The Probabilistic Method

Due on or before Monday, 7 June, noon.
There are two bonus problem towards the end. These problem are a little harder, so partial answers here will get partial credit.

Problem 1: The Van der Waerden number\(^1\) \(W(2, k)\) is the least integer \(n\) such that the following holds. For any 2-coloring of \([n]\), there is a monochrome arithmetic progression of length \(k\). Show that

(a) \(W(2, k) > 2^{k/2}\).
(b) Improve this to \(W(2, k) > \frac{2^k}{2e}\).

Problem 2: Let \(G = (V, E)\) be a simple graph and suppose to each \(v \in V\) is associated, a set \(S(v)\) of colors of size at least \(10d\), where \(d \geq 1\). Suppose, in addition, that for each \(v \in V\) and \(c \in S(v)\) there are at most \(d\) neighbors \(u\) of \(v\) such that \(c\) lies in \(S(u)\). Prove that there is a proper coloring of \(G\) assigning to each vertex \(v\), a color from its class \(S(v)\).

Problem 3: Prove that there is a positive constant \(c\) so that every \(d\)-regular graph, where \(d \geq 2\), contains a spanning subgraph in which every connected component is a star with at least \(c \frac{d}{\log d}\) leaves.

Problem 4: Prove that there exists a positive constant \(\delta > 0\) and an integer \(n_0 = n_0(\delta)\) so that the following holds: For all \(n > n_0\) and every collection \(S_1, S_2, \ldots, S_m\) (with \(m \leq 2^n\)) of subsets of \([2n]\) satisfying \(|S_i| = n\) for all \(i\), there is a function \(f : [2n] \to [n]\) such that for every \(i, 1 \leq i \leq m\), we have \(0.63n \leq |f(S_i)| \leq 0.64n\). (Observe that \(0.63 < 1 - \frac{1}{e} < 0.64\).)

Problem 5: Let \(G_1, G_2, \ldots, G_m\) be \(m\) graphs on the same set of vertices \([n]\), and suppose that the chromatic number of each graph \(G_i\) is exactly \(k\). Show that there is a partition of the set of vertices \([n]\) into two disjoint sets \(A_1, A_2\) so that for every \(i, 1 \leq i \leq m\), and for every \(j, 1 \leq j \leq 2\), the chromatic number of the induced subgraph of \(G_i\) on \(A_j\) is at least \(k/2 - \sqrt{2k \log(2m)}\). (Hint: Use an appropriate martingale and show that with positive probability, a random partition will do the job.)

\(^1\)Van der Waerden proved that \(W(2, k) < \infty\), so the notion makes sense. His proof of this fact is a non-trivial inductive argument.
Bonus Problems:

Problem 1*: Show that for any $d$ there is $c(d)$ such that the edges of any bipartite graph with maximum degree $d$ in which every cycle has length at least $c(d)$ can be $(d + 1)$-colored so that no two adjacent edges have the same color and no cycle is 2-coloured.

Problem 2*: For a given $\epsilon > 0$, prove that there exists a constant $C = C(\epsilon)$ such that the following holds: For any set $S \subseteq \mathbb{F}_3^n$ with $|S| \geq \epsilon 3^n$, there exist $x, y, z$ in $S$ such that the (Hamming) distance between any pair of them is at least $n - C\sqrt{n}$. (Hint: Let $S_{\text{near}} := \{x \in \mathbb{F}_3^n | d(x, S) \leq C\sqrt{n}/2\}$. Set up a martingale and show that $|S_{\text{near}}| \geq (2/3)^n$.)