MA/ACM 144a HW 4. (Due Date: Dec. 11. 4PM)

Turn your HW into the homework box.

Homework Rules: Collaboration on problem sets is encouraged, but

a) Write up each problem independently.

b) Write on **each** problem with whom you consulted and the sources you used. If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

c) It is illegal to consult materials from previous quarters.

1. Suppose that U is a uniform random variable on $[0,1]$ and a coin with probability U of heads is minted. The coin is tossed repeatedly, and the outcomes are $X_1, X_2, \cdots$ with $X_n = 1$ denoting heads in the nth toss. Let $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$ and B_n denote the number of heads in the first n tosses. Find the conditional expectation $\mathbb{E}[U|\mathcal{F}_n]$.

(Hint: Compute $\mathbb{P}\{B_n = k | B_{n-1} = k-1\}$.)

2. Let $X_1, X_2, \cdots$ be independent nonnegative random variables with $\mathbb{E}[X_j] = 1$. Define $M_0 := 1$ and $M_n := \prod_{j=1}^n X_j$ for $n \in \mathbb{N}$. Then M is a nonnegative martingale, so that $M_\infty := \lim_{n\to\infty} M_n$ exists a.s. Show that if $\prod_{n=1}^\infty \mathbb{E}[X_n^{1/p}] > 0$ for some $p > 1$ then

$$M_n \to M_\infty \text{ in } L^1.$$

(Hint: Let $a_n := \mathbb{E}[X_n^{1/p}]$. Define $N_n = \prod_{j=1}^n X_j^{1/p}/a_j$. What can you say about N?)

3. (Exercise 5.3) Let $\Omega = [0,1), I_{k,n} = [k2^{-n}, (k+1)2^{-n}), \mathcal{F}_n = \sigma(I_{k,n} : 0 \le k < 2^n), \mathbb{P} =$ Lebesgue measure on Ω. Suppose f is measurable and Lipschitz continuous on Ω , i.e., $|f(t) - f(s)| \le K|t-s|$ for $0 \le s, t < 1$. Show that $X_n = (f((k+1)2^{-n}) - f(k2^{-n}))/2^{-n}$ on $I_{k,n}$ defines a martingale, $X_n \to X_\infty$ a.s. and in L^1, and

$$f(b) - f(a) = \int_a^b X_\infty(\omega)\, d\omega.$$

4. (Exercise 7.3) Let S_n be a symmetric simple random walk starting 0, and let $T = \inf\{n : S_n \notin (-a, a)\}$ where a is an integer. (i) Use the fact that $S_n^2 - n$ is a martingale to show that $\mathbb{E}[T] = a^2$. (ii) Find constants b and c so that $Y_n = S_n^4 - 6ns_n^2 + bn^2 + cn$ is a martingale and use this to compute $\mathbb{E}[T^2]$.

5. Suppose there are N job interviewees and let X_j be the jth interviewee's suitability for the job. We assume that X_j are i.i.d. uniformly distributed R.V.'s on $[0,1]$. The boss interviews each in turn and can determine the value of X_j perfectly. He must immediately decide whether to choose the interviewee or not. No recall of rejected interviewees is possible. Find a stopping time T that maximizes $\mathbb{E}[X_T]$. To do so, let $a_N = 0$ and $a_{n-1} = (1 + a_n^2)/2$ for $n = N-1, \cdots, 1$.

(a) Show that for any stopping time T, the process Y defined by

$$Y_0 = a_0 \text{ and } Y_n = X_n^T \vee a_n \text{ for } n \geq 1,$$

is a supermartingale. (Hint: Consider $\mathbb{E}[Y_n|\mathcal{F}_{n-1}]$ on $\{T \leq n-1\}$ and $\{T \geq n\}$, respectively.)

(b) For the stopping $T^* = \inf\{n > 0 : X_n > a_n\}$, the process Y defined above is a martingale.

(c) Show that for any stopping time T, $\mathbb{E}[X_T] \leq \mathbb{E}[X_{T^*}]$.

6. Consider a Markov Chain $\{X_n\}$ having probability transition matrix

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}.$$

Note that $\{X_n\}$ has stationary distribution $\pi = (1/3, 1/3, 1/3)$. Using the argument in the proof of the Basic Limit Theorem, show that, no matter what the initial distribution π_0 of X_0 is,

$$\|\pi_n - \pi\| \leq \frac{2}{3}\left(\frac{11}{16}\right)^n$$

for all n.