The Probabilistic Method

Due on or before Friday, 21 May.
There are two bonus problems towards the end. These problems are a little harder, so partial answers here will get partial credit.

**Problem 1:** Let $X$ be a random variable taking integral non-negative values and suppose $\mathbb{E}(X^2) < \infty$. Prove\(^\text{1}\) that
\[
\mathbb{P}(X = 0) \leq \frac{\mathbb{V}(X)}{\mathbb{E}(X^2)}.
\]

**Problem 2:** Let $v_1 := (x_1, y_1), v_2 := (x_2, y_2), \ldots, v_n = (x_n, y_n)$ be $n$ vectors with each $x_i, y_i \in \mathbb{Z}$. Suppose further, that $|x_i|, |y_i| \leq \frac{2n^2}{100\sqrt{n}}$ for all $i$. Prove that there exist disjoint sets $I, J \subseteq [n]$ such that
\[
\sum_{i \in I} v_i = \sum_{j \in J} v_j.
\]

**Problem 3:** By a cyclic interval in $\mathbb{F}_p$ of size $r$ for $r \in \mathbb{N}$, we mean a set of the form $\{a + 1, a + 2, \ldots, a + r\}$ for some $a \in \mathbb{F}_p$. Prove that for every set $X$ of at least $4k^2$ distinct elements in $\mathbb{F}_p$ for an odd prime $p$ there exist $a, b \in \mathbb{F}_p$ such that $aX + b := \{ax + b | x \in X\}$ intersects every cyclic interval in $\mathbb{F}_p$ of length at least $p/k$. (Hint: Consider a fixed partition $\mathcal{I}$ of $\mathbb{F}_p$ into $2k$ cyclic intervals of equal length. Observe that a set $X$ intersects every cyclic interval in $\mathbb{F}_p$ iff it intersects every $I \in \mathcal{I}$).

**Problem 4:** For an integer $m > 0$, let $\mathbb{Z}_m$ denote the set of integers modulo $m$. For any two subsets $A, B$ of $\mathbb{Z}_m$, and $x \in \mathbb{Z}_m$, denote
\[
s(A, B, x) := |\{(a, b)|a \in A, b \in B, a + b = x\}|.
\]

For any partition $(A, B)$ of $\mathbb{Z}_m$, denote
\[
c(A, B) := \max_{x \in \mathbb{Z}_m} |s(A, A, x) + s(B, B, x) - 2s(A, B, x)|.
\]

Prove that for every $m$ odd, there is a partition of $\mathbb{Z}_m$ into two (disjoint) parts $(A, B)$ such that $c(A, B) = O(\sqrt{m \log m})$.

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\(^1\)Note that this improves upon Chebyshev’s inequality as Chebyshev only proves $\mathbb{P}(X = 0) \leq \frac{\mathbb{V}(X)}{\mathbb{E}(X^2)}$.\n
Problem 5: Show that if $G = G_{n,1/2}$ is a random graph, then for any $\epsilon > 0$, asymptotically almost surely $G$ has no bipartite subgraph with at least $(1/2 + \epsilon)e(G)$ edges.

Bonus Problems:

Problem 1*: Let $G = (V, E)$ be a graph with $n$ vertices and minimum degree $\delta > 10$. Prove that there is a partition of $V$ into two disjoint subsets $A$ and $B$ such that $|A| = O(n \log \delta/\delta)$, and each vertex of $B$ has at least one neighbor in $A$ and at least one neighbor in $B$.

Problem 2*: Call a family $\mathcal{F}$ of subsets of $[n]$ Distinguishing if for every two distinct subsets $A, B$ of $[n]$ there exists $F \in \mathcal{F}$ so that $|A \cap F| \neq |B \cap F|$. Show that there exists such a distinguishing family $\mathcal{F}$ of $[n]$ of size $|\mathcal{F}| \leq (2 + o(1))\frac{n}{\log_3 n}$. 