On the Intersection of Secant Sets in the Hill Cap

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Abstract

The Hill cap is a unique, up to equivalence, 56-cap $K \subset PG(5,3)$. In the original 1973 paper [5], outlining the construction this cap, a theorem of Hill implies that for any vector $P \in PG(5,3) \backslash K$ there are exactly 10 lines through it which also intersect the Hill cap in two points. We call the set of vectors in K that are intersected by these 10 lines a secant set of K and denote it $A(P)$. Since each line intersects K in 2 vectors, we can easily see that $|A(P)| = 20$. In a 2000 paper [6], Hill posed the problem of giving a theoretical proof to the computer result that for any two distinct vectors $P, Q \in PG(5,3) \backslash K$, $|A(P) \cap A(Q)| \leq 8$. Here we give that proof.

1 Introduction

Let's begin by introducing the concept of a cap set first. Let $\mathbb{F}_q^n$ be the n-dimensional vector space over the finite field of characteristic q where $q = p^k$ for $p, k \in \mathbb{N}_+$ and p prime. A cap set is a subset $K \subset \mathbb{F}_q^n$ such that no three points in K are collinear. That is equivalent to saying that K contains no 3-arithmetic progressions. We can check if three points form an arithmetic progression if for $a, b, c \in \mathbb{F}_q^n$ the relation,

$$\frac{a+c}{2} = b$$

holds. The above equation can be applied to any field, but, in this paper, we are mainly concerned about $\mathbb{F}_3$, so we can reduce it as follows,

$$a + c = 2b \Rightarrow$$

$$a + b + c = 0 \tag{1}$$

We call a k-cap a cap set $K \subset \mathbb{F}_q^n$ such that $|K| = k$ and a maximal cap, or ovaloid, a cap set such that there is no $(|K| + c)$-cap for $\forall\, c \in \mathbb{N}_+$. While caps are defined as subsets of $\mathbb{F}_q^n$, they can also be found in the respective projective space. In fact, the main cap we will be working with, the Hill cap, in contained within the projective space of $\mathbb{F}_3^6$ which we describe in the next section.

2 The Projective Space

We define a n-dimensional projective space as the projection of a $(n+1)$-dimensional vector space with the orgin removed. We write this as follows,

$$PG(n,q) = \{\mathbb{F}_q^{n+1} \backslash \{0\}\} \backslash \sim$$

Where the equivalence relation $\sim$ defines equality for some $x, y \in PG(n,q)$ as $x \sim y$ if and only if $x = \lambda y$ for $\lambda \neq 0 \in \mathbb{F}_q$. In the case of $\mathbb{F}_3$, we can see that $\lambda \in \{1,2\}$, but multiplying by 1 changes nothing, so our equivalence relation becomes, $x \sim y$ if and only if $x = 2y$. With this equivalence, we can work backwards from (1)

$$a + b = -c \Rightarrow$$

$$a + b = 2c \Rightarrow$$

$$a + b = c \tag{2}$$

Using (2), we can check if three points are collinear by adding two of their representitives together and checking that they give the third.

Due to this equivalence relation, an n-dimensional projective space has less points than its respective $(n+1)$-dimensional vector space. We show this in the following lemma.

Lemma 1. $|PG(n,q)| = \frac{q^{n+1}-1}{q-1}$

Proof. Take a point in our projective space given as $(a_1, a_2, ..., a_n, a_{n+1})$, where a_{n+1} is non-zero. We divide by that point to get $(\frac{a_1}{a_{n+1}}, \frac{a_2}{a_{n+1}}, ..., \frac{a_n}{a_{n+1}}, 1)$. There are q^n such points. Now take the point where $a_{n+1} = 0$, $(a_1, a_2, ...a_n, 0)$, but $a_n \neq 0$. We do the same and divide by a_n to get $(\frac{a_1}{a_n}, \frac{a_2}{a_n}, ..., 1, 0)$. There are q^{n-1} such points. We continue this, and, by induction, we obtain $|PG(n,q)| = q^n + q^{n-1} + ... + q + 1 = \frac{q^{n+1}-1}{q-1}$. □

We now introduce the notion of supports. For a given vector $x = (a_1, a_2, ..., a_n)$, its set of supports is $S = \{1, 2, ..., n\}$. We call a support of x a subset $G \subseteq S$ such that for every element in G, the corresponding entry in x is non-zero and every other entry is. We have that $|G| = k$ for $k \leq n$, so we call it a k-support of x. We will give an example. Let $G = \{1, 3, 5\}$, then, in $PG(5, 3)$, there are 4 projectively distinct vectors (8 total) with G as a support. They are explicitly given as

$$(1, 0, 1, 0, 1, 0)$$

$$(1, 0, 1, 0, 2, 0)$$

$$(1, 0, 2, 0, 2, 0)$$

$$(1, 0, 2, 0, 1, 0)$$

We can also write the set of vectors with support G as $(*, 0, *, 0, *, 0)$ where $* \neq 0 \in \mathbb{F}_3$. We can count the number of vectors with a given support as follows

Lemma 2. *In $PG(n,q)$, there are $(q-1)^{w-1}$ distinct vectors with a fixed w-support.*

Proof. Since each of the positions is fixed, there is a choice of $q-1$ possible non-zero values, giving $(q-1)^w$ possible vectors. $PG(n,q)$ has the equivalence relation that for some $P \in PG(n,q)$, $P = a \cdot P$ where a is a scalar such that $a \neq 0 \in \mathbb{F}_q$. This gives us $q-1$ possible choices for a and because for any of those choices we get an equivalent vector, there are $\frac{(q-1)^w}{q-1} = (q-1)^{w-1}$ distinct vectors. □

From the notion of supports, we can define the notion of weight of a vector. We say a vector x with support G has a weight w given by $w = |G|$. An equivalent definition of w is the number of non-zero entries in x. If we want to count the number of distinct vectors for all w-suports, then there are $\binom{n}{w}$ possible combinations of different positions, so we just proved following more general lemma.

Lemma 3. *In $PG(n,q)$, thre are $\binom{n}{w} \cdot (q-1)^{w-1}$ distinct vectors of weight w.*

3 The Construction of Hill

In his original 1973 paper [5], Hill contructed a maximal 56-cap in $PG(5,3)$. However, before we detail that construction, we will define the notion of secants, or 2-lines, and secant sets. Let $K \subset PG(n,q)$ be a cap and $P \in PG(n,q)\backslash K$. If a line through P intersects K in the two points $a_1, a_2 \in K$, then we call the line formed by a_1 and a_2 a secant of K and denote it (a_1, a_2). A secant set of P, denoted $A(P)$, is the set of all points in K that are secants of P. We call $|A(P)|$ the secant weight of P. These definitions bring us to a result of Bose.

Lemma 4 (**Bose**)**.** *Let $K \subset PG(n,q)$ be a k-cap and u_i be the number of secants of K for each $P_i \in PG(n,q)\backslash K$ $\forall$ $i \in \{1,2,...,m\}$ and $m = |PG(n,q)| - k$. Then*

$$\sum_{i=1}^{m} u_i = k(k-1)$$

Proof. The proof of this result is given in Lemma 1 of [1]. □

With this result we can now prove a lemma which is the basis of our intersection theorem.

Lemma 5. *If $K \subset PG(5,3)$ is a 56-cap, then $|A(P_i)| = 20$ $\forall$ $P_i \in PG(5,3)\backslash K$ where $i \in \{1,2,...,308\}$.*

Proof. Let u_i denote the number of secants of K through P_i. By Lemma 4,

$$\sum_{i=1}^{308} u_i = (56)(55) = 3080$$

Now, due to high transitivity in the automorphism group of K that is cited by Hill here [5], we must have that $u_i = u_j$, $\forall$ $i,j \in \{1,2,...,308\}$. We can re-write our sum as follows,

$$\sum_{i=1}^{308} u_i = 308u_i = 3080 \Rightarrow$$

$$u_i = 10$$

Since each P_i has 10 secants through it and each secant has 2 points in K, $|A(P_i)| = 20$. □

We can now now give the construction of this 56-cap as originally done by Hill. First let f be the quadric form

$$f(P,Q) = \sum_{i=1}^{6}(x_i y_i) + \sum_{i=1}^{5}(x_i y_{i+1} + y_i x_{i+1})$$

for $P = (x_1, x_2, ..., x_6)$ and $Q = (y_1, y_2, ..., y_6)$ in $PG(5,3)$. Now we let X be points of the elliptic quadric associated with f,

$$X = \{P \in PG(5,3) \mid f(P,P) = 0\}$$

It is easy to check that $|X| = 112$. Now let $PO^-(6,3)$ denote the orthogonal group of isometries of f which is a subgroup of $PGL(6,3)$ and t be the following linear map

$$t = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

Since $f(t(P), t(Q)) = f(P,Q)$, $t \in PO^-(6,3)$. Under the natural action of t, we can split X into 16 orbits each with 7 points such that

$$X = \bigcup_{i=1}^{16} X_i$$

We can now simply pick 8 of these orbits whose union gives a 56-cap, $K = \bigcup_{i=1}^{8} X_i$. This set is known as the Hill cap, and Hill [4] showed that it is maximal and unique up to equivalence.

4 The Construction of Calderbank and Fishburn

Before we outline the construction of Calderbank and Fishburn which was originally written here [7], we need to describe the notion of block designs. Let X be a set of k elements. Now we take subsets of X each of size $n < k$ and we make sure that every λ number of subsets has t points in commons. We call these subsets a $t-(k,n,\lambda)$ block design. To give an example of this, we will use the block design used by Calderbank and Fishburn in the Hill cap construction.

Let $X = \{1,2,3,4,5,6\}$ be the set of supports of $PG(5,3)$. We consturuct the following $2-(6,3,2)$ block design and denote it $B(X)$,

$$B(X) = \{(3,4,5), (4,5,6), (2,5,6), (2,3,6), (2,3,4), (1,2,4), (1,3,5)(1,4,6), (1,2,5), (1,3,6)\}$$

Each of these blocks gives $(3-1)^(3-1) = 4$ (by Lemma 2) projectively distinct vectors for a total of 40. We will write them using the * notation as it's easier to work with in the intersection theorem. Based on the above blocks, we get the following set of 40 vectors,

$$(0,0,*,*,*,0)$$
$$(0,0,0,*,*,*)$$
$$(0,*,0,0,*,*)$$
$$(0,*,*,0,0,*)$$
$$(0,*,*,*,0,0)$$
$$(*,*,0,*,0,0)$$
$$(*,0,*,0,*,0)$$
$$(*,0,0,*,0,*)$$
$$(*,*,0,0,*,0)$$
$$(*,0,*,0,0,*)$$

Now we take the set of all vectors of weight 6 of which there are $\binom{6}{6}\cdot(3-1)^(6-1) = 32$ (by Lemma 3) projectively distinct ones. We split this set of 32 into two sets of 16 one which constains all the weight 6 vectors with an even number of one entiries and the other with an odd number of one entries. The union of our set of 40 vectors from the block design and the 16 weight 6 vectors with an odd number of one entries gives the Hill cap.

5 The Intersection Theorem

Before we state and prove our theorem, we will show how the secant sets $A(P)$ can be constructed for P taking on several different weight. The way we construct $A(P)$ is by looking at all the possible combinations of sums of points in the Hill cap and by (2) if such a sum gives P then the two points in that sum belong to $A(P)$.We will divide each of these constructions into the sum of 2 weight 6 vectors, 1 weight 6 and 1 weight 3 vectors, and 2 weight 3 vectors, as these are all the possible sums of vectors in the Hill cap that could give us P.

P = (*,0,0,0,0,0)
There are no possible vectors of weight 6 + weight 6 that could give us P. There are also no possible vectors of weight 3 + weight 6 that could give us P. This leaves us only with all the blocks of the design that have a non-zero first entry, and each one can combine twice with itself to give P. There are 5 such blocks with 4 vectors each, making $|A(P)| = 5 \cdot 4 = 20$. The same is true if the non-zero entry is in any position, there will always be exactly 5 blocks that match that position.

P = (*,*,0,0,0,0)
There is one vector of weight 6 with 0 one entries and 1 vector of weight 6 with 4 one entries that could combine with each other to give P. Then there are 6 more weight 6 vectors with 2 one entries (we simply permute the position of the 1s) that could give us P. There are no weight 6 + weight 3 vectors that could give P. Now we have the blocks $(*,*,0,*,0,0)$ and $(*,*,0,0,*,0)$ that can combine once with themselves for 4 more vectors. Then there are the block $(0,*,*,0,0,*)$ and $(*,0,*,0,0,*)$ that fully combine with each other for 8 more

vectors, making $|A(P)| = 2+6+4+8 = 20$. Further constructing the permutations of this weight, if $P = (0,0,*,*,0,0)$ or $(0,0,0,*,*,0)$ or $(0,0,0,0,*,*)$, the distribution of vectors making $A(P)$ will stay the same, but which blocks are used changes based on the location of the non-zero entries.

P = (*,*,*,*,0,0)

There is one vector of weight 6 with 0 one entries and 1 vector of weight 6 with 2 one entries that could combine with each other to give P. Then there are the blocks $(0,0,0,*,*,*)$ and $(0,*,0,0,*,*)$ that can combine with 4 different vectors of weight 6 each with 4 one entries to give P. This leaves us with the following block pairs: $(0,0,*,*,*,0)$ and $(*,*,0,0,*,0)$, $(0,*,*,0,0,*)$ and $(*,0,0,*,0,*)$, $(0,*,*,*,0,0)$ and $(*,*,0,*,0,0)$. The first two pairs can combine twice with themselves for 8 vectors, and the last pair can only combine once for 2 more vectors, making $|A(P)| = 2+8+8+2| = 20$. Similarly, if we were to construct $P = (0,0,*,*,*,*)$, the distribution of vectors stays the same, but which blocks are used changes, based on the position of the 0 entires.

P = (*,*,*,*,*,0)

No two vectors of weight 6 can combine to give P. The following 5 blocks, however, can each combine with 1 weight 6 vector with 4 one entries to give P: $(0,0,0,*,*,*)$, $(0,*,0,0,*,*)$, $(0,*,*,0,0,*)$, $(*,0,0,*,0,0)$, $(*,0,*,0,0,*)$. This leaves the following pairs of blocks where they only combine once with themselves for 2 vectors each: $(0,0,*,*,*,0)$ and $(*,*,0,0,*,0)$, $(0,0,*,*,*,0)$ and $(*,*,0,*,0,0)$, $(0,*,*,*,0,0)$ and $(*,0,*,0,*,0)$, $(0,*,*,*,0,0)$ and $(*,*,0,0,*,0)$, $(*,*,0,*,0,0)$ and $(*,0,*,0,*,0)$. These are all the possible blocks that could give P, making $|A(P)| = 10+10 = 20$.

P = (*,*,*,*,*,*)

These are all the vectors of weight 6 with an odd number of one entries. We construct $A(P)$ by taking one point from every single block and combining it with a weight 6 vector with even entry ones.

Theorem 1. *Let K be the Hill cap, then for any two distinct vectors $P, Q \in PG(5,3)\backslash K$, $|A(P) \cap A(Q)| \leq 8$.*

Proof. To prove this, we will split it into different cases where P and Q will take on values of vectors with different weights.

Case: $P = (*,0,0,0,0,0)$ and $Q = (*,*,0,0,0,0)$

Since $A(P)$ has no vectors of weight 6, none can intersect. Now the block $(*,0,*,0,0,*)$ contains 4 vectors in both $A(P)$ and $A(Q)$, so those can intersect. The blocks $(*,*,0,*,0,0)$ and $(*,*,0,0,*,0)$ each contain 4 vectors in $A(P)$ and 2 vectors in $A(Q)$, so 4 total vectors can intersect. No other blocks match, so we conclude that $|A(P) \cap A(Q)| \leq 8$.

Case: $P = (*,0,0,0,0,0)$ and $Q = (*,*,*,*,0,0)$

Since $A(P)$ has no vectors of weight 6, none can intersect. Now we have the blocks $(*,*,0,0,*,0)$ and $(*,0,0,*,0,*)$ each of which contain 4 vectors in $A(P)$ and 2 vectors in $A(Q)$, giving us 4 total vectors that could intersect. This only leaves the block $(*,*,0,*,0,0)$ which contains 1 vector in $A(Q)$, so we conclude that $|A(P) \cap A(Q)| \leq 5$.

Case: $P = (*,0,0,0,0,0)$ and $Q = (*,*,*,*,*,0)$

Since $A(P)$ has no vectors of weight 6, none can intersect. Now the blocks $(*,*,0,0,*,0)$, $(*,*,0,*,0,0)$, and $(*,0,*,0,*,0)$ each have 4 vectors in $A(P)$ and 2 vectors in $A(Q)$, for a total of 6 vectors that could intersect. This leaves the blocks $(*,0,0,*,0,*)$ and $(*,0,*,0,0,*)$ each of which contain 4 vectors in $A(P)$, but only 1 vector in $A(Q)$, for 2 more possible vectors of intersection. These are all the matching blocks, so we conclude that $|A(P) \cap A(Q)| \leq 8$.

Case: $P = (*,*,0,0,0,0)$ and $Q = (*,*,*,*,0,0)$

First we see that $A(P)$ has the following weight 6 vectors: 1 with 0 one entries, 6 with 2 one entires, and 1 with 4 one entries. $A(Q)$, on the other hand, has the following weight 6 vectors: 1 with 0 one entires, 1 with 2 one entries, and 4 with 4 one entries. From this it is obvious that only 3 weight 6 weight vectors could intersect. Now the block $(0,*,*,0,0,*)$ has 4 vectors in $A(P)$, but only 2 vectors in $A(Q)$, so 2 will intersect. This leaves the block $(*,*,0,0,*,0)$ which has 2 vectors in both $A(P)$ and $A(Q)$ as well as the block $(*,*,0,*,0,0)$ which has 2 vectors in $A(P)$ and 1 vector in $A(Q)$, giving 3 more possible intersections. Since these are all the blocks that match, we conclude that $|A(P) \cap A(Q)| \leq 8$.

Case: $P = (*,*,0,0,0,0)$ and $Q = (*,*,*,*,*,0)$

Since $A(Q)$ has only 5 weight 6 vectors each with 4 one entries and $A(P)$ has only 1 such vector, we have only 1 possible intersection. Now we look at the blocks $(*,*,0,0,*,0)$ and $(*,*,0,*,0,0)$ both of which have 2 vectors in $A(P)$ and $A(Q)$, giving 4 more possible intersection vectors. The only matching blocks left are $(0,*,*,0,0,*)$ and $(*,0,*,0,0,*)$ both of which have only 1 vector contained in $A(Q)$, so we conclude that $|A(P) \cap A(Q)| \leq 7$.

Case: $P = (*,*,*,*,0,0)$ and $Q = (*,*,*,*,*,0)$

First we notice that Q can only have 5 vectors of weight 6 all of which contain four 1 entries. P only has 4 vectors of weight 6 with four 1 entries, so at most 4 of the weight 6 vectors can intersect. Now we look at which blocks can intersect. There are the blocks $(0,0,0,*,*,*)$ and $(0,*,0,0,*,*)$ each of which has 2 vectors in P and 1 vector in Q. Since there is only 1 of each in Q, and they combine with weight 6 vectors, they can only be pciked 1 way. So if these two blocks intersect in two vectors then 2 of the weight 6 vectors will not, bringing us back to an intersection of at most 4. Similarly, we have the blocks $(0,*,*,0,0,*)$, and $(*,0,0,*,0,*)$, which can only intesect at 1 vector as Q contains one of each, but, if they do, the corresponding weight 6 vectors will not. Now, in the construction of Q, the design $(0,0,*,*,*,0)$ can be matched with the designs $(*,*,0,0,*,0)$ and $(*,*,0,*,0,0)$ for 4 total vectors, meaning the two later blocks can only be picked 1 way, so if $(0,0,*,*,*,0)$ intersects, the other two blocks will not, bringing us to a total of 6 possible intersection vectors. This leaves the block $(0,*,*,*,0,0)$ which can intersect at 1 vector, so we conclude that $|A(P) \cap A(Q)| \leq 7$.

Case: $P = (*,*,0,0,0,0)$ and $Q = (0,0,*,0,0,0)$

We can only have the blocks with a non-zero third entry match, which, in the case of $A(P)$, are $(0,*,*,0,0,*)$ and $(*,0,*,0,0,*)$ for 4 vectors each, so $|A(P) \cap A(Q)| \leq 8$.

Case: $P = (*,*,*,*,0,0)$ and $Q = (0,0,0,*,0,0)$

We can only have the blocks with a non-zero fourth entry match, which, in the case of $A(P)$, are $(0,0,0,*,*,*)$, $(0,*,0,0,*,*)$, $(0,*,*,*,0,0)$, and $(*,*,0,0,*,0)$. All 4 blocks contribute two vectors to $A(P)$, so $|A(P) \cap A(Q)| \leq 8$.

Case: $P = (*,*,*,*,*,0)$ and $Q = (0,0,0,0,0,*)$

We can only have the blocks with a non-zero sixth entry match, which, in the case of $A(P)$, are all blocks that combine with a weight 6 vector. There are 5 of these block, and each

contributes only 1 vector, so $|A(P) \cap A(Q)| \leq 5$.
Case: $P = (*, *, *, *, 0, 0)$ and $Q = (0, 0, 0, *, *, 0)$
First we look at the weight 6 vectors, of which, based on the number of 1 entires, only 3 could intersect. Now we look at the blocks $(*, *, 0, *, 0, 0)$ and $(*, *, 0, 0, *, 0)$, which $A(Q)$ contains all 8 of, while $A(P)$ contains 3, so we are guaranteed an intersection of 3. There is also the block $(0, 0, *, *, *, 0)$ which contains 2 points in both sets that could possible intersect. This leaves us with the block $(0, 0, 0, *, *, *)$ which, in $A(P)$ combines with 2 weight 6 vectors, while, in A(Q), combines with itself. Points from this block could never intersect because the entries in the first 2 non-zero postions will always be opposite, as one combines with itself and the other with weight 6 vectors, so $|A(P) \cap A(Q)| \leq 8$.
Case: $P = (*, *, *, *, 0, 0)$ and $Q = (0, 0, 0, 0, *, *)$
Same as the previous case, we can have only 3 vectors of weight 6 match. The only two matching blocks are $(0, 0, 0, *, *, *)$ and $0, *, 0, 0, *, *$, both of which have 2 points in $A(P)$, so $|A(P) \cap A(Q)| \leq 7$.
Case: $P = (*, *, *, *, *, 0)$ and $Q = (0, 0, 0, 0, *, *)$
Since all weight 6 vectors contained in $A(P)$ have 4 one entires, we can only have 1 in the intersection. This leaves the matching block $(0, 0, 0, *, *, *)$, $(0, *, 0, 0, *, *)$, $(*, 0, *, 0, 0, *)$ all of which only contribute 1 point to $A(P)$ and the block $(*, 0, *, 0, *, 0)$ which contributes 2, so $|A(P) \cap A(Q)| \leq 6$.
Case: $P = (*, 0, 0, 0, 0, 0)$ and $Q = (0, *, 0, 0, 0, 0)$
$A(P)$ contains 5 blocks with a non-zero first entry, while $A(Q)$ contains 5 blocks with a non-zero second entry. Two of these blocks match up, so $|A(P) \cap A(Q)| \leq 8$.
Case: $P = (*, *, 0, 0, 0, 0)$ and $Q = (*, *, 0, 0, 0, 0)$
There are only two such points and the 8 weight 6 vectors that construct one are the compliment of the ones that construct the other. The same is true for the blocks $(*, *, 0, *, 0, 0)$ and $(*, *, 0, 0, *, 0)$ as they combine once with themselves, so two vectors combine to give P and the other two combine to give Q. This leaves us with the blocks $(0, *, *, 0, 0, *)$ and $(*, 0, *, 0, 0, *)$ both of which are fully contained in $A(P)$ and $A(Q)$, so $|A(P) \cap A(Q)| \leq 8$.
Case: $P = (*, *, 0, 0, 0, 0)$ and $Q = (0, *, *, 0, 0, 0)$
First we start with the matching blocks; we have $(0, *, *, 0, 0, *)$ which is fully contained in $A(P)$ and contributes 2 vectors to $A(Q)$, so those are in the intersection. The same happens with the block $(*, *, 0, 0, *, 0)$ which is fully contained within $A(Q)$ and contributes 2 vectors to $A(P)$, so they are in the intersection. Then we look at the 1 weight 6 vector with 0 one entries. If the $*$(s) in both P and Q are the same entry, then this vector is contained within the intersection, but, if such is the case, the vector with 4 one entries cannot intersect due to the different positionings of the 2 entries. This leaves 6 weight 6 vectors with 2 one entries. The only ones which could intersect start with 2s in the first 2 entries, of which there are at most 4 when the $*$s in P and Q are different and 3 when they are the same, so $|A(P) \cap A(Q)| \leq 8$.
Case: $P = (*, *, 0, 0, 0, 0)$ and $Q = (0, 0, *, *, 0, 0)$
The only matching block is $(*, 0, *, 0, 0, *)$ which is fully contained within $A(P)$ and $A(Q)$, so all of it intersects. The argument for which weight 6 vectors intersect is the same as the previous case, so $|A(P) \cap A(Q)| \leq 8$.
Case: $P = (*, 0, 0, 0, 0, 0)$ and $Q = (*, *, *, *, *, *)$
Since every block is used in the construction of $A(Q)$, then there will be exactly 5 blocks

which match for 1 point each, so $|A(P) \cap A(Q)| \leq 5$.
Case: P = weight 3/6 vectors and Q = weight 6/3 vectors
As was stated here [6], if since P and Q belong to the elliptic quadric, the intersection of the sets $A(P)$ and $A(Q)$ is the set of points in a solid given by the intersection of two hyperplanes each of which is a $(20, 8)$-set, so $|A(P) \cap A(Q)| \leq 8$.

□

6 Conclusion

While our proof is exhaustive and we hope it brings some insight to the sysmetries of the Hill cap, this problem is still lacking a simpler geometric proof. Ruther research can be conducted to obtain such an argument.

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