Problem 1.

Combine Turán’s theorem and the Szemerédi Regularity Lemma to prove the following: For every $H$ fixed with $\chi(H) = r + 1$ and $\epsilon > 0$, there exists an integer $n_0 = n_0(H, \beta)$ such that every $G_n$ with $n \geq n_0$ and satisfying $e(G_n) \geq (1 - \frac{1}{r} + \beta)(\epsilon n)^{r+1}$ contains a copy of $H$.

Hint: Use the Key Lemma with $d = \beta/2$, $\epsilon = (\beta/6)^{(r+1)t}$ for a suitable $t$.

Solution. As $\chi(H) = r + 1$, $H$ can be partitioned into $r+1$ independent sets. Let $t$ be the size of the greatest part. Then $H \subset K_{r+1}(t)$.

By Szemerédi Regularity Lemma, there exists an integer $T(\epsilon, \frac{1}{r})$ such that for any $G_n$ with $n \geq T$, there exists a partition $V = (V_0, V_1, \ldots, V_k)$ with $\frac{1}{r} \leq k \leq T$ such that $|V_0| \leq \epsilon n$, $V_1, \ldots, V_k$ have the same size $c$ and the number of non-$\epsilon$-regular pair is at most $\epsilon c^2$.

Purify this graph by deleting the following edges:

1. all edges connected with $V_0$: the number is $\leq \epsilon n \times n = \epsilon n^2$;
2. all internal edges in $V_1, \ldots, V_k$: the number is $\leq \frac{c^2}{2} \times k \leq \frac{1}{2r} n^2 \leq \frac{\epsilon}{2} n^2$;
3. all edges between non-$\epsilon$-regular pairs: the number is $\leq \epsilon k^2 \times c^2 \leq \epsilon n^2$;
4. all edges between pairs with density less than $d$: the number is $\leq d c^2 \times \frac{k^2}{2} \leq \frac{\epsilon}{2} n^2$.

After purification, the number of edges left is $\geq (1 - \frac{1}{r} + \beta)(\frac{n}{2}) - (\frac{\epsilon}{2} n^2 + \frac{\epsilon}{2} n^2) = (1 - \frac{1}{r} + \frac{\beta}{2}) \frac{n^2}{2}$.

Then, $e(R) \geq [(1 - \frac{1}{r} + \frac{\beta}{2}) \frac{n^2}{2}] / c^2 \geq [(1 - \frac{1}{r} + \frac{\beta}{2}) \frac{n^2}{2}] \times \frac{k^2}{2} = (1 - \frac{1}{r} + \frac{\beta}{2}) \frac{k^2}{2}$. By Turán’s theorem, $K_{r+1}(t) \subset R$.

Now, $H \subset K_{r+1}(t) \subset R(t)$, $\Delta(H) \leq (r + 1)t$, and $\delta = d - \epsilon = \frac{\beta}{2} - (\frac{\beta}{6})^{(r+1)t}$. If $\frac{\beta}{6}^{(r+1)t} = \epsilon \leq \epsilon_0 = \frac{\beta^{\Delta(H)}}{2 + \Delta(H)}$, $t - 1 \leq \epsilon_0 c$, then by the Key Lemma, $G_n$ contains a copy of $H$.

$\epsilon_0 \geq (\frac{\beta}{2} - (\frac{\beta}{6})^{(r+1)t}) \frac{(r+1)^t}{2 + (r+1)t} \geq \frac{(\frac{\beta}{2})^{(r+1)t}}{\epsilon^{(r+1)t}} = (\frac{\beta}{2 \times \epsilon})^{(r+1)t} \geq (\frac{\beta}{6})^{(r+1)t} = \epsilon$,

$\epsilon_0 \geq \frac{\epsilon_0 (1 - \epsilon)n}{k} \geq \epsilon_0 (1 - \epsilon)n \geq t - 1$ if $n \geq M(H, \beta)$. Therefore, if $n \geq n_0(H, \beta)$, we are done.

Problem 2.

Show that there exists $\delta = \delta(\epsilon) > 0$ so that if a graph $G_n$ is $\epsilon$-far from being triangle-free, then $G$ contains at least $\delta n^3$ triangles.
Solution. Let \( t = \lceil \frac{\epsilon}{4} \rceil \). Then by Szemerédi Regularity Lemma, there is a partition \( V = (V_0, V_1, \ldots, V_k) \) with \( t \leq k \leq M(\epsilon) \) such that \( |V_0| \leq \frac{\epsilon}{4} n \), \( V_1, \ldots, V_k \) have the same size \( c \) and the number of non-\( \frac{\epsilon}{4} \)-regular pairs is at most \( \frac{\epsilon}{4} k^2 \). Purify this graph by deleting the following edges:

- all edges connected with \( V_0 \): the number is \( \leq \frac{\epsilon}{4} n \times n = \frac{\epsilon}{4} n^2 \);
- all internal edges in \( V_1, \ldots, V_k \): the number is \( \leq \frac{c^2}{2} \times k \leq \frac{1}{2k} n^2 \leq \frac{1}{2} n^2 \leq \frac{\epsilon}{8} n^2 \);
- all edges between non-\( \frac{\epsilon}{4} \)-regular pairs: the number is \( \leq \frac{\epsilon}{4} k^2 \times c^2 \leq \frac{\epsilon}{4} n^2 \);
- all edges between pairs with density less than \( \frac{\epsilon}{2} \): the number is \( \leq \frac{\epsilon}{2} c^2 \times \frac{c^2}{2} \leq \frac{\epsilon}{4} n^2 \).

As our graph is \( \epsilon \)-far from being triangle-free, after purification, \( G \) is still not triangle-free. Assume that a triangle is formed among \( V_1, V_2, V_3 \). Then \( \{V_1, V_2\}, \{V_2, V_3\}, \{V_1, V_3\} \) are \( \frac{\epsilon}{4} \)-regular with density at least \( \frac{\epsilon}{2} \).

Let \( U_i = \{v \in V_i : |\Gamma(v) \cap V_i| < \frac{\epsilon}{4} c\} \), \( i = 2 \) or 3. By \( \frac{\epsilon}{4} \)-regularity, \( |U_i| < \frac{\epsilon}{4} c \) for both \( i = 2 \) and 3, implying that \( |(U_2 \cup U_3)^c| \geq (1 - \frac{\epsilon}{2}) c \).

For each \( v \in (U_2 \cup U_3)^c \), since \( |\Gamma(v) \cap V_i| \geq \frac{\epsilon}{4} c \) for both \( i = 2 \) and 3, by \( \frac{\epsilon}{4} \)-regularity, there are at least \( \frac{\epsilon}{4} \times (\frac{\epsilon}{4} c)^2 \) edges between \( \Gamma(v) \cap V_2 \) and \( \Gamma(v) \cap V_3 \), forming \( \frac{\epsilon^3}{64} c^2 \) triangles at vertex \( v \). As \( |(U_2 \cup U_3)^c| \geq (1 - \frac{\epsilon}{2}) c \), the number of triangles in \( G \) is at least \( \frac{\epsilon^3}{64} (1 - \frac{\epsilon}{2}) c^3 \geq \frac{\epsilon^3}{64} (1 - \frac{\epsilon}{2})(1 - \frac{\epsilon}{8}) n^3 \).

**Problem 3.**

Let \( \mathbb{A} \) be the set of all integers of the form \( 4^i + 4^j \) with \( 0 \leq i < j \). Show

1. Every \( n \in \mathbb{N} \) has at most 3 different representations of the form

\[ a_p + a_q \] with \( a_p < a_q \) and \( a_p, a_q \in \mathbb{A} \).

**Solution.** Every \( n \in \mathbb{N} \) has at most 1 way to express as \( n = 4^i + 4^j + 4^k + 4^l \), where \( i, j, k, l \) are nonnegative and not all equal. Once written in this form, there are at most 3 ways to be split into \( a_p \) and \( a_q \) in \( \mathbb{A} \), \( a_p < a_q \), such that \( n = a_p + a_q \).

2. For any partition \( \mathbb{A} := \mathbb{A}_1 \cup \mathbb{A}_2 \cup \cdots \cup \mathbb{A}_r \) there exists \( 1 \leq s \leq r \) such that infinitely many \( n \in \mathbb{N} \) can be written as \( a'_i + a'_j \) with \( a'_i < a'_j \), \( a'_i, a'_j \in \mathbb{A}_s \), in at least 3 different ways.
Solution. Let $4^\mathbb{N} = \{4^1, 4^2, \ldots\}$. Color \( \binom{4^\mathbb{N}}{2} \) with $r$ colors such that if $4^i + 4^j \in A_s$, then we color $\{4^i, 4^j\}$ with color $s$. By infinite Ramsey theory, there exists an infinite $S \subset 4^\mathbb{N}$ such that all pairs in $S$ have the same color $s$. Hence, if an $n \in \mathbb{N}$ can be written as $4^i + 4^j + 4^k + 4^l$, where $4^i, 4^j, 4^k, 4^l \in S$ are distinct, then this $n$ can be written as $a_p + a_q$ with $a_p < a_q$, $a_p, a_q \in A_s$, in at least 3 different ways. Note that there are infinitely many such $n$, so we are done.

Problem 4.
Show that there is an infinite subset $A := \{a_1 < a_2 < \cdots\}$ of $\mathbb{N}$ such that for all $i \neq j$, $a_i + a_j$ has an even number of prime factors counted without multiplicity.

Solution. Let $\mathbb{P}$ be the set of all primes. Color \( \binom{\mathbb{P}}{2} \) with 2 colors as follows: for any $\{p_i, p_j\} \subset \mathbb{P}$, if $p_i + p_j$ has an odd number of prime factors, then color it with 1; if $p_i + p_j$ has an even number of prime factors, then color it with 2. By infinite Ramsey theory, there exists an infinite $S \subset \mathbb{P}$ such that all pairs in $S$ have the same color. If all pairs in $S$ have color 1, note that either $T = \{p_i \in S : p_i \equiv 1 \pmod{3}\}$ or $U = \{p_i \in S : p_i \equiv 2 \pmod{3}\}$ is infinite, since we are only looking at primes. If $T$ is infinite, define $A = 3T$; if not, then define $A = 3U$, and we are done.

Problem 5.
Define the Cochromatic Number of a graph $G$ as the smallest size of a partition of the vertex set of $G$ in which each part is either an independent set or induces a complete subgraph in $G$. Suppose $\chi(G) = \omega(G) = n$. Then show that there exists a subgraph $H$ of $G$ whose cochromatic number is at least $\frac{n}{2\log_2 n}$.

Solution. There exists $k$ such that $R(k, k) \leq n < R(k + 1, k + 1)$. As $n < R(k + 1, k + 1)$, there exists $H \subset G$ such that all complete subgraphs and all independent sets are of size $\leq k$. Note that $R(k, k) > 2^k \Rightarrow 2\log_2 R(k, k) > k$. As a result, $co(H) =$ size of partition $\geq \frac{n}{k} > \frac{n}{2\log_2 R(k, k)} \geq \frac{n}{2\log_2 n}$. 
Problem 6.

Determine all \( k \in \mathbb{N} \) for which the following statement holds. There exists \( n_0(k) \in \mathbb{N} \) such that for all \( n \geq n_0(k) \), any connected graph on \( n \) vertices contains an \textbf{induced} subgraph \( H \) with \( e(H) = k \).

Solution. If \( k \neq \binom{r}{2} \) for any \( r \), then for all \( n \in \mathbb{N} \), \( K_n \) does not contain an induced subgraph \( H \) with \( e(H) = k \).

Now, assume \( k = \binom{r}{2} \) for some \( r \). If there exist vertices \( u, v \) such that the distance between \( u \) and \( v \) is \( k \), then the shortest path from \( u \) to \( v \) is an induced subgraph \( H \) with \( e(H) = k \) since there cannot be any shortcut edge along this path. If there exists a vertex \( u \) with degree at least \( R(r-1,k) \), then in \( \Gamma(u) \), there is either a clique of size \( r-1 \) or an independent set of size \( k \). A clique of size \( r-1 \) with form a clique of size \( r \) with vertex \( u \), obtaining an induced subgraph \( H \) with \( e(H) = k \).

An independent set of size \( k \) will form a star of size \( k+1 \) with vertex \( u \), obtaining an induced subgraph \( H \) with \( e(H) = k \).

If the diameter \( D \) of the graph is less than \( k \), and the maximum degree \( \Delta \) is less than \( R(r-1,k) \), then the number of vertices \( n \) will be less than \( 1 + \Delta + \Delta^2 + \cdots + \Delta^D \). Hence, if \( n \geq n_0(k) = 1 + R(r-1,k) + \cdots + R(r-1,k)^k \), then any connected graph on \( n \) vertices contains an induced subgraph \( H \) with \( e(H) = k \).