Incentive Contract for a Long-term Project with Moral Hazard

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Abstract

In this paper, I study the optimal contract problem when a firm faces a long-term project. In many cases, a firm cannot pursue several projects simultaneously because of limited resources. Particularly, the firm might need to choose between long and short-term projects. I consider a short-term project as one that generates an instantaneous profit to the firm without any effect on the future, as analyzed by DeMarzo and Sannikov (2006). The firm can also invest in a long-term project: one that requires an indefinite amount of time to complete its objective. I assume that the long-term project generates profits once it is accomplished. Using a continuous-time moral hazard model, I characterize the incentive compatibility condition in a relatively general contracting space. Moreover, I find a unique optimal contract under a restricted contracting space which consists of the two components, the termination level and the completion payment. Comparison of optimal contracts for long and short-term projects provides an interesting insight to managerial short-termism: the firm not the agent could prefer a short-term project to a long-term project if there is a moral hazard problem.

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1 Introduction

In many cases, a firm’s project requires long-term investment or R&D before it starts generating a positive cash flow to the firm. For example, an automobile company invests its resources in order to develop a new model, and the new car provides a profit after the development is completed. A hired manager is in charge of such a project. However, it is difficult for investors to observe the manager’s effort level. The manager may enjoy some private benefits instead of exerting effort. In this paper, I analyze the optimal contract problem under a moral hazard setting where an agent controls the investment process for a project.

I study a continuous-time moral hazard model in order to examine the optimal contract. This modeling is desirable since it is tractable and easily comparable to the existing literature. Specifically, I model a long-term investment by an arithmetic Brownian motion. An agent’s effort choice is reflected in the drift term in the investment process. That is, the agent’s effort helps the investment process complete more quickly on average. This investment does not generate any cash flow before it reaches a fixed threshold. However, once the project is completed, it delivers a stream of positive cash flow to the investors without agent’s further effort.

First, I characterize the incentive compatibility condition in a general contracting space. Also, I study two essential elements of the incentive contract, the completion payment and the termination level. However, the presence of two state variables, the agents continuation value and the investment level, makes the model intractable. Hence, I focus on a restricted contracting space for the baseline model. In the baseline model, contracting space is restricted to include only the termination level and the final completion payment fixed at the beginning of the contract. Under these restrictions, I find a unique incentive-compatible contract maximizing the investors’ profit. Despite the inherent limitation of the contract, it can serve as a benchmark for a more complex contracting space. In addition, I extend the baseline model to allow for one-time adjustment of the termination level with an intermediate compensation. In my extended model, the payoff for the investors slightly increases compared to the baseline model.

Moreover, I compare my model with DeMarzo and Sannikov (2006). In their model, the agent directly controls the drift of the cash flow process. I will call such projects as short-
term projects hereafter since such projects do not require any time interval between the agent’s effort and cash flow. Under a comparison rule, I compare the expected profit the investors obtain from the two different types of projects. More specifically, I fix a set of parameters that make the two projects yield the same profit when there is no information asymmetry. However, the comparison of two projects with moral hazard problem shows that the short-term project can be preferred to the long-term project by the investors. This implies that the short-termism (often criticized as a moral hazard problem of agent) can actually arise for the sake of the principal. I expect that the results can provide a new insight to the short-termism issue in the literature.

The remainder of the paper is as follows. Section 2 reviews the related literature. Section 3 analyzes the agent’s incentive compatibility and essential components of the incentive-compatible contract under a general contracting space. In Section 4, I examine the optimal contract under a restricted contracting space and provide some comparative statics. Section 5 compares the optimal contract for long and short-term projects. Section 6 concludes.

2 Related Literature

This paper is closely related to several streams of the literature. First of all, this paper builds on the literature of a hidden-action principal-agent problem, introduced by Hölmstrom (1979). Among many works in this literature, Spear and Srivastava (1987) and Rogerson (1985) are related to my paper. They analyze a dynamic principal-agent model in discrete-time setting. Since the seminal work of Sannikov (2008), many researchers follow the novel technique to analyze an agency problem in continuous-time setting. For example, DeMarzo and Sannikov (2006), Zhu (2013b), Biais et al. (2007), and He (2009) have used the methodology in order to analyze a dynamic principal-agent model where the agent controls a cash flow process. While the existing literature has full tractability since they consider only one state variable, my model requires two state variables, which makes the problem intractable. One notable exception is

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2Cvitanić et al. (2009) use a stochastic maximum principle approach in order to characterize optimal contract in a similar setting.
Cvitanic et al. (2013) which analyzes moral hazard and adverse selection in continuous-time setting. In this paper, they consider two states variables, the agent’s continuation value and temptation value.

Also, there is a literature on optimal contracting problem for a long-term project. For example, Zhu (2013a) studies a myopic agency problem where there exists a tension between a short-term benefit and long-term cost. On the other hand, Sannikov (2013) studies the situation where the agent’s effort has a persistent effect on the future output. My modeling of long term project is closely related to Georgiadis (2014). However, Georgiadis (2014) focuses more on agents’ behavior in a group although he also considers a simple contracting problem. In this paper, I consider an extended contracting space compared to the one in Georgiadis (2014).

Another closely related research area is experimentation. For instance, Manso (2011), Hörner and Samuelson (2013), and Guo (2014) study the contracting problem when players do not know the profitability of a risky project. The main difference between this literature and my model is that players know the quality or profitability of the project in my model. However, the profitability is unknown in the literature of experimentation. While the agent’s past behavior is reflected in the posterior belief on the quality of the project in the experimentation literature, it is directly reflected in the current investment level in my model. Hence, the two models have different implications. If players’ main concern is the unknown quality of project, the experimentation model would be more appropriate. However, if the main concern is the accumulated effort or development to complete a project, my model specification would be more suitable.

### 3 The Model

I consider a continuous-time principal-agent model, where a principal or investors need to hire an agent in order to operate an investment process or a R&D process. If the principal decides not to hire the agent, both players receive their reservation values. The firm’s cash flow process evolves according to

\[ dY_t = \kappa \mathbb{1}_{\{\tau = \tau_0, \tau \leq t\}} dt, \]
where $\kappa$ is a constant, and $\tau$ is a stopping time depending on the investment process. Specifically, the stopping time $\tau = \min[\tau_u, \tau_d]$ is decided by the investment process $\{I_t\}_{0 \leq t \leq \tau}$ such that

$$\tau_u = \inf\{s | I_s \geq \bar{I} \text{ for } s \in [0, \infty)\} \text{ and } \tau_d = \inf\{s | I_s \leq I(H_s) \text{ for } s \in [0, \infty)\},$$

where $\bar{I}$ is exogenously given but $I(t)$ is determined by investors. Also, $H_s$ denotes a history of investment process until time $s$. Therefore, $\tau$ is a $I$-measurable stopping time. $\bar{I}$ and $I(H_t)$ represent the completion level and the termination level of the investment, respectively. Therefore, the term $\kappa \mathbb{1}_{\tau = \tau_u}$ means that if the investment process reaches the completion level before the principal terminates the project, the successful project starts generating cash flow at rate $\kappa$ from the moment without any agency problem. Note that $I(H_t)$ could be a negative infinity for every $H_t$. In this case, the principal never terminates the investment. In this paper, I model the publicly observable investment process by the following arithmetic Brownian motion

$$dI_t = a_t dt + \sigma dZ_t,$$

where $\sigma$ is a constant and $Z = \{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard Brownian motion. The drift term $a_t \in \{0, \mu\}$, where $\mu > 0$, is decided by the agent’s binary effort choice. Each choice gives a different cost to the agent. If the agent chooses “shirking” ($a_t = 0$), then she enjoys private benefit $\phi dt$ for each time $t$. On the other hand, if she chooses “working” ($a_t = \mu$), then there is no private benefit.

Under this environment, a contract $\Gamma = (C, \bar{I}, B, A)$ specifies a cumulative intermediate compensation $C = \{C_t\}_{t \geq 0}$ to the agent, a lower bound $\underline{I}$, the bonus payment $B = (B_u, B_d)$ at time $\tau$, where $B_u$ is compensated to the agent if $\tau = \tau_u$ and $B_d$ is provided if $\tau = \tau_d$, and a recommended effort process $A = \{a_t\}_{t \geq 0}$. All four components are adapted to $I$.

Two players, the principal and the agent, are both risk-neutral. The principal or investors discount the future at rate $r > 0$, and the agent discounts at $\rho > r$. The agent is protected by limited liability. This implies $dC_t \geq 0$ for all $t$ and $B \geq 0$. For simplicity, assume that both players’ reservation values are 0. Also, I assume that investors possess full bargaining power.

In this paper, I say that a contract $\Gamma$ is incentive-compatible if it induces the agent to work
until completion or termination. That is, a contract \( \Gamma \) is incentive-compatible if \( \mathcal{A} = \{a_t = \mu \}_{0 \leq t < \tau} \) is a solution to the following agent’s problem:

\[
\max_{a = \{a_t \in \{0, \mu\} \mid 0 \leq t < \tau\}} E^a \left[ \int_0^\tau e^{-\rho t} \left( dC_t + \phi \left( 1 - \frac{a_t}{\mu} \right) dt \right) + e^{-\rho \tau} (\mathcal{B}_u \mathbb{1}_{\{\tau = \tau_u\}} + \mathcal{B}_d \mathbb{1}_{\{\tau = \tau_d\}}) \right].
\]

Note that the expectation depends on the effort process \( a = \{a_t \in \{0, \mu\} \mid 0 \leq t < \tau\} \). From now on, I suppress \( a \) in the expectation operator if the effort process is \( \mathcal{A} = \{a_t = \mu \}_{0 \leq t < \tau} \) for brevity. Moreover, I assume that parameters \( \kappa \) and \( \phi \) satisfies \( \kappa > \phi \). This is a necessary condition for the incentive-compatible contract to be socially optimal.

The principal’s problem is to find an incentive-compatible contract \( \Gamma \) maximizing his discounted expected profit

\[
E \left[ -\int_0^\tau e^{-\rho t} dC_t + e^{-\rho \tau} \left( \frac{\kappa}{\rho} \mathbb{1}_{\{\tau = \tau_u\}} - \mathcal{B}_u \mathbb{1}_{\{\tau = \tau_u\}} - \mathcal{B}_d \mathbb{1}_{\{\tau = \tau_d\}} \right) \right] - C_0,
\]

where a constant \( C_0 \) is the setup cost for the project. Note that if \( \phi = 0 \), the principal can achieve the first best profit by choosing \( I = -\infty \), \( \{C_t = 0\}_{0 \leq t < \tau} \), and \( \mathcal{B}_u = \mathcal{B}_d = 0 \), and the agent always exerts effort until the completion.\(^3\) This policy gives the profit

\[
\exp \left( -\mu + \frac{\sqrt{\mu^2 + 2r \sigma^2}}{\sigma^2} (I - I_0) \right) \frac{\kappa}{\rho} - C_0
\]

to the principal and the reservation value to the agent. From now on, I call this profit the first best profit.

4 Incentive Compatibility

In this section, I characterize the agent’s incentive compatibility condition and two essential components of the optimal incentive-compatible contract, \( I \) and \( \mathcal{B}_u \). Among two components, \( \mathcal{B}_u \) does not appear in DeMarzo and Sannikov (2006) since the contract is only terminated when the agent’s continuation value reaches zero in their problem. On the other hand, the possibility of a finite \( I \) is not considered in Georgiadis (2014).

\(^3\)I implicitly assume that the agent works if both actions give the same utility to the agent.
Before I analyze the incentive compatibility condition, I put some restriction on the choice of $C$ and $I$ in order to obtain tractability. Specifically, I only allow a finite number of intermediate compensation and a finite number of termination level updating based on the investment level. Denote $K_c$ and $K_d$ as the number of intermediate compensation and the number of termination level updating. Then, $i$-th intermediate compensation $C^i$ is provided according to the threshold $I_{c,i}$ such that

$$dC_t = C^i \text{ if } t = \inf\{s | I_s \geq I_{c,i} \text{ for } s \in [\tau_{c,i-1}, \infty)\} \text{ and } t < \inf\{s | I_s \leq I(s) \text{ for } s \in [0, \infty)\},$$

where

$$\tau_{c,i} \equiv \inf\{s | I_s \geq I_{c,i} \text{ for } s \in [\tau_{c,i-1}, \infty)\} \text{ for } i = 1, 2, 3, \ldots, K_c \text{ and } \tau_{c,0} = 0.$$

On the other hand, the $j$-th termination level $I_j$ is adjusted by the thresholds $I_{d,j}$ such that

$$I(H_t) = I_j \text{ if } t = \inf\{s | I_s \geq I_{d,j} \text{ for } s \in [\tau_{d,j-1}, \infty)\} \text{ and } t < \inf\{s | I_s \leq I(s) \text{ for } s \in [0, \infty)\},$$

where

$$\tau_{d,j} \equiv \inf\{s | I_s \geq I_{d,j} \text{ for } s \in [\tau_{d,j-1}, \infty)\} \text{ for } i = 1, 2, 3, \ldots, K_d \text{ and } \tau_{d,0} = 0.$$

Later, I demonstrate that the second restriction is closely related to the first restriction. For brevity, I define a threshold $I_{u,k} \in \{I_{c,1}, \ldots, I_{c,K_c}, I_{d,1}, \ldots, I_{d,K_d}\}$ such that

$$dC_t = C^k, \text{ and } I(H_t) = I_k$$

when

$$t = \inf\{s | I_s \geq I_{u,k} \text{ for } s \in [\tau_{u,k-1}, \infty)\} \text{ and } t < \inf\{s | I_s \leq I(H_s) \text{ for } s \in [0, \infty)\},$$

where

$$\tau_{u,k} \equiv \inf\{s | I_s \geq I_{u,k} \text{ for } s \in [\tau_{u,k-1}, \infty)\} \text{ for } i = 1, 2, 3, \ldots, K_u \text{ and } \tau_{u,0} = 0$$

allowing $C^k = 0$ and $I_k = I_{k+1}$, where $K_u$ is the number of any intermediate compensation or termination level updating. Therefore, $K_u \leq K_c + K_d$. Therefore, $I_{u,k}$ represents the updating point of the intermediate compensation or the termination level.
First, I analyze the agent’s problem and find the incentive-compatible condition. I denote the agent’s continuation value at time $t$ by $W(I_t | I, C, B, a)$. That is,
\[
W(I_t | I, C, B, a) = E_t^a \left[ \int_t^\tau e^{-\rho(s-t)} \left( dC_s + \phi \left( 1 - \frac{a_s}{\mu} \right) ds \right) + e^{-\rho \tau} \left( B_u \mathbb{1}_{\{\tau = \tau_u\}} + B_d \mathbb{1}_{\{\tau = \tau_d\}} \right) \right].
\]

I can rewrite this continuation value using $\tau_{u,k}$. Denote $W_k(I_t) = W^k(I_t | I, C, B, a)$ as the agent’s continuation value for $I_t \in [I_{u,k-1}, I_{u,k})$. That is,
\[
W_k(I_t) = E_t^a \left[ \int_t^{\hat{\tau}} e^{-\rho(s-t)} \phi \left( 1 - \frac{a_s}{\mu} \right) ds + e^{-\rho \hat{\tau}} \left( (C^{k+1} + W^{k+1}(I_{u,k})) \mathbb{1}_{\{\hat{\tau} = \tau_{u,k}\}} + B_d \mathbb{1}_{\{\hat{\tau} = \tau_d\}} \right) \right],
\]
where $\hat{\tau} = \min[\tau_{u,k}, \tau_d]$ is a stopping time.

The agent chooses her effort level maximizing her continuation value each time $t$. This maximization problem satisfies the Hamilton-Jacobi-Bellman equation
\[
\rho W^k dt = \max_a \left[ \phi \left( 1 - \frac{a}{\mu} \right) dt + a \frac{\partial W^k}{\partial I} dt + \frac{1}{2} \sigma^2 \frac{\partial^2 W^k}{\partial I^2} dt \right]
\]
subject to the boundary conditions
\[
W^k(I_{k-1}) = B_d \text{ and } W^k(I_{u,k}) = C^{k+1} + W^{k+1}(I_{u,k})
\]
for $k = 1, 2, \ldots, K_u$.

I can characterize the incentive-compatibility condition using the equation.

**Proposition 1** The contract $\Gamma$ is incentive-compatible if
\[
\frac{\partial W^k}{\partial I} \geq \frac{\phi}{\mu} \text{ for every } I_{k-1} < I < I_{u,k} \text{ and } k = 1, 2, \ldots, K_u.
\]

Intuitively, if the agent shirks at time $t$, she obtains a private benefit $\phi dt$. However, she loses $\mu W_t dt$ since the drift term is 0. Hence, by setting $\mu \partial W^k / \partial I \geq \phi$, the principal can incentivize the agent. Combining the incentive-compatibility condition and the limited liability condition yields the following proposition.

**Proposition 2** There is no incentive-compatible contract satisfying $B_u = 0$. 7
This says that the principal has to compensate for the completion of the investment in order to incentivize the agent. Unless the principal provides the payment for completion, the agent has an incentive to delay the completion since she can only enjoy the private benefit before the completion. Note that the limited liability excludes a negative payment which can make the agent incentivized with $B_u = 0$. In addition to this bonus payment, Proposition 3 shows that a finite lower bound $I$ is the other essential component for the incentive compatibility and a positive profit to investors.

**Proposition 3** The optimal incentive-compatible contract $\Gamma$ satisfies $I_k > -\infty$ for every $k$.

Although the probability of completion is equal to one if $I(H_t) = \infty$ for every $H_t$, it is not optimal for the principal to set $I(H_t)$ as such. If the current investment level is really low at time $t$, it takes long time to complete the investment. Therefore, for the agent, it would be better to enjoy the private benefit than to exert effort to complete the project unless she will be compensated big enough in the future. However, the compensation that makes the agent work yields a negative profit to the principal at the very low level $I$. Also, the investment level can get to the low level with a positive probability. Hence, the principal cannot achieve both objectives, incentive-compatibility and positive profit, simultaneously if the principal does not set a finite termination level.

**Proposition 2** and 3 indicate that the optimal incentive-compatible contract has to include a positive $B_u$ and a finite $I(H_t)$ for every $t$. Generally, those $B_u$ and $I(H_t)$ can change as the investment level and the agent’s continuation value change. However, this general case is difficult to analyze since this problem requires to solve a partial differential equation instead of an ordinary differential equation. Therefore, I characterize the incentive compatibility condition by imposing some restrictions on the dependence of $C$ and $I$ upon the investment level.

## 5 Lower Bound

It is difficult to find a general optimal contract since the principal’s value function depends on two state variables $I$ and $W$. In this section, I restrict the contracting space and find the optimal
contract under that restriction. Formally, I restrict the contracting space to \( \Gamma = (I, B, \mathcal{A}) \) such that \( I \) and \( B \) are constant. That is, I do not allow any intermediate compensation, adjustable lower bound, and adjustable bonus payment. This contract can be interpreted as a lower bound for the general optimal contract since the restricted contracting space includes two essential components in the simplest way. Although this is very restrictive, the contract could be close to the optimal one in some cases. For instance, companies may not have enough cash or budget to provide any intermediate compensation. Also, they may not be able to update the initial agreement for some reasons. In these cases, investors may focus on the final payment or fix the termination level at the beginning of employment.

5.1 Optimal Contract under the Restricted Contract Space

Under this restricted contracting space, the agent’s continuation value is

\[
W_t(I, B, a) = E_t^a \left[ \int_t^\tau e^{-\rho s} \left( 1 - \frac{\alpha_s}{\mu} \right) ds + e^{-\rho \tau} (B_u\mathbb{1}_{\{\tau=\tau_u\}} + B_d\mathbb{1}_{\{\tau=\tau_d\}}) \right].
\]

If the effort process is \( A = \{a_t = \mu\}_{0 \leq t < \tau} \), \( W_t(I, B, a) \) can be written as

\[
W_t(I, B, A) = \frac{\exp(\eta^-I + \eta^+I_t) - \exp(\eta^-I + \eta^+I)}{\exp(\eta^-I + \eta^+I) - \exp(\eta^-I + \eta^+I)} B_u + \frac{\exp(\eta^-I + \eta^+I) - \exp(\eta^-I + \eta^+I)}{\exp(\eta^-I + \eta^+I) - \exp(\eta^-I + \eta^+I)} B_d,
\]

where

\[
\eta^- = -\mu - \sqrt{\frac{\mu^2 + 2\rho \sigma^2}{\sigma^2}} \quad \text{and} \quad \eta^+ = -\mu + \sqrt{\frac{\mu^2 + 2\rho \sigma^2}{\sigma^2}}.
\]

This equation enables one to find the final payment in a closed form. I characterize it later.

Under this condition, the principal’s problem is expressed by

\[
\max_{(I, B_u, B_d)} E \left[ e^{-\tau\rho} \left( \frac{\kappa}{r} \mathbb{1}_{\{\tau=\tau_u\}} - B_u\mathbb{1}_{\{\tau=\tau_u\}} - B_d\mathbb{1}_{\{\tau=\tau_d\}} \right) \right]
\]

\[
= \max_{(L, B_u, B_d)} \left[ \frac{\exp(\nu^-I + \nu^+I_0) - \exp(\nu^-I_0 + \nu^+I)}{\exp(\nu^-I + \nu^+I) - \exp(\nu^-I + \nu^+I)} \left( \frac{\kappa}{r} - B_u \right) \right]
\]

\[
- \frac{\exp(\nu^-I_0 + \nu^+I) - \exp(\nu^-I + \nu^+I_0)}{\exp(\nu^-I + \nu^+I) - \exp(\nu^-I + \nu^+I)} B_d,
\]

4The related mathematical result is stated in Appendix B.
where
\[ \nu^- = \frac{-\mu - \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}, \quad \nu^+ = \frac{-\mu + \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}, \]

and IC means the set of \((L, B_u, B_d)\) satisfying the incentive-compatibility condition. Since the setup cost does not affect the principal’s choice of the optimal contract if he hires the agent, I suppress \(C_0\) in this section.\(^5\) Now, I characterize each component of the optimal contract.

**Lemma 1** The optimal \(B_d\) is equal to zero.

When the principal can’t adjust the lower bound and bonus payments, \(B_d\) only makes the incentivization more difficult since \(B_d\) gives an incentive to shirk. Hence, I define the principal’s choice set as \((L, B_u)\) without loss of generality.

**Lemma 2** For given \(L\), the optimal bonus payment is
\[
B_u(L) = \frac{\exp(\eta^- L + \eta^+ \bar{I}) - \exp(\eta^- \bar{I} + \eta^+ L)}{\eta^+ \exp(\eta^- L + \eta^+ \bar{I}) - \eta^- \exp(\eta^- \bar{I} + \eta^+ L)} \frac{\phi}{\mu},
\]

where
\[
I^* = \min \left[ \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) + \bar{L}, \bar{I} \right].
\]

In order to incentivize the agent with any \(I\) on \([L, \bar{I}]\), the slope of the continuation value with respect to \(I\) must be greater than \(\frac{\phi}{\mu}\) for all \(I \in [L, \bar{I}]\) by Proposition 1. For given \(L\), the unique completion payment comes from the strict convexity of \(\frac{\partial W}{\partial I}\) and the compactness of \([L, \bar{I}]\). Here, one can see why the lower bound is essential part for the optimal contract. For a given \(L\), \(\frac{\partial W}{\partial I}\) is minimized at the point \(I^*\). If \(L\) approaches to negative infinity, \(I^*\) also goes to negative infinity. That is, there is no finite completion payment \(B_u\) providing incentives since \(B_u\) goes to positive infinity as \(I\) approaches to negative infinity. It is worth mentioning that \(I^*\) is strictly greater than \(L\). This means that \(\frac{\partial W}{\partial I}\) is strictly greater than \(\frac{\phi}{\mu}\) for \(I \in [L, I^*)\). That is, for \(I\) in that region, the completion payment is provided more than needed to incentivize the agent at each point. This means that when \(I\) decreases the principal can incentivize the agent

\(^5\)Unless the principal’s discounted expected profit is greater than \(C_0\), the principal does not hire the agent. However, if he decides to hire the agent, \(C_0\) does not affect the choice of \(L\) and \(B\) since \(C_0\) is a sunk cost. Hence, without loss of generality, I ignore \(C_0\) or assume \(C_0 = 0\) in the analysis.
without increasing the completion payment in this region. On the other hand, for $I \in (I^*, \tilde{I})$, the principal has to increase the completion payment as $I$ decreases. Mathematically, this is reflected in the convexity of $W$ for $I \in (I^*, \tilde{I})$, while $W$ is concave in the region, $[\tilde{I}, I^*)$. Therefore, when the investment level falls close to $\tilde{I}$, there is a “self-incentive” effect. This effect does not arise if $I = -\infty$. In this case, $W$ is convex on the whole region. Now, the principal’s problem is reduced to find the optimal $I$ maximizing his discounted expected profit at time 0.

**Proposition 4** For given parameters $(r > 0, \rho > r, \mu > 0, \sigma > 0, \kappa > 0, \phi, \tilde{I}, I_0)$ such that

$$\frac{\kappa}{r} - B_u(I_0) > 0,$$

(1)

there exists a unique optimal contract $(\tilde{I}, B_u)$ providing a positive expected discounted profit to the principal. In this contract, $B_u$ satisfies

$$B_u(I) = \frac{\exp(\eta^- I + \eta^+ \tilde{I}) - \exp(\eta^- \tilde{I} + \eta^+ I)}{\eta^+ \exp(\eta^- I + \eta^+ I^*) - \eta^- \exp(\eta^- I^* + \eta^+ I)} \frac{\phi}{\mu},$$

where

$$I^* = \min \left[ \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) + \tilde{I}, \tilde{I} \right],$$

and $\tilde{I}$ is the solution to the equation

$$\mathcal{P}'(\tilde{I}) \left( \frac{\kappa}{r} - B_u(\tilde{I}) \right) - \mathcal{P}(\tilde{I}) \frac{\partial B_u(\tilde{I})}{\partial \tilde{I}} = 0,$$

where

$$\mathcal{P}(\tilde{I}) = \frac{\exp(\nu^- I + \nu^+ I_0) - \exp(\nu^- I_0 + \nu^+ I)}{\exp(\nu^- I + \nu^+ I) - \exp(\nu^- I_0 + \nu^+ I)},$$

and

$$\mathcal{P}'(\tilde{I}) = -\frac{2 \sqrt{\mu^2 + 2r \sigma^2}}{\sigma^2} \exp \left( -\frac{2\mu}{\sigma^2} \right) \frac{\exp(\nu^- I_0 + \nu^+ \tilde{I}) - \exp(\nu^- \tilde{I} + \nu^+ I_0)}{[\exp(\nu^- I + \nu^+ I) - \exp(\nu^- I_0 + \nu^+ I)]^2}.$$

If the condition (1) does not hold, there is no incentive-compatible contract providing a positive profit to the principal.

In the remaining paper, I call this contract the baseline contract. **Figure 1** shows the optimal choice of $I$ when $(r = 0.1, \rho = 0.15, \mu = 10, \sigma = 3, \kappa = 15, \phi = 3, I_0 = 0, \tilde{I} = 50)$.  

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Although I drop the possibility of an intermediate payment to the agent in the baseline contract, this restriction is not severe if the principal can’t adjust the termination level in the middle of investment for some reason. For example, if the verification of investment performance incurs some cost, the principal may want to avoid a frequent update of the termination level. The following corollary formally states that the intermediate compensation is closely related to the adjustment of the termination level.

**Corollary 1** For a fixed $I$, a finite intermediate payment is not optimal in the set of incentive-compatible contract.

**Corollary 1** has two implications. First, it says that if the principal can’t adjust the termination level after the initiation of the contract, he has to focus on the completion payment. Intuitively, the intermediate compensation has no role for incentivization after it is paid to the agent. Hence, the principal has to provide $B_u(I)$ eventually in order to fully incentivize the agent. This means that the intermediate compensation only increases the principal’s cost if the termination level is fixed. Second, this implies that if the principal provides an intermediate
compensation, he has to modify the termination level. In subsection 5.3, I examine how the contract changes in this case.

### 5.2 Comparative Statics

In this section, I analyze some comparative statics of the optimal contract under the restricted contracting space as in the previous section. The main interests are the optimal termination level, the completion probability, and the players’ discounted expected utilities. First, I define the completion probability as a function of $I_0$ and $I$. Formally,

$$P(I_0, I) = Pr(\tau = \tau_0 | I_0) = \frac{\exp(-\delta I) - \exp(-\delta I_0)}{\exp(-\delta I) - \exp(-\delta I)},$$

where $\delta = \frac{2\mu}{\sigma}$. Table 1 summarizes the analytical findings of comparative statics.

<table>
<thead>
<tr>
<th></th>
<th>$\partial I^*$</th>
<th>$\partial U_0$</th>
<th>$\partial W_0$</th>
<th>$\partial P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial \phi$</td>
<td>$+$</td>
<td>$-$</td>
<td>$\pm$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\partial I_0$</td>
<td>$+$</td>
<td>$+$</td>
<td>$\pm$</td>
<td>$\pm$</td>
</tr>
</tbody>
</table>

Table 1: Comparative Statics for the Baseline Contract

It is particularly interesting to understand the effect of the private benefit and the initial investment level because they play an important role in the following section where I compare my model with another model based on a different specification. First, I analyze the effect of the private benefit the agent can enjoy. As the benefit increases, the optimal termination level increases and principal’s expected profit decreases. The intuition is clear. The completion probability also decreases as the termination level increases. On the other hand, the agent’s expected utility can go either way. Note that as $\phi$ increases above a certain level, the expected profit is negative and the agent will not be employed in the first place. Therefore, if $\phi$ is big enough, the agent’s expected utility will be zero. For $\phi = 0$, it gives zero utility to the agent while the principal obtains the first-best profit. Hence, the effect depends on the value of $\phi$.

---

6See Appendix B for the related mathematical result.
Second, I also examine the effect of the initial investment level \( I_0 \). As the initial investment level gets close to \( \bar{I} \), the project is getting closer to the completion. Therefore, the optimal termination level and the principal’s expected profit increase. On the other hand, the direction of the agent’s discounted expected utility and completion probability can be positive or negative. The completion probability decreases if the principal increases the optimal termination level significantly compared to the increment of the initial investment level. The decrease of the agent’s utility can be demonstrated indirectly through the principal’s first-best profit. Denote the first-best value of the profit by \( FB(\mu, I_0) \). That is,

\[
FB(\mu, I_0) = \exp \left( -\nu^+(\bar{I} - I_0) \right). \]

Then, doing simple calculations gives

\[
\frac{\partial FB(\mu, I_0)}{\partial \mu} = \frac{1}{\sqrt{\mu^2 + 2r\sigma^2}}\nu^+(\bar{I} - I_0)FB(\mu, I_0) > 0, \quad \text{and} \\
\frac{\partial^2 FB(\mu, I_0)}{\partial \mu \partial I_0} = \frac{1}{\sqrt{\mu^2 + 2r\sigma^2}}\nu^+FB(\mu, I_0)(\nu^+(\bar{I} - I_0) - 1) < 0 \quad \text{if and only if} \quad \nu^+(\bar{I} - I_0) < 1.
\]

As one can see, the effect of \( \mu \) decreases if the initial level is really close to \( \bar{I} \). This implies that the role of the agent’s effort decreases when the initial level is close to \( \bar{I} \). Hence, the agent’s utility can decrease. Note that, in the extreme case, \( I_0 = \bar{I} \), there is no need to hire the agent for the completion. Hence, in this case, the agent obtains zero utility. Figure 2 and Figure 3 illustrate the comparative static results for the case \( (r = 0.1, \rho = 0.15, \mu = 10, \sigma = 3, \kappa = 15, \phi = 3, I_0 = 0, \bar{I} = 50) \) varying parameters \( I_0 \) and \( \phi \). One thing to note is that the agent’s discounted expected utility sharply decreases when \( I_0 \) is really close to \( \bar{I} \). From this result, I can anticipate that the agent prefers the project which is moderately far from the completion level. That is, she may prefer the project taking more time to the one that is closer to the completion. However, she does not prefer a project which is far from the completion. This result can be connected to the short-termism problem in the literature; agents prefer a moderately short-term project compared to a long-term project, but they prefer a moderately short-term project to a very short-term project.
Figure 2: Comparative Statics with respect to $I_0$

Figure 3: Comparative Statics with respect to $\phi$
5.3 One Step Further

In this section, I extend the contracting space. In contrast to the baseline contract, I allow one time adjustment of the termination level with an intermediate compensation denoted by $C$, which can be zero. In this case, the principal’s problem is much more complex since the final payment and the intermediate compensation depend on three thresholds $(\tilde{I}, I_1, I_2)$, where $\tilde{I}$, $I_1$, and $I_2$ denote the updating point, the first termination level, and the updated termination level, respectively. More specifically, the first termination level is set at $I_1$. However, if the investment level reaches $\tilde{I}$ before it drops to the first termination level, the termination level is adjusted to $I_2$ paying the intermediate compensation $C$ to the agent. The following proposition shows the existence of the optimal contract and illustrates the optimal payments as the function of three threshold levels.

**Proposition 5** Suppose that parameters $(r > 0, \rho > r, \mu > 0, \sigma > 0, \kappa > 0, \phi, \tilde{I}, I_0)$ satisfy the condition

$$\frac{\kappa}{r} - B_u(I_0) > 0.$$ 

Then, there exists a solution $(\tilde{I}, I_1, I_2, C, B)$ to the principal’s problem, and the optimal $B$ and $C$ are given as follows:

$$\begin{align*}
B(I_2) &\equiv \frac{\exp(\eta^- I_2 + \eta^+ \tilde{I}) - \exp(\eta^- \tilde{I} + \eta^+ I_2)}{\eta^+ \exp(\eta^- I_2 + \eta^+ I^{**}) - \eta^- \exp(\eta^- I^{**} + \eta^+ I_2)} \phi \\
C(\tilde{I}, I_1, I_2) &\equiv \max \left[ \frac{\phi}{\mu} \frac{\exp(\eta^- I_1 + \eta^+ \tilde{I}) - \exp(\eta^- \tilde{I} + \eta^+ I_1)}{\eta^+ \exp(\eta^- I_1 + \eta^+ I^*) - \eta^- \exp(\eta^- I^* + \eta^+ I_1)} - \tilde{P}_A(\tilde{I}, I_1, I_2)B(I_2), 0 \right],
\end{align*}$$

where

$$I^{**} = \min \left[ \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) + I_2, \tilde{I} \right],$$

and

$$I^* = \min \left[ \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) + I_1, \tilde{I} \right].$$

From now on, I call this contract the extended contract. Note that this contract includes the baseline contract since setting $\tilde{I} = \bar{I}$, $I_1 = I^*$, and $I_2 = \bar{I}$ gives the same profit to the
principal as \((I^*, B_u(I^*))\). Hence, this extended contract must provide at least the same profit as the baseline contract. Figure 4 illustrates the principal’s discounted expected profit when \((r = 0.1, \rho = 0.15, \mu = 10, \sigma = 3, \kappa = 15, \phi = 3, \bar{I} = 50)\) varying \(I_0\) from 0 to 50.

![Figure 4: The baseline and extended contracts - Principal](image)

This example shows that the extended contract provides slightly higher profit to the principal. According to the numerical example, the extended contract gives 1.29 percent higher profit to the principal when \(I_0\) is equal to zero. At this point, the optimal \((\hat{I}, L_1, L_2)\) is \((24.2084, -2.7348, 20.3077)\), and \((C, B)\) is \((8.1294, 29.1332)\). On the other hand, the baseline contract is \((I, B_u) = (-2.6729, 41.0295)\) with the same parameters. Also, the extended contract provides 2.85 percent higher profit to the principal when \(I_0 = 48.78\) with \((\hat{I}, L_1, L_2) = (49.3877, 47.1187, 47.7064)\) and \((C, B) = (0, 10.7773)\). At this point, the baseline contract is \((I, B_u) = (47.1143, 16.3975)\). Surprisingly, the principal can achieve higher profit by adjusting the termination level without any intermediate payment. However, in this case, two termination levels should be very close in order to fully incentivize the agent. For the future research, it would be interesting to analyze a more extended case where frequent updating is allowed.
6 Comparison with DeMarzo and Sannikov (2006)

DeMarzo and Sannikov (2006) find the optimal contract when the agent controls the drift of a cash flow process. That is, the cash flow process evolves as

\[ dY_t = \hat{a}_t dt + \sigma d\tilde{Z}_t, \]

where \( \hat{a}_t \) could be \( \hat{\mu} \) or zero according to the agent’s effort choice and \( \mathcal{Z} = \{ \tilde{Z}_t, \tilde{F}_t; 0 \leq t < \infty \} \) is a standard Brownian motion. If the agent shirks \( \hat{a}_t = 0 \), she enjoys the private benefit \( \phi dt \).

The following Proposition rephrases the result in DeMarzo and Sannikov (2006), which characterizes the optimal contract implementing high effort until the termination.

**Proposition 6 (Proposition 1 and 7 in DeMarzo and Sannikov (2006))**  The contract that maximizes the principal’s profit and delivers the value \( W_0 \in [0, \hat{W}] \) to the agent takes the following form: \( W_t \) evolves according to:

\[ dW_t = \rho W_t dt - dC_t - \phi \left( 1 - \frac{\hat{a}_t}{\hat{\mu}} \right) + \phi (dY_t - \hat{\mu}dt). \]

When \( W_t \in [0, \hat{W}] \), \( dC_t = 0 \). When \( W_t = \hat{W} \), payments \( dC_t \) cause \( W_t \) to reflect at \( \hat{W} \). If \( W_0 > \hat{W} \), an immediate payment \( W_0 - \hat{W} \) is made. The contract is terminated at time \( \tau \) when \( W_t \) reaches 0. The principal’s expected payoff at any point is given by a concave function \( b(W_t) \), which satisfies

\[ rb(W) = \hat{\mu} + \rho Wb'(W) + \frac{1}{2} \phi^2 \dot{\sigma}^2 b''(W) \]

on the interval \([0, \hat{W}]\), \( b'(W) = -1 \) for \( W \geq \hat{W} \), and boundary conditions \( b(0) = 0 \) and \( rb(\hat{W}) = \hat{\mu} - \rho \hat{W} \).

In contrast to my model, DeMarzo and Sannikov (2006) consider a project in which the agent controls the cash flow directly. For instance, in an automobile company, a manager may control sales or production of existing models. Such tasks can be understood as a short-term project. On the other hand, development of a new model is considered as a long-term
investment project. If the company faces limited resources, it should decide between short-
term and long-term projects. Hence, it is worth comparing two different types of projects to
see which project the firm will pursue.

The difference between DeMarzo and Sannikov (2006) and my model has an important
implication. There are two main differences. First of all, the current investment level \( I_t \) plays a
key role in my model. It measures the distance to the completion level. The lower the current
level is, the longer the project remains incomplete on average. Secondly, the long-term project
can be ended by both completion and termination. If the long-term project is completed, then
it generates the cash flow without an agent. Hence, the firm does not suffer from an agency
problem after it finishes the project. On the other hand, the firm pursuing a short-term project
constantly encounters the agency problem unless they fire the agent giving up the additional
cash flow. Hence, one can see that there is a tension between the length of agency problem and
its intensity when a principal chooses its project. Specifically, if \( I_0 \) is closer to \( \bar{I} \), the expected
time to completion will be shorter. On the other hand, if \( I_0 \) is far from \( \bar{I} \), it is expected to take
a long time to finish the project. Therefore, the length of agency problem is determined by the
distance between \( I_0 \) and \( \bar{I} \). However, in the long term project, the agency problem can be more
severe. Since the agent can’t enjoy the private benefit once the project is completed, the agent
has higher incentive to shirk delaying the completion of the project. When it comes to the
short-term project, this type of incentive does not exist since the contract runs out only when
the firm fires the agent. This implies that the long-term project can be worse than the short-
term project with respect to the agency problem. Based on these implications, I numerically
compare two projects from the principal’s perspective in the following.

Before I compare the two contracts, I need to decide how to specify parameters because the
comparison depends on the way I set the parameters. First, note that parameters \( (r, \rho, \mu(\hat{\mu}), \phi, \sigma(\hat{\sigma})) \)
appear in both specifications. In order to reduce complexity in comparison, I use the same val-
ues of those parameters for both models. Note that the long-term project also depends on four
additional variables; \( \kappa, C_0, I_0, \) and \( \bar{I} \). Since only the distance between \( I_0 \) and \( \bar{I} \) matters, I fix
\( I_0 \) as zero without loss of generality. After that, I specify \( \kappa \) and \( C_0 \) according to the following
The equation (2) says that both specifications provide the same profit to the principal when there is no moral hazard problem. On the other hand, the equation (3) means that both projects provide the same profit if \( a_t = \hat{a}_t = 0 \) all the time. That is, if the principal can operate each project without a manager or agent, both projects give the same profit. These two conditions make the two project comparable in the sense that they have the same net present value ignoring the cost and the benefit related to the agent.

Figure 5, Figure 6, and Figure 7 show the numerical results based on parameters \( (r = 0.1, \rho = 0.15, \sigma = \tilde{\sigma} = 12, \mu = \hat{\mu} = 10, I_0 = 0) \) varying the degree of agency problem \( (\phi) \) for three different \( \bar{I} \), 10, 50, and 100. The results show that the long-term project could provide higher profit to the principal when the completion level is close enough to the initial level and the agent’s private benefit is not high enough. However, if \( \bar{I} \) is very far from the initial investment level, the long-term project provides lower profit than the short-term for all \( \phi \) values. These results provide one important implication regarding “short-termism” : If one take into account the agency problem, the short-term project could be the best choice for the principal or investors.

This comparison has two critical limitations. First of all, the parameter \( \sigma \) \( (\tilde{\sigma}) \) does not have the same effect on both specifications. In DeMarzo and Sannikov (2006), \( \tilde{\sigma} \) only affects the principal negatively since it reflects the unobservability of the agent’s action. On the other hand, in my specification, higher \( \sigma \) can provide higher profit to the principal because a higher volatility can help the investment process to reach the completion level. This property is reflected in the equation (3). That is, the long-term project provides a positive profit to the principal although the drift term is equal to zero if there is no setup cost. This is different from the short-term project which gives zero profit if the drift term of cash flow process is equal to zero. The other limitation arises from the setup cost \( C_0 \). If \( C_0 \) is fixed, the principal’s profit is a strictly increasing function in \( I_0 \) by the comparative static result. However, if \( C_0 \) satis-
fied the condition (3), the principal’s profit can decrease as $I_0$ increases since the setup cost is also an increasing function in $I_0$. Figure 8 shows that this could happen under the parameters $(r = 0.1, \rho = 0.15, \sigma = 12, \mu = 10, \bar{I} = 100)$ varying $I_0$ from 0 to $\bar{I}$.

In summary, two observations (a higher $\sigma$ can increase the principal’s profit and $I_0$ closer to $\bar{I}$ can decrease the principal’s profit) make the interpretation not clear. Nonetheless, this result can provide a research direction regarding the short-termism issues.

![Figure 5: Comparison between the short-term and long-term contract when $\bar{I} = 10$.](image-url)
Figure 6: Comparison between the short-term and long-term contract when $I = 50$.

Figure 7: Comparison between the short-term and long-term contract when $I = 100$. 
7 Conclusion

In this paper, I examine the optimal contract problem when the agent controls a long-term investment process. I characterize the incentive compatibility condition in the general contracting space. The characterization shows that there are two essential components for the optimal incentive-compatible contract, a termination level and a completion payment. Based on these results, I find the optimal contract under a restricted contracting space which includes the two components in a tractable way. This result shows that the principal can obtain a positive profit while fully incentivizing the agent. Moreover, the comparative static results demonstrate that the agent prefers a project which is moderately far from the completion level while the principal always prefers the project closer to the completion. Also, I extend the contracting space by adding a one-time intermediate compensation and termination level updating. The numerical results show that this extension makes the principal slightly better off. Finally, I compare my result with DeMarzo and Sannikov (2006). This comparison gives an interesting insight regarding the short-termism problem. That is, the principal herself could prefer a short-term
project to a long-term project if there is an agency problem.

In the future research, a more complex or general contracting space can be considered. This analysis can provide a tighter comparison between a long-term and short-term project. Also, one can incorporate a repeated relation between the principal and the agent. That is, after the completion of the long-term project, the principal may re-hire the manager by assigning another long-term investment or short-term project to her. I expect that a repeated relation would reduce the agency cost. In the aspect of model specification of a long-term project, one can introduce a time-varying cost or $\kappa$. These settings will make the problem more difficult since the value of the project changes over time. Another interesting direction is to combine my model with the short-term project model such that an agent can assign her time or effort between two projects in order to maximize her utility according to the contract. To characterize optimal contract with multi-tasks is an interesting topic for the future research.
Appendix A  Proofs

A.1 Proof of Proposition 1

If the agent chooses $a = 0$,

$$\rho W^k dt = \phi dt + \frac{1}{2} \sigma^2 \frac{\partial W^k}{\partial I^2} dt.$$  

On the other hand, choosing $a = \mu$ yields

$$\rho W^k dt = \mu \frac{\partial W^k}{\partial I} dt + \frac{1}{2} \sigma^2 \frac{\partial W^k}{\partial I^2} dt.$$  

Therefore, the contract is incentive-compatible if

$$\mu \frac{\partial W^k}{\partial I} dt \geq \phi dt$$

for every $I$ and $k$.

A.2 Proof of Proposition 2

By the Proposition 1, the incentive-compatible contract must satisfy

$$\frac{\partial W^k}{\partial I} \geq \frac{\phi}{\mu} \text{ for all } I \in (I_{k-1}, I_{u,k}).$$

Also, the limited liability condition implies that $W(I) \geq 0$ for all $I \in [I, \bar{I}]$. Note that the payment $B_u$ is the same with $W(\bar{I})$. Suppose that $W(\bar{I}) = 0$. The condition $\partial W^k(I)/\partial I \geq \phi/\mu > 0$, for all $I \in (I_{k-1}, I_{u,k})$ and $k$, implies that $W(I) < 0$ for $I < \bar{I}$. This violates the limited liability condition.

A.3 Proof of Proposition 3.

It is enough to show that the principal can’t incentivize the agent by transferring all output from the project to the agent since it is the maximum transfer for the principal to the agent.
without loss. First, suppose that $I_k = -\infty$, then every $-\infty < I_t < I_{u,k+1}$ can be reached with a positive probability. Hence, if there exists $I_t$ such that

$$E_t^\varepsilon \left[ e^{-\rho t \kappa \frac{\mu}{\rho}} | I_t \right] < E_t^\varepsilon \left[ \int_t^\tau e^{-\rho(s-t)} \phi ds + e^{-\rho t \kappa \frac{\mu}{\rho}} | I_t \right],$$

where $\varepsilon = \{a_s = \mu\}_{t \leq s \leq \tau}$ and $\hat{\varepsilon} = \{a_s = 0\}_{t \leq s \leq \tau}$, then the proof is done. The difference between them is:

$$E_t^\varepsilon \left[ e^{-\rho t \kappa \frac{\mu}{\rho}} | I_t \right] - E_t^\varepsilon \left[ \int_t^\tau e^{-\rho(s-t)} \phi ds + e^{-\rho t \kappa \frac{\mu}{\rho}} | I_t \right] = e^{-\eta^+(I-I_t) \kappa \frac{\mu}{\rho}} - e^{-\frac{\varepsilon + \eta^+ I_{t}}{\eta^-(I-I_t)} \left( \kappa - \phi \right) - \frac{\phi}{\rho}}. $$

Since $\lim_{I_t \to -\infty} f(I_t) = -\phi/\rho < 0$ and $f(I) = 0$, there is $I_t$ such that $f(I_t) < 0$.

### A.4 Proof of Lemma 1

Since

$$\frac{\eta^- \exp(\eta^- I_t + \eta^+ \bar{I}) - \eta^+ \exp(\eta^- \bar{I} + \eta^+ I_t)}{\exp(\eta^- I + \eta^+ \bar{I}) - \exp(\eta^- \bar{I} + \eta^+ I)} B_d < 0$$

for $B_d > 0$ and $\underline{I} \leq I_t \leq \bar{I}$,

$$\left| \frac{\partial W_t(I_t, B_u, B_d)}{\partial I_t} \right|_{B_d=0} > \left| \frac{\partial W_t(I_t, B_u, B_d)}{\partial I_t} \right|_{B_d>0}.$$

Therefore, by setting $B_d = 0$, the principal can achieve the incentive compatibility condition with strictly lower $B_u$ for every $\underline{I} \leq I_t \leq \bar{I}$.

### A.5 Proof of Lemma 2

For given $\underline{I}$, the principal has to set

$$B_u \geq \frac{\exp(\eta^- \underline{I} + \eta^+ \bar{I}) - \exp(\eta^- \bar{I} + \eta^+ \underline{I})}{\eta^+ \exp(\eta^- \underline{I} + \eta^+ \bar{I}^*) - \eta^- \exp(\eta^- \bar{I}^* + \eta^+ \underline{I})} \frac{\phi}{\mu},$$

where

$$I^* = \min \left[ \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} + \underline{I}, \bar{I} \right) \right].$$
in order to satisfy the incentive compatibility condition for all $I \leq I_t \leq \bar{I}$. Notice that
\[
\frac{\partial^2 W}{\partial I^2} = \frac{(\eta^+)^2 \exp(\eta^- I + \eta^+ I) - (\eta^-)^2 \exp(\eta^- I + \eta^+ I)}{\exp(\eta^- I + \eta^+ I) - \exp(\eta^- I + \eta^+ I)} B_u.
\]
This second derivative is equal to zero if
\[
I = \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) + \bar{I} \equiv I^{**}.
\]
Also, the second derivative is strictly greater (less) than 0 if $I > (<) I^{**}$.

Since the agent’s expected discounted profit
\[
E \left[ e^{-rT} \left( \kappa \mathbb{1}_{\{\tau = \tau_u\}} - B_u \mathbb{1}_{\{\tau = \tau_u\}} \right) \right] = \frac{\exp(\nu^- I + \nu^+ I_0) - \exp(\nu^- I_0 + \nu^+ I)}{\exp(\nu^- I + \nu^+ I) - \exp(\nu^- I + \nu^+ I)} \left( \frac{\kappa}{r} - B_u \right)
\]
is a strictly decreasing function in $B_u$, it is optimal for him to set $B_u$ as the minimum value in the set of $B_u$’s satisfying the incentive compatibility condition.

A.6 Proof of Proposition 4

Recall that the principal’s problem is
\[
\max_{I} \mathcal{P}(I) \left( \frac{\kappa}{r} - B_u(I) \right),
\]
where
\[
\mathcal{P}(I) = \frac{\exp(\nu^- I + \nu^+ I_0) - \exp(\nu^- I_0 + \nu^+ I)}{\exp(\nu^- I + \nu^+ I) - \exp(\nu^- I + \nu^+ I)} \quad \text{and}
\]
\[
B_u(I) = \frac{\exp(\eta^- I + \eta^+ \bar{I}) - \exp(\eta^- \bar{I} + \eta^+ I)}{\eta^+ \exp(\eta^- I + \eta^+ I^*) - \eta^- \exp(\eta^- I^* + \eta^+ I) \mu} \phi.
\]

First, I show that there is a unique $I^{**}$ such that if $I < I^{**}$, the principal’s discounted expected profit is less than 0. There are two possibilities. When $I^* = \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) + \bar{I}$, I can rewrite $B_u(I)$ by
\[
B_u(I) = \frac{\exp(\eta^+ (\bar{I} - \bar{I})) - \exp(\eta^- (\bar{I} - \bar{I}))}{(\eta^- - \eta^+) \frac{\eta^-}{\eta^+} \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) \frac{\eta^+}{\eta^-} \mu} \phi.
\]
Notice that this is a strictly decreasing function in $\underline{I}$ and it is equal to zero when $\underline{I} = \bar{I}$. Therefore, there is a unique $I^{**}$ such that $B_u(I^{**}) = \frac{\kappa}{r}$. Therefore, choosing $\underline{I}$ lower than $I^{**}$ gives a negative profit to the principal. On the other hand, if $I^* = \bar{I}$,

$$B_u(I) = \frac{1 - \exp(-(\eta^+ - \eta^-)(\bar{I} - L))}{\eta^+ - \eta^- \exp(-(\eta^+ - \eta^-)(\bar{I} - L))} \mu$$

is a strictly decreasing function in $L$ and $B_u = 0$ when $\bar{I} = L$. Therefore, $B_u$ has the maximum value when

$$\underline{I} = \bar{I} - \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right).$$

After some algebra, I can obtain the maximum value of $B_u(L)$

$$B_u = \frac{\phi}{\rho}$$

Hence, when $I^* = \bar{I}$, $B_u(I)$ is always strictly less than $\frac{\kappa}{r}$ since I assume that $\kappa > \phi$ and $\rho > r$.

Second, $B_u(I)$ must be greater than $B_u(I_0)$ since $L$ can’t be greater than $I_0$ and $B_u$ is a strictly decreasing function in $L$. From now on, I focus on the range $I^{**} \leq L \leq I_0$ without loss of generality.

There are three possible cases.

1. $(\frac{\kappa}{r} - B_u(I_0) \leq 0)$

   In this case, it is impossible for the principal to achieve a positive utility regardless of the choice of $\underline{I}$ since

   $$\frac{\partial B_u(I)}{\partial L} = \begin{cases} -\frac{\phi \exp(\eta^+ I + \eta^+ \bar{I} - \eta^- \exp(\eta^- I + \eta^+ I))}{\eta^+ \exp(\eta^- I + \eta^+ \bar{I} - \eta^- \exp(\eta^- I + \eta^+ I))} < 0 & \text{if } I^* \neq \bar{I} \\ -\frac{\phi e^{-\frac{\mu}{2} (I + L)} \frac{1 - (\eta^+ - \eta^-) \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right)}{\eta^+ \exp(\eta^- I + \eta^+ I) - \eta^- \exp(\eta^- I + \eta^+ I)}} < 0 & \text{if } I^* = \bar{I} \end{cases}$$

2. $(\frac{\kappa}{r} - B_u(I_0) > 0$ and $\bar{I} - \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) \geq I_0)$

   In this case, $I^* = \underline{I} + \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right)$ for all $\underline{I} \in [I^{**}, I_0]$. Under this circumstance, there is a unique $\underline{I}$ maximizing the principal’s discounted expected profit since

   $$\frac{\partial U_0(L)}{\partial \underline{I}} = \mathcal{P}(L) \left( \frac{\kappa}{r} - B_u(L) \right) - \mathcal{P}(L) \frac{\partial B_u(L)}{\partial \underline{I}} = \begin{cases} \mathcal{P}'(I_0) \left( \frac{\kappa}{r} - B_u(I_0) \right) < 0 & \text{if } \underline{I} = I_0 \\ -\mathcal{P}(I^{**}) \frac{\partial B_u(L)}{\partial \underline{I}} \bigg|_{L=I^{**}} > 0 & \text{if } \underline{I} = I^{**} \end{cases}$$
and
\[
\frac{\partial^2 U_0(I)}{\partial I^2} = \mathcal{P}''(I) \left( \frac{\kappa}{r} - \mathcal{B}_u(I) \right) - 2\mathcal{P}'(I) \frac{\partial \mathcal{B}_u(I)}{\partial I} - \mathcal{P}(I) \frac{\partial^2 \mathcal{B}_u(I)}{\partial I^2} < 0
\]
for \( I^* < I < I_0 \), where
\[
U_0(I) = \mathcal{P}(I) \left( \frac{\kappa}{r} - \mathcal{B}_u(I) \right),
\]
\[
\mathcal{P}'(I) = -\frac{2\sqrt{\mu^2 + 2r^2}}{\sigma^2} \exp \left( -\frac{2\mu}{\sigma^2 I} \right) \frac{\exp(\nu I_0 + \nu^+ I) - \exp(\nu I_0 - \nu^+ I)}{[\exp(\nu I_0 + \nu^+ I) - \exp(\nu I_0 + \nu^+ I)]^2},
\]
\[
\mathcal{P}''(I) = -\frac{4\mu^2 + 2r^2}{\sigma^4} \exp \left( -\frac{2\mu}{\sigma^2 I} \right) \frac{\exp(\nu I + \nu^+ I) + \exp(\nu I - \nu^+ I)}{[\exp(\nu I + \nu^+ I) - \exp(\nu I - \nu^+ I)]^2},
\]
\[
\frac{\partial^2 \mathcal{B}_u(I)}{\partial I^2} = -\frac{\phi (\eta^-)^2 \exp(\eta I + \eta^+ I) - (\eta^+)^2 \exp(\eta I + \eta^+ I)}{\mu \eta^+ \exp(\eta I + \eta^+ I) - \eta^- \exp(\eta I + \eta^+ I)}.\]

3. \( \left( \frac{\kappa}{r} - \mathcal{B}_u(I_0) > 0 \right) \) and \( I - \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) < I_0 \)

Now, \( I^* \) could be \( I + \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) \) or \( \bar{I} \) depending on \( I \). In order to analyze this situation, consider the following Lemma.

**Lemma 3** Denote \( \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) \) by \( \mathcal{T} \). When \( \frac{\kappa}{r} - \mathcal{B}_u(I_0) > 0 \), if there exists \( \bar{I} \in (\bar{I} - \mathcal{T}, I_0) \) such that
\[
\left. \frac{\partial U_0(I)}{\partial I} \right|_{I=\bar{I}} = 0,
\]
then
\[
\left. \frac{\partial^2 U_0(I)}{\partial I^2} \right|_{I=\bar{I}} < 0.
\]

**Proof:** By the condition \( \left. \frac{\partial U_0(I)}{\partial I} \right|_{I=\bar{I}} = 0, \)
\[
\mathcal{P}'(\bar{I}) \left( \frac{\kappa}{r} - \mathcal{B}_u(\bar{I}) \right) - \mathcal{P}(\bar{I}) \mathcal{B}'_u(\bar{I}) = 0.
\]
Therefore,
\[
\frac{\partial^2 U_0(I)}{\partial I^2} \bigg|_{I=\bar{I}} = \mathcal{P}''(\bar{I}) \left( \frac{\kappa}{r} - B_u(\bar{I}) \right) - 2\mathcal{P}'(\bar{I})B'_u(\bar{I}) - \mathcal{P}(\bar{I})B''_u(\bar{I})
\]

\[
= \mathcal{P}'' \frac{\mathcal{P}(\bar{I})B'_u(\bar{I})}{\mathcal{P}'(\bar{I})} - 2\mathcal{P}'(\bar{I})B'_u(\bar{I}) - \mathcal{P}(\bar{I})B''_u(\bar{I})
\]

\[
= -2\mathcal{P}(\bar{I})B'_u(\bar{I}) \frac{1}{\sigma^2} \left( \sqrt{\mu^2 + 2\rho^2} \frac{1 + \exp((\nu^+ - \nu^-)(I_0 - \bar{I}))}{1 - \exp((\nu^+ - \nu^-)(I_0 - \bar{I}))} + \sqrt{\mu^2 + 2\rho^2} \eta^+ + \eta^- \exp(-\eta^+ - \eta^-)(\bar{I} - \bar{I}) \right).
\]

I need to show that
\[
\sqrt{\mu^2 + 2\rho^2} \frac{1 + \exp((\nu^+ - \nu^-)(I_0 - \bar{I}))}{1 - \exp((\nu^+ - \nu^-)(I_0 - \bar{I}))} + \sqrt{\mu^2 + 2\rho^2} \eta^+ + \eta^- \exp(-\eta^+ - \eta^-)(\bar{I} - \bar{I}) < 0.
\]

Since the LHS is a strictly decreasing function in \(\bar{I}\), it is enough to show that
\[
\sqrt{\mu^2 + 2\rho^2} \frac{1 + \exp((\nu^+ - \nu^-)(I_0 - \bar{I} + \bar{T}))}{1 - \exp((\nu^+ - \nu^-)(I_0 - \bar{I} + \bar{T}))} + \sqrt{\mu^2 + 2\rho^2} \eta^+ + \eta^- \exp(-\eta^+ - \eta^-)(\bar{I} - \bar{I}) \frac{1 + \exp((\nu^+ - \nu^-)(I_0 - \bar{I} + \bar{T}))}{1 - \exp((\nu^+ - \nu^-)(I_0 - \bar{I} + \bar{T}))} + \mu
\]

\[
< 0,
\]

where the last inequality holds since
\[
\sqrt{\mu^2 + 2\rho^2} > \mu \quad \text{and} \quad \frac{1 + \exp((\nu^+ - \nu^-)(I_0 - \bar{I} + \bar{T}))}{1 - \exp((\nu^+ - \nu^-)(I_0 - \bar{I} + \bar{T}))} < -1.
\]

Since \(\frac{\partial U_0(I)}{\partial I} \bigg|_{I=I_0} = \mathcal{P}'(I_0) \left( \frac{\kappa}{r} - B_u(I_0) \right) < 0\) and \(U_0(I)\) is a strict convex function for \(I \in [I^{**}, \bar{I} - \bar{T}]\), if \(\frac{\partial U_0(I)}{\partial I} \bigg|_{I=I_0} \leq 0\), there exists a unique \(I \in [I^{**}, \bar{I} - \bar{T}]\) maximizing the principal’s profit. Note that \(\frac{\partial U_0(I)}{\partial I} \bigg|_{I=I_0} \leq 0\) implies \(\frac{\partial U_0(I)}{\partial I} < 0\) for \(I \in (\bar{I} - \bar{T}, I_0]\) by Lemma 3. Also, if \(\frac{\partial U_0(I)}{\partial I} \bigg|_{I=I_0} > 0\), there exists a unique \(I \in (\bar{I} - \bar{T}, I_0]\) maximizing the principal’s profit by Lemma 3.
A.7 Proof of Corollary 1

Denote the project level right after the final intermediate payment by $I_t$. Then, the bonus payment $B_u$ must be less than $B^*_u$ which is the optimal bonus payment without any intermediate payment. (If not, the compensation scheme with intermediate payments makes the principal worse off.) Recall that there is a $I^* \in (I, I)$ such that $\frac{\partial W(I, B^*_u, A)}{\partial I} \bigg|_{I = I^*} = \frac{\phi}{\mu}$. Hence, $\frac{\partial W(I, B_u, A)}{\partial I} \bigg|_{I = I^*} < \frac{\phi}{\mu}$, where $A = \{a_t = \mu\}_{0 \leq t < \tau}$. Since $I^*$ is reached with a positive probability, the compensation scheme with intermediate payments is not incentive-compatible.

A.8 Proof of Comparative Statics

Lemma 4 For given parameters, the optimal termination level $I^*$ is a strictly increasing function in $I_0$. That is,

$$\frac{\partial I^*}{\partial I_0} > 0.$$ 

Proof: Recall that

$$\frac{\partial U_0(I)}{\partial I} \bigg|_{I = I^*} = \mathcal{P}'(I^*) \left( \frac{\kappa}{r} - B_u(I^*) \right) - \mathcal{P}(I^*) B'_u(I^*) = 0.$$ 

By the implicit function theorem,

$$\frac{\partial I^*}{\partial I_0} = -\left( \frac{\partial^2 U_0(I)}{\partial I^2} \bigg|_{I = I^*} \right)^{-1} \left[ \frac{\partial \mathcal{P}'(I^*)}{\partial I_0} \left( \frac{\kappa}{r} - B_u(I^*) \right) - \frac{\partial \mathcal{P}(I^*)}{\partial I_0} B'_u(I^*) \right] > 0$$

since

$$\frac{\partial \mathcal{P}'(I^*)}{\partial I_0} = \frac{\nu^+ \exp(\nu^- I^* + \nu^+ I_0) - \nu^- \exp(\nu^- I_0 + \nu^+ I^*)}{\exp(\nu^- I^* + \nu^+ I) - \exp(\nu^- I + \nu^+ I^*)} > 0,$$

and

$$\frac{\partial \mathcal{P}(I^*)}{\partial I_0} = -2 \frac{\sqrt{\mu^2 + 2r \sigma^2}}{\sigma^2} \exp \left( -\frac{2 \mu I}{\sigma^2} \right) \frac{\nu^- \exp(\nu^- I_0 + \nu^+ I) - \nu^+ \exp(\nu^- I + \nu^+ I_0)}{(\exp(\nu^- I^* + \nu^+ I) - \exp(\nu^- I + \nu^+ I^*))^2} > 0.$$ 

Note that this implies that

$$\frac{\partial B_u(I^*)}{\partial I_0} < 0$$

since $B_u(I)$ is a strictly decreasing function in $I$.
Lemma 5 For given parameters, the optimal termination time $I^*$ is a strictly increasing function in $\phi$. That is,

$$\frac{\partial I^*}{\partial \phi} > 0.$$ 

Proof: Again, by the implicit function theorem,

$$\frac{\partial I^*}{\partial \phi} = \left( \frac{\partial^2 U_0(I)}{\partial I^2} \bigg|_{I=I^*} \right)^{-1} \left[ (P'(I^*)) \frac{\partial B_u(I^*)}{\partial \phi} + (P(I^*)) \frac{\partial B_u'(I^*)}{\partial \phi} \right] > 0,$$

since

$$\frac{\partial B_u(I^*)}{\partial \phi} = \frac{B_u(I^*)}{\phi} > 0,$$

$$\frac{\partial B_u'(I^*)}{\partial \phi} = \frac{B_u'(I^*)}{\phi} < 0.$$

Lemma 6 For given parameters, the principal’s discounted expected utility is a strictly increasing function in $I_0$. That is,

$$\frac{\partial U_0(I^*)}{\partial I_0} > 0.$$ 

Proof: By the envelope theorem,

$$\frac{\partial U_0(I^*)}{\partial I_0} = \frac{\partial P(I^*)}{\partial I_0} \left( \frac{\kappa}{r} - B_u(I^*) \right) - (P(I^*)) \frac{\partial B_u(I^*)}{\partial I_0} > 0.$$ 

The inequality holds by Lemma 4.

Lemma 7 Consider the case where $I^* = \frac{1}{\eta^+-\eta^-} \log \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) + I_*^* < \bar{I}$. Under this circumstance, the agent’s discounted expected utility at time 0 is a strictly increasing (decreasing) function in $I_0$ if

$$\frac{\partial I^*}{\partial I_0} < (>) 1.$$ 

On the other hand, if $I^* = \bar{I}$, the agent’s discounted expected utility at time 0 is a strictly increasing (decreasing) function in $I_0$ if

$$\frac{\partial I^*}{\partial I_0} < (>) \frac{(\eta^- e^{-I_0+\eta^+ (\eta^-)} - \eta^+ e^{\eta^- (I^*+\eta^+ I_0)})(\eta^- e^{\eta^- (I^*+\eta^+ I_0)} - \eta^+ e^{-I^*+\eta^+ I_0})}{(\eta^+ - \eta^-)(\eta^- e^{-I^*+(I+\bar{I})} + \eta^+ (I+\bar{I}^*) - \eta^+ e^{-I^*+(I+\bar{I})} + \eta^- (I+\bar{I}^*))} \leq 1.$$
Proof: Note that when \( I^* = \frac{1}{\eta^+ - \eta^-} \log \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) + \bar{I} < \bar{I}, \)

\[
B_u(I^*) = \frac{\exp(\eta^-\bar{I} + \eta^+\bar{I}) - \exp(\eta^-I + \eta^+I)}{\eta^+ \exp(\eta^-\bar{I} + \eta^+\bar{I}) - \eta^- \exp(\eta^-I + \eta^+I)} \phi.
\]

Therefore, the agent’s discounted expected utility at time 0 is

\[
W_0(I_0) = \frac{\exp(\eta^-I^* + \eta^+I_0) - \exp(\eta^-I_0 + \eta^+I^*)}{\eta^+ \exp(\eta^-I^* + \eta^+\bar{I}) - \eta^- \exp(\eta^-I + \eta^+I^*)} \phi.
\]

Differentiation this with respect to \( I_0 \) yields

\[
\frac{\partial W_0(I_0)}{\partial I_0} = \frac{-\eta^+ \exp(\eta^-I^* + \eta^+I_0) + \eta^- \exp(\eta^-I_0 + \eta^+I^*)}{\eta^+ \exp(\eta^-I^* + \eta^+\bar{I}) - \eta^- \exp(\eta^-I + \eta^+I^*)} \left( \frac{\partial I^*}{\partial I_0} - 1 \right) \frac{\phi}{\mu},
\]

Hence,

\[
\frac{\partial W_0(I_0)}{\partial I_0} > (\cdot) 0 \text{ if and only if } \frac{\partial I^*}{\partial I_0} < (\cdot) 1.
\]

When \( I^* = \bar{I}, \)

\[
\frac{\partial W_0(I_0)}{\partial I_0} = \frac{1}{\eta^+ e^{\eta^-\bar{I} + \eta^+\bar{I}} - \eta^- e^{\eta^-\bar{I} + \eta^+\bar{I}}} \mu \cdot \left[ \frac{(\eta^+ - \eta^-)(\eta^- e^{\eta^-\bar{I} + \eta^+\bar{I}} + \eta^+ e^{\eta^-\bar{I} + \eta^+\bar{I}}) - \eta^+ e^{\eta^-\bar{I} + \eta^+\bar{I}}}{\eta^+ \exp(\eta^-I^* + \eta^+\bar{I}) - \eta^- \exp(\eta^-I + \eta^+I^*)} \frac{\partial I^*}{\partial I_0} 
+ \eta^+ \exp(\eta^-I^* + \eta^+I_0) - \eta^- \exp(\eta^-I_0 + \eta^+I^*) \right].
\]

Hence, \( \frac{\partial W_0(I_0)}{\partial I_0} > 0 \) if the term in the square bracket is positive, and this holds if the condition in Lemma holds. Also, the difference between the numerator and denominator in the condition in Lemma is :

Numerator–Denominator = \((e^{\eta^-\bar{I} + \eta^+I_0} - e^{\eta^-I + \eta^+I})e^{\eta^-\bar{I} + \eta^+\bar{I}}[(\eta^-)^2 - (\eta^- - \eta^+)(\bar{I} - \bar{I}) - (\eta^+)^2].
\]

Since \( \bar{I} - \bar{I} < \frac{1}{\eta^+ - \eta^-} \log \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right), \) the difference is greater than 0. Hence, the threshold is less than 1. (Note that both numerator and denominator are negative.)

Lemma 8 For a given parameters, the principal’s discounted expected utility is a strictly decreasing function in \( \phi. \) That is,

\[
\frac{\partial U_0(I^*)}{\partial \phi} < 0.
\]
Proof: By the envelope theorem,

\[
\frac{\partial U_0(I^*)}{\partial \phi} = -\mathcal{P}(I^*) \frac{\partial B_u(I^*)}{\partial \phi} \\
= -\mathcal{P}(I^*) \frac{\exp(\eta^{-I + \eta^+ I}) - \exp(\eta^{-I + \eta^+ I})}{\eta^+ \exp(\eta^{-I + \eta^+ I}) - \eta^- \exp(\eta^{-I^* + \eta^+ I^*})} \mu < 0.
\]

Lemma 9 The agent’s discounted expected utility at time 0 can be increasing or decreasing in \(\phi\), depending on the parameter values.

Proof: Note that if \(\phi = 0\), the agent’s discounted expected utility is 0 since \(B_u(I) = 0\) regardless of the value of \(I\). On the other hand, there is \(\bar{\phi}\) such that \(B_u(I_0) > \frac{\mu}{\gamma}\) if \(\phi \geq \bar{\phi}\) since \(B_u(I_0)\) is a strictly increasing function in \(\phi\). Hence, the agent’s discounted expected utility is 0 if \(\phi \geq \bar{\phi}\). That is, the principal does not hire the agent or sets \(I = I_0\). Combining this with the limited liability gives us the desired result.

Lemma 10 The completion probability can be increasing or decreasing in \(\phi\), depending on the parameter values.

Proof: Note that

\[
\frac{\partial \mathcal{P}}{\partial I_0} \geq 0
\]

if and only if

\[
\frac{\partial I^*}{\partial I_0} \leq \frac{1 - e^{-\delta(I^*-I_0)}}{1 - e^{-\delta(I^*-I_0)}},
\]

Lemma 11 The completion probability is a strictly decreasing function in \(\phi\). That is,

\[
\frac{\partial \mathcal{P}}{\partial \phi} < 0.
\]
**Proof:** This holds since \( \frac{\partial I^*}{\partial \phi} > 0 \) and

\[
\frac{\partial P}{\partial \phi} = \delta e^{-\delta I^*} \frac{e^{-\delta I^*} - e^{-\delta I_0}}{[e^{-\delta I^*} - e^{-\delta I_0}]^2} \frac{\partial I^*}{\partial \phi} < 0.
\]

\[\square\]

### A.9 Proof of Proposition 5

Note that the principal’s problem is

\[
\max_{(\widehat{I}, \bar{I}, L_1, L_2, C, B)} P(I_0, \widehat{I}, L_1) \left[ -C + \widehat{P}(\widehat{I}, \bar{I}, L_2) \left( \frac{K}{r} - B \right) \right]
\]

subject to

\[
\frac{\partial P_A(I_t, \widehat{I}, L_1)}{\partial I_t} (C + \widehat{P}(\widehat{I}, \bar{I}, L_2)B) \geq \frac{\phi}{\mu} \text{ for } I_t \in [I_1, \widehat{I}],
\]

\[
\frac{\partial P_A(I_t, \bar{I}, L_2)}{\partial I_t} B \geq \frac{\phi}{\mu} \text{ for } I_t \in [L_2, \bar{I}], \text{ and }
\]

\[
C \geq 0,
\]

where

\[
P(I, \widehat{I}, L_1) = \exp(\nu^- L_1 + \nu^+ I) - \exp(\nu^- I + \nu^+ L_1),
\]

\[
\widehat{P}(I, \widehat{I}, L_2) = \exp(\nu^- L_2 + \nu^+ \widehat{I}) - \exp(\nu^- \widehat{I} + \nu^+ L_2),
\]

\[
P_A(I, \widehat{I}, L_1) = \frac{\exp(\eta^- L_1 + \eta^+ I) - \exp(\eta^- I + \eta^+ L_1)}{\exp(\eta^- L_1 + \eta^+ L_1) - \exp(\eta^- I + \eta^+ I)}, \text{ and }
\]

\[
\widehat{P}_A(I, \widehat{I}, L_2) = \frac{\exp(\eta^- L_2 + \eta^+ \widehat{I}) - \exp(\eta^- \widehat{I} + \eta^+ L_2)}{\exp(\eta^- \widehat{I} + \eta^+ I) - \exp(\eta^- L_2 + \eta^+ L_2)}.
\]

**Claim 1**

\[
\frac{\partial \widehat{P}_A(I_t, \bar{I}, L_2)}{\partial I_t} B = \frac{\phi}{\mu} \text{ for some } I_t \in [L_2, \bar{I}], \text{ and }
\]

\[
\frac{\partial P_A(I_t, \widehat{I}, L_1)}{\partial I_t} (C + \widehat{P}_A(\widehat{I}, \bar{I}, L_2)B) = \frac{\phi}{\mu} \text{ for some } I_t \in [I_1, \widehat{I}].
\]
Proof: First, suppose that both conditions do not bind. Then, the principal can obtain a higher profit by reducing $B$ since

\[
\frac{\partial \mathcal{P}_A(I_t, \hat{I}, L_2)}{\partial I_t} > 0 \quad \text{and} \quad \frac{\partial \mathcal{P}_A(I_t, \hat{I}, L_1)}{\partial I_t} > 0.
\]

Let’s assume that the first condition binds but the second does not. There are two possibilities, $C > 0$ and $C = 0$. If $C > 0$, the principal increases his profit by reducing $C$. On the other hand, if $C = 0$, lowering $L_1$ by $\epsilon > 0$ yields a higher profit to the principal since

\[
\frac{\partial \mathcal{P}(I_0, \hat{I}, L_1)}{\partial L_1} < 0 \quad \text{and} \quad \frac{\partial^2 \mathcal{P}(I_t, \hat{I}, L_1)}{\partial I_t \partial L_1} > 0.
\]

Therefore, the second condition must bind.

Suppose that the first condition does not bind but the second does. Then, choosing \( \hat{B} = B - \Delta, \) and \( \hat{C} = C + \tilde{P}_A(\hat{I}, \hat{I}, L_2)\Delta \)
does not affect the first constraint and still satisfies the second constraint, where

\[
\min_{I_t \in [L_2, I]} \frac{\partial \tilde{P}_A(I_t, \hat{I}, L_2)}{\partial I_t} B - \frac{\phi}{\mu} \equiv \Delta > 0.
\]

This compensation scheme gives a higher profit to the principal since

\[
-\mathcal{P}(I_0, \hat{I}, L_1)\tilde{P}_A(\hat{I}, \hat{I}, L_2)\Delta + \mathcal{P}(I_0, \hat{I}, L_1)\tilde{P}(\hat{I}, \hat{I}, L_2)\Delta = \mathcal{P}(I_0, \hat{I}, L_1)\Delta[\tilde{P}(\hat{I}, \hat{I}, L_2) - \tilde{P}_A(\hat{I}, \hat{I}, L_2)]
\]
is strictly greater than zero by Theorem 2.

This claim directly provides the following results.

Claim 2 For given \((\hat{I}, L_1, L_2)\), the optimal $B$ is

\[
B(L_2) \equiv \frac{\exp(\eta^- L_2 + \eta^+ \hat{I}) - \exp(\eta^- \hat{I} + \eta^+ L_2)}{\eta^+ \exp(\eta^- L_2 + \eta^+ I^{**}) - \eta^- \exp(\eta^- I^{**} + \eta^+ L_2)} \frac{\phi}{\mu},
\]

where

\[
I^{**} = \min \left[ \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) + L_2, \hat{I} \right],
\]
and the optimal $C$ is
\[ C(\tilde{I}, I_1, I_2) \equiv \max \left[ \frac{\phi}{\mu} \frac{\exp(\eta^+ L_1 + \eta^+ \tilde{I}) - \exp(\eta^+ \tilde{I} + \eta^+ L_1)}{\eta^+ \exp(\eta^+ L_1 + \eta^+ I^*) - \eta^- \exp(\eta^- I^* + \eta^+ L_1)} - \tilde{P}_A(\tilde{I}, I_1, I_2)B(I_2), 0 \right], \]

where
\[ I^* = \min \left[ \frac{1}{\eta^+ - \eta^-} \ln \left( \frac{(\eta^-)^2}{(\eta^+)^2} \right) + I_1, \tilde{I} \right]. \]

**Proof:** Convexity of $\frac{\partial \tilde{P}_A(I_1, I_2)}{\partial I_1}$ and $\frac{\partial \tilde{P}_A(I_1, I_2)}{\partial I_2}$ implies that
\[ B \geq \frac{\exp(\eta^- I_2 + \eta^+ \tilde{I}) - \exp(\eta^- \tilde{I} + \eta^+ L_2)}{\eta^+ \exp(\eta^- I_2 + \eta^+ I^*) - \eta^- \exp(\eta^- I^* + \eta^+ L_1)} \frac{\phi}{\mu} \]

and
\[ C \geq \frac{\phi}{\mu} \frac{\exp(\eta^- I_1 + \eta^+ \tilde{I}) - \exp(\eta^- \tilde{I} + \eta^+ L_1)}{\eta^+ \exp(\eta^- I_1 + \eta^+ I^*) - \eta^- \exp(\eta^- I^* + \eta^+ L_1)} - \tilde{P}_A(I_1, I_2)B(I_2). \]

Since $C \geq 0$ and the principal’s profit is a strictly decreasing function in $C$ and $B$, the claim holds.

**Claim 3** $I_2$ is greater than $I_1$.

**Proof:** Suppose not. That is, $I_1 > I_2$. This implies that $C(\tilde{I}, I_1, I_2) = 0$ since
\[ \frac{\phi}{\mu} \frac{\exp(\eta^- I_1 + \eta^+ \tilde{I}) - \exp(\eta^- \tilde{I} + \eta^+ L_1)}{\eta^+ \exp(\eta^- I_1 + \eta^+ I^*) - \eta^- \exp(\eta^- I^* + \eta^+ L_1)} < \tilde{P}_A(I_1, I_2)B(I_2) \]

regardless of $\tilde{I}$. Now, I show that the constraints can’t bind if $I_1 > I_2$. Without loss of generality, assume that
\[ B = \frac{\exp(\eta^- I_2 + \eta^+ \tilde{I}) - \exp(\eta^- \tilde{I} + \eta^+ L_2)}{\eta^+ \exp(\eta^- I_2 + \eta^+ I^*) - \eta^- \exp(\eta^- I^* + \eta^+ L_2)} \frac{\phi}{\mu}. \]

Recall that the other constraint is
\[ \frac{\partial \tilde{P}_A(I_1, I, I_2)}{\partial I_1} \tilde{P}_A(I, I_1, I_2)B \geq \frac{\phi}{\mu} \text{ for } I_1 \in [I_1, \tilde{I}]. \]

This is equivalent to
\[ \frac{\eta^+ \exp(\eta^- L_1 + \eta^+ I^*) - \eta^- \exp(\eta^- I^* + \eta^+ L_1)}{\eta^+ \exp(\eta^- L_2 + \eta^+ I^*) - \eta^- \exp(\eta^- I^* + \eta^+ L_2)} \frac{\exp(\eta^- L_2 + \eta^+ \tilde{I}) - \exp(\eta^- \tilde{I} + \eta^+ L_2)}{\eta^+ \exp(\eta^- L_1 + \eta^+ I^*) - \eta^- \exp(\eta^- I^* + \eta^+ L_1)} \geq 1. \]

This condition does not hold since the left hand side is a strictly increasing function in $L_1$ and the left hand side is equal to one if $I_1 = L = 2$. Therefore, $I_1 \leq I_2$. 

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Claim 4 There is $I^*$ such that $I_1 \geq I^*$.

Proof: Since $\frac{\partial B(L_1)}{\partial L_1} < 0$ and $\lim_{L_2 \to -\infty} B(L_2) = \infty$, there is $I^{**}$ such that $B(I^{**}) = \frac{\kappa}{r}$. Therefore, $L_2 \geq I^{**}$. For given $I^{**}$, there exist $I^*$ such that

$$\frac{\phi}{\mu} \frac{\exp(\eta - I^* + \eta^+ \hat{I}) - \exp(\eta - \hat{I} + \eta^+ I^*)}{\eta^+ \exp(\eta - I^* + \eta^+ I^*) - \eta^+ \exp(\eta - I^* + \eta^+ I^*)} - \tilde{P}_A(\hat{I}, I, I^{**}) B(I^{**}) = \frac{\kappa}{r}$$

since

$$\frac{\exp(\eta - I_1 + \eta^+ \hat{I}) - \exp(\eta - \hat{I} + \eta^+ L_1)}{\eta^+ \exp(\eta - I_1 + \eta^+ I^*) - \eta^+ \exp(\eta - I^* + \eta^+ I^*)}$$

is a strictly decreasing function in $I_1$ and goes to infinity as $I_1 \to -\infty$. Hence, if $I_1 < I^*$, the principal never obtains a non-negative profit. 

Therefore, the principal’s problem is :

$$\max_{(I, L_1, L_2, B)} P(I_0, \hat{I}, I_1) \left[ -C(\hat{I}, L_1, L_2) + \tilde{P}(\hat{I}, I, L_2) \left( \frac{\kappa}{r} - B(L_2) \right) \right]$$

subject to $C(\hat{I}, L_1, L_2) = \max \left[ \frac{\phi}{\mu} \frac{\exp(\eta - I_1 + \eta^+ \hat{I}) - \exp(\eta - \hat{I} + \eta^+ L_1)}{\eta^+ \exp(\eta - I_1 + \eta^+ I^*) - \eta^+ \exp(\eta - I^* + \eta^+ I^*)} - \tilde{P}_A(\hat{I}, I, L_2) B(L_2), 0 \right]$, and

$$B(L_2) = \frac{\exp(\eta - L_2 + \eta^+ \hat{I}) - \exp(\eta - \hat{I} + \eta^+ L_2)}{\eta^+ \exp(\eta - L_2 + \eta^+ I^{**}) - \eta^+ \exp(\eta - I^{**} + \eta^+ L_2)} \mu.$$

The compactness of $(\hat{I}, L_1, L_2) \in ([I_0, \hat{I}], [L^*, I_0], [L_1, \hat{I}])$ and the continuity of the objective function on the region guarantee the existence of a solution to the principal’s problem.

Appendix B Additional Mathematical Results

Theorem 1 Let $X$ be a $(\mu, \sigma^2)$ Brownian motion with initial condition $x \in [b, B]$. Let $\tau$ be the stopping time $\tau = \min[\tau_b, \tau_B]$, and $r > 0$. If $\sigma^2 > 0$, then

$$E_x[e^{-\tau r}]X(\tau) = b]P_x(X(\tau) = b) = \frac{e^{R_1 x e^{R_2 B}} - e^{R_2 x e^{R_3 B}}}{e^{R_1 b e^{R_2 B}} - e^{R_2 b e^{R_1 B}}}.$$

7Proposition 5.3. in Stokey (2008)
\[ E_x[e^{-rt}|X(\tau) = B]P_x(X(\tau) = B) = \frac{e^{R_1b}e^{R_2x} - e^{R_2b}e^{R_1x}}{e^{R_1b}e^{R_2B} - e^{R_2b}e^{R_1B}}, \]

where

\[ R_1 = -\frac{\mu - \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2} \quad \text{and} \quad R_2 = -\frac{\mu + \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}. \]

**Theorem 2** Let \( X \) be a \((\mu, \sigma^2)\) Brownian motion with initial condition \( x \in [b, B] \). Let \( \tau \) be the stopping time \( \tau = \min[\tau_b, \tau_B] \), and \( r > 0 \). If \( \sigma^2 > 0 \), then

\[ \frac{\partial}{\partial r} \left\{ E_x[e^{-rt}|X(\tau) = B]P_x(X(\tau) = B) \right\} < 0. \]

**Proof**: First, notice that

\[ \frac{\partial R_1}{\partial r} = -\frac{1}{\sqrt{\mu^2 + 2r\sigma^2}} \quad \text{and} \quad \frac{\partial R_2}{\partial r} = \frac{1}{\sqrt{\mu^2 + 2r\sigma^2}}. \]

After some algebra, I can obtain:

\[
\frac{\partial}{\partial r} \left\{ E_x[e^{-rt}|X(\tau) = B]P_x(X(\tau) = B) \right\} = \left( \frac{\sqrt{\mu^2 + 2r\sigma^2}}{e^{R_1b}e^{R_2B} - e^{R_2b}e^{R_1B}} \right)^{R_1b + R_2(B+x)} \left[ (1 - e^{-(R_2-R_1)(B+x-2b)})(x-B) + (e^{-(R_2-R_1)(x-b)} - e^{-(R_2-R_1)(B-b)})(B+x-2b) \right].
\]

Since \( e^{2R_1b + R_2(B+x)} > 0 \), it is enough to show that \( f(x) < 0 \) for \( b < x < B \). Notice that \( f(B) = f(b) = 0 \). Now, I show that \( f(x) \) is a strictly convex function on \( b < x < B \). The second derivative of \( f(x) \) is:

\[ f''(x) = a^2e^{-a(B+x-2b)}(B-x) + a^2e^{-a(x-b)}(B+x-2b) + 2ae^{-a(B+x-2b)} + 2ae^{-a(x-b)}, \]

where \( a \equiv (R_2 - R_1) > 0 \). Since \( (B-b) > 0 \) and \( a > 0 \),

\[
f''(x) = ae^{-a(x-b)}[a^2e^{-a(B-b)}(B-b) + 2ae^{-a(B-b)} - ae^{-a(B-b)}(x-b) + a(B+x-2b) - 2]
= ae^{-a(x-b)}a(x-b)(1 - e^{-a(B-b)}) > 0.
\]

\[ \square \]
Theorem 3 \(^8\) Let \(X\) be a \((\mu, \sigma^2)\) Brownian motion with initial condition \(x \in [b, B]\), and let \(\tau\) be the stopping time \(\tau = \min[\tau_b, \tau_B]\). If \(\sigma^2 > 0\), then

\[
P_x(X(\tau) = b) = \frac{e^{-\delta B} - e^{-\delta x}}{e^{-\delta B} - e^{-\delta b}},
\]

\[
P_x(X(\tau) = B) = \frac{e^{-\delta x} - e^{-\delta b}}{e^{-\delta B} - e^{-\delta b}}, \quad \text{if } \mu \neq 0,
\]

where \(\delta \equiv 2\mu/\sigma^2\), and

\[
P_x(X(\tau) = b) = \frac{B - x}{B - b},
\]

\[
P_x(X(\tau) = B) = \frac{x - b}{B - b}, \quad \text{if } \mu = 0.
\]

\(^8\)Proposition 5.4. in Stokey (2008)
References


Sannikov, Yuliy. 2013. “Moral hazard and long-run incentives.”


