The Probabilistic Method

Due Monday 26th April, Noon.

**Problem 1:** Suppose \( n \geq 4 \) and \( \mathcal{H} \) is an \( n \)-uniform hypergraph with at most \( \frac{4^n - 1}{3^n} \) edges. Show that one can color the vertices of \( \mathcal{H} \) using 4 colors such that every edge has at least one vertex of each color.

**Problem 2:** Prove that there is an absolute constant \( c > 0 \) with the following property. Let \( A \) be an \( n \times n \) matrix with pairwise distinct entries. Then there is a permutation of the rows of \( A \) so that no column in the permuted matrix contains an increasing subsequence of length at least \( c \sqrt{n} \).

**Problem 3:** If \( \{(A_i, B_i)\}_{1 \leq i \leq h} \) is a family of pairs of subsets of a set \( S \) such that

(i) \( |A_i| = k, |B_i| = l \) for all \( i \).

(ii) For all \( i \), we have \( A_i \cap B_i = \emptyset \).

(iii) \( (A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset \) if \( i \neq j \).

Show that \( h \leq \frac{(k+l)^k+l}{k^l} \).

**Problem 4** Let \( k > 1 \) and suppose \( G \) is a graph without loops satisfying \( e(G) = m, \Delta(G) \leq 2k - 1 \). Show that there exists a bipartite subgraph \( H \subset G \) such that \( e(H) \geq \frac{km}{2k - 1} \).

**Problem 5** A tournament \( T_n \) with \( n \) players is said to possess an *Absolute Winner-Loser Pair* (abbreviate AWLP) of size \( k \) if there are disjoint sets of players \( A, B \) such that \( |A| = |B| = k \) and every player of \( A \) beats every player of \( B \). Prove that for any given integer \( k \) and sufficiently large \( n \), every tournament with \( n \) players has an AWLP of size \( k \). In fact, if \( n(k) \) denotes the minimum \( n \) such that any tournament on \( n \geq n(k) \) players has an AWLP of size \( k \) then \( n(k) \leq k(2 + o(1))^k \).

**Problem 6:** Prove that there is a positive constant \( c \) so that for all \( n > 1 \) and every \( n \)-uniform hypergraph \( \mathcal{H} \) with at most \( cn^{1/4}2n \) edges, there is an ordering of the vertices of \( \mathcal{H} \) such that there are no two edges \( A \) and \( B \) that intersect in a unique element, and all members of \( A \setminus B \) precede all those of \( B \setminus A \), while the unique element in \( A \cap B \) appears

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1. Hint: See theorem 1.3.3 of the text for a similar problem.
after all those of $A \setminus B$ and before all those of $B \setminus A$. Hence conclude that for $c, n$, and $\mathcal{H}$ as above, $\mathcal{H}$ is two-colorable.\footnote{Do NOT use the Beck or R-S improvement for the lower bound for $m(n)$.