}$$

Furniture, plumbing and fixtures are only a small fraction of the house's 10 tons unless he really has a lot of books, thick granite countertops and tile floors.

[2]For details of water use in various types of farming, crop yield dependence on transpiration, the differences between water demands in vegetarian and high meat-and-dairy diets, and the resulting limits on human population, see D. Molden, C. de Fraiture, F. Rijsberman 2007 *Issues in Science and Technology*, Vol 22, issue 4, p. 39.

The volume of a balloon is $V_b = 4\pi/3r^3$. The total number of balloons we need is N, which should satisfy the following equation due to Archimedes

$$\boxed{m_{house} = \bar{\rho}_{house} L^3 = N V_b (\rho_{air} - \rho_{He})} \tag{8}$$

which yields $\boxed{N \simeq 10^6}$ and $\boxed{V_b \sim L^3 \bar{\rho}_{house}/(\rho_{air} - \rho_{He}) \sim 35 L^3}$

The size of the balloon cluster is several times of that of the house: $(2R/L \simeq 4)$. This is a lower limit. Including furniture, ballon weight and string weight would require a modestly larger cluster.

The movie trailer picture http://disney.go.com/disneypictures/up/ shows balloons occupying a region 7 times the diameter of the house, which is actually fairly realistic.

b) Before it is filled with Helium, the surface area of the balloon skin is about $\pi(5\text{cm})^2 \times 2 = 150\ \text{cm}^2$. The thickness of the skin is about 1/100cm (I use the thickness of a piece of paper as an estimate). So the weight of the balloon skin is

$$m_{bs} \simeq \rho_f \times 150\text{cm}^2 \times 1/100\text{cm} \simeq 1.5\text{g}. \tag{9}$$

So the weight of the balloon skins is about 1.5×10^6g, only about 10% of the weight of the house, so it was OK to neglect their weight.

The length of the string is roughly the radius of the cluster. Suppose we use cheap cotton string to tie the ballons: the radius of the cross section of the string is about half a millimeter, so the weight of the strings would be

$$M_s = N R \pi (0.05\text{cm})^2 \times \rho_s \simeq 0.75 \times 10^{-2} N (R/\text{cm})\text{g} \simeq 0.1 N^{4/3}\ \text{g}. \tag{10}$$

So the total weight of strings would be about 10^7g, comparable to $m_{house} = 1.2 \times 10^7$g. Clearly this would prevent the whole operation since the balloons couldn't even lift their own strings (a familar problem to anyone who has brought home a party balloon and looked at it a few hours later)!! Cotton string is a poor choice. Better to use thinner nylon or kevlar line to hold those balloons. The minimum thickness for these will be something you'll be able to estimate a bit later in the course.

c) The answer is already in part (a), $2R/L \simeq 4$ plus a bit for furniture, strings, etc.

4 Too many cars

Let's assume there are $\sim 10^9$ rich-world folks in the world, half of whom drive the 10,000 miles = $16,000$km per year assumed in manufacturer's car maintenance recommendations. Then the total distance driven by all the cars in the world is increasing at

$$\frac{1}{2} \times 10^9 \times 16,000\text{km y}^{-1}/3 \times 10^7\text{s y}^{-1} = 3 \times 10^5\text{km s}^{-1} = c$$

Considering that in the US there is slightly *more* than one car per adult, not half of one, and there are lots of vehicles outside the US, Europe and Japan, our estimate is more likely to be on the low side than the high side, and the distance is probably already growing a bit faster than the speed of light.