INDICATORS, CHAINS, ANTICHAINS, RAMSEY PROPERTY

MIODRAG SOKIČ

Abstract. We introduce a list of classes of relational structures obtained from chains, antichains, and indicators, and we show which of them are Fraïssé classes. In addition to this, we classify these classes according to the Ramsey property.

1. Introduction

The purpose of this paper is to introduce a list of classes of finite ordered structures and verify the Ramsey property for those classes. Most examples of Ramsey classes are classes of structures with linear orderings; see [11], [12], [13], [14]. A connection between Ramsey classes of ordered finite structures and topological dynamics was established in [8]. The technique developed in [8] was applied to classes of finite posets, obtained from Schmerl list [16], by adding linear orderings in [17], [18] and [19]. It appears that some of these classes do not satisfy the Ramsey property, and in this paper we add unary relations to such structures in order to get new classes which do satisfy the Ramsey property.

Let $\mathbb{Q}$ be the set of rational numbers, let $n \geq 1$ be a natural number, and let $B_n = C_n = \{1, \ldots, n\} \times \mathbb{Q}$. We use $\leq$ to denote the natural orderings on $\mathbb{Q}$, $\mathbb{N}$, and $\mathbb{R}$. We define partial orderings $\leq_{B_n}$ and $\leq_{C_n}$ on the sets $B_n$ and $C_n$, respectively, such that for all $(i, x), (j, y) \in B_n = C_n$ we have:

$$(i, x) \leq_{B_n} (j, y) \Leftrightarrow (i = j \text{ and } x \leq y);$$

$$(i, x) \leq_{C_n} (j, y) \Leftrightarrow (i < j \text{ or } i = j \text{ and } x = y).$$

Therefore we have posets $\mathbb{B'}_n = (B_n, \leq_{B_n})$ and $\mathbb{C'}_n = (C_n, \leq_{C_n})$. Note that $\mathbb{B'}_n$ is the union of $n$ copies of $(\mathbb{Q}, \leq)$ such that elements from different copies are incomparable. The structure $\mathbb{C'}_n$ is obtained by replacing each point from $\mathbb{Q}$ with an antichain of size $n$. There are automorphisms of the structure $\mathbb{B'}_n$ which permute the incomparable copies of rationals, and there are automorphisms of $\mathbb{C'}_n$ which permute the incomparable chains. In order to avoid such automorphisms we consider structures $\mathbb{B}_n = (B_n, \leq_{B_n}, (I_i^{B_n})_{i=1}^n)$ and $\mathbb{C}_n = (C_n, \leq_{C_n}, (I_i^{C_n})_{i=1}^n)$ defined such that for all $1 \leq i \neq j \leq n$ we have:

1. $(I_i^{B_n})_{i=1}^n$ and $(I_i^{C_n})_{i=1}^n$ are unary relations on $B_n$ and $C_n$, respectively;
2. $B_n = \bigcup_{i=1}^n \{x : I_i^{B_n}(x)\}$ and $C_n = \bigcup_{i=1}^n \{x : I_i^{C_n}(x)\}$;
3. $\{x : I_i^{B_n}(x)\} \cap \{y : I_j^{B_n}(y)\} = \emptyset = \{x : I_i^{C_n}(x)\} \cap \{y : I_j^{C_n}(y)\}$;
4. $\{x : I_i^{B_n}(x)\}$ is maximal chain with respect to $\leq_{B_n}$;
5. $\{x : I_i^{C_n}(x)\}$ is maximal chain with respect to $\leq_{C_n}$.

For a given structure $\mathcal{A}$, we denote the class of finite substructures embeddable into $\mathcal{A}$ by $\text{Age}(\mathcal{A})$. We consider the following classes:

$B_n = \text{Age}(\mathbb{B}_n)$ and $C_n = \text{Age}(\mathbb{C}_n)$.

We assume that all classes of finite structures are closed under isomorphic images.

For a given set $A$, we denote the collection of all linear orderings on $A$ by $\text{lo}(A)$. If $\sqsubseteq$ is a partial ordering on $A$ then the collection of all linear extension of $\sqsubseteq$ is denoted by $\text{le}(\sqsubseteq)$. By adding
arbitrary linear orderings to structures from $\mathcal{B}_n$ and $\mathcal{C}_n$ we get the following classes:

$$\mathcal{OB}_n = \{ (A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A) : (A, \leq^A, (I_i^A)_{i=1}^n) \in \mathcal{B}_n \& \ \preceq^A \in \text{lo}(A) \},$$

$$\mathcal{OC}_n = \{ (A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A) : (A, \leq^A, (I_i^A)_{i=1}^n) \in \mathcal{C}_n \& \ \preceq^A \in \text{lo}(A) \}.$$ 

If we add only linear extension of partial ordering we get the following classes:

$$\mathcal{EB}_n = \{ (A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A) \in \mathcal{OB}_n : \preceq^A \in \text{le}(A) \},$$

$$\mathcal{EC}_n = \{ (A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A) \in \mathcal{OC}_n : \preceq^A \in \text{le}(A) \}.$$ 

Let $\mathcal{A} = (A, \leq^A, (I_i^A)_{i=1}^n)$ be a structure from $\mathcal{B}_n$ or $\mathcal{C}_n$. We say that $\preceq \in \text{lo}(A)$ is convex on $\mathcal{A}$ if for all $1 \leq i \leq n$ and all $x, y, z$ from $A$ we have:

$$I_i^A(x), \ I_i^A(z), x \preceq y \leq z \Rightarrow I_i^A(y).$$

The set of convex linear orderings on $\mathcal{A}$ we denote by $\text{co}(\mathcal{A})$. Adding arbitrary linear orderings which are convex, we have the following classes:

$$\mathcal{COB}_n = \{ (A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A) \in \mathcal{OB}_n : \preceq^A \in \text{co}(\mathcal{A}) \},$$

$$\mathcal{OC}_n = \{ (A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A) \in \mathcal{OC}_n : \preceq^A \in \text{co}(\mathcal{A}) \}.$$ 

If we add a linear extension which is convex we get:

$$\mathcal{CEB}_n = \{ (A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A) \in \mathcal{EB}_n : \preceq^A \in \text{co}(\mathcal{A}) \}.$$ 

Note that adding a convex linear extension to structures from $\mathcal{OC}_n$ which contains two antichains of size at least two is not possible, so we consider the class:

$$\mathcal{CEC}_n = \{ (A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A) \in \mathcal{OC}_n :$$

$$(\forall i \leq n)(\forall x, y \in A)x \preceq^A y \iff x \leq^A y \}.$$ 

Let $\mathcal{A}$ and $\mathcal{B}$ be given structures in a signature $L$, and let $\mathcal{K}$ be a class of finite structures in a signature $L$. If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, then we write $\mathcal{A} \cong \mathcal{B}$. If there is an embedding from $\mathcal{A}$ into $\mathcal{B}$, we write $\mathcal{A} \hookrightarrow \mathcal{B}$, and if $\mathcal{A}$ is a substructure of $\mathcal{B}$, then we write $\mathcal{A} \leq \mathcal{B}$. The collection of all substructures of $\mathcal{B}$ isomorphic to $\mathcal{A}$ is denoted by:

$$\left( \begin{array}{c} \mathcal{B} \\ \mathcal{A} \end{array} \right) = \{ \mathcal{C} \leq \mathcal{B} : \mathcal{C} \cong \mathcal{A} \}.$$ 

If $\mathcal{C} \in \mathcal{K}$ is such that for any coloring

$$c : \left( \begin{array}{c} \mathcal{B} \\ \mathcal{A} \end{array} \right) \to \{ 1, \ldots, r \},$$

there is $\mathcal{B}' \in \left( \begin{array}{c} \mathcal{C} \\ \mathcal{A} \end{array} \right)$ such that

$$c \mid \left( \begin{array}{c} \mathcal{B}' \\ \mathcal{A} \end{array} \right) = \text{const},$$

( $\mid$ denotes restriction of a function to a subset) then we write:

$$\mathcal{C} \to (\mathcal{B})^\mathcal{A}_r.$$ 

We say that the class $\mathcal{K}$ satisfies the Ramsey property (RP) if for any two structures $\mathcal{B}$ and $\mathcal{A}$ from $\mathcal{K}$ and any natural number $r$ there is a structure $\mathcal{C} \in \mathcal{K}$ such that $\mathcal{C} \to (\mathcal{B})^\mathcal{A}_r$.

In this paper we prove the following.

**Theorem 1.** Classes 

$$\mathcal{OB}_n, \mathcal{EB}_n, \mathcal{COB}_n, \mathcal{CEB}_n, \mathcal{OC}_n, \mathcal{CEC}_n,$$

satisfy the Ramsey property, and the class 

$$\mathcal{OC}_n, \mathcal{CEC}_n$$

do not satisfy the Ramsey property.
Our proofs are based on the cross-construction developed in [17]. The idea is to construct structures on a product, where each coordinate gives some information about the structures. Note that some of this proof can be conducted by using the partite construction developed in [13] and [14].

In Section 2 we examine classes according to Fraïssé theory; see [7] and [5]. In addition to this we extend the list of Fraïssé classes. Section 3 is dedicated to an examination of the Ramsey property for the class obtained by adding linear orderings which are linear extensions of the partial orderings. In Section 4 we introduce the class $T$ and verify that it has the Ramsey property. This will be used in Section 6. Classes of structures with convex linear ordering are examined in Section 5 and Section 6. Structures with arbitrary linear orderings are examined in Section 7. We finish with an examination of classes with respect to the ordering property, which is closely related to the RP, and an application to topological dynamics.

2. Fraïssé classes

Let $\mathcal{K}$ be a class of finite structures in a signature $L$. Then $\mathcal{K}$ satisfies:

- **Hereditary property (HP):** If whenever $A \rightarrow B$ and $B \in \mathcal{K}$ implies $A \in \mathcal{K}$.
- **Joint embedding property (JEP):** If for all $A, B \in \mathcal{K}$ there is $C \in \mathcal{K}$ such that $A \rightarrow C$ and $B \rightarrow C$.
- **Amalgamation property (AP):** If for all $A, B, C \in \mathcal{K}$ and all embeddings $f : A \rightarrow B, g : A \rightarrow C$ there is $D \in \mathcal{K}$ and embeddings $f' : B \rightarrow D, g' : C \rightarrow D$ such that $f' \circ f = g' \circ f$.

A class $\mathcal{K}$ of finite structures in a countable signature $L$ is called a Fraïssé class if it contains structures of arbitrary large finite cardinality, is countable (it contains only countably many isomorphic types), and satisfies HP, JEP and AP. A structure $K$ is **ultrahomogeneous** if every automorphism between its finite substructures can be extended to an automorphism of $K$. A countable infinite structure $K$ is called a Fraïssé structure if all of its finitely generated substructures are finite and if it is ultrahomogeneous. For a class $\mathcal{K}$ we denote the class of all finite substructures that are isomorphic to a substructure of $\mathcal{K}$ by $\text{Age}(\mathcal{K})$.

**Theorem 2.** [7] (i) If $A$ is a Fraïssé structure then $\text{Age}(A)$ is a Fraïssé class.

(ii) If $A$ is a Fraïssé class then there is a unique, up to isomorphism, Fraïssé structure $A$ such that $\text{Age}(A) = A$.

The structure $A$ which is given by the second part of the previous Theorem is called a **Fraïssé limit** of the class $\mathcal{A}$, $A = \text{F lim}(\mathcal{A})$. Therefore, we have a bijection between Fraïssé classes and Fraïssé structures given by:

$$\mathcal{A} \mapsto \text{F lim}(\mathcal{A}), A \mapsto \text{Age}(A).$$

Let $L$ and $L'$ be two signatures such that $L \subset L'$. Let $A$ and $A'$ be structures in $L$ and $L'$, respectively, defined on the same set. If the interpretation of the symbols from $L$ is the same in both structures, then we say that $A$ is a **reduct** of $A'$ or that $A'$ is an expansion of $A$, $A = A'[L]$. If $K'$ is a class of structures in $L'$ and

$$K = \{A'[L] : A' \in K'\},$$

then we say that $K$ is a reduct of the class $K'$ or that $K'$ is an expansion of the class $K$, $K = K'[L]$.

**Lemma 1.** Let $n \geq 2$ be a natural number.

(i) $\mathcal{B}_n, \mathcal{C}_n, \mathcal{O}_n$ and $\mathcal{E}_n$ are Fraïssé classes.

(ii) $\mathcal{E}_n$ and $\mathcal{O}_n$ satisfy HP and JEP, but they do not satisfy AP.

(iii) $\mathcal{C}_n, \mathcal{O}_n$ and $\mathcal{E}_n$ satisfy HP, but they do not satisfy JEP and AP.

**Proof.** (i) This is easy exercise, see [16] and [17].

(ii) It is easy to verify HP and JEP, so we give a counterexample for AP. Moreover the same example works for $\mathcal{E}_n$ and $\mathcal{O}_n$. Let $A = (A, \leq_A, (I_i^A)_{i=1}^n, \preceq_A)$ be a structure from $\mathcal{E}_n$ such that $A = \{a\}$ and $I_1^A(a)$. Let $B = (B, \leq_B, (I_i^B)_{i=1}^n, \preceq_B)$ and $C = (C, \leq_C, (I_i^C)_{i=1}^n, \preceq_C)$ be structures from $\mathcal{E}_n$ such that $B = \{b_1, b_2\}, C = \{c_1, c_2\}, I_1^B(a_1), I_2^B(a_2), I_1^C(c_1), I_2^C(c_2), b_1 \preceq_B b_2, c_2 \preceq_C c_1,$
$b_1$ and $b_2$ are incomparable with respect to $\leq^B$, $c_1$ and $c_2$ are incomparable with respect to $\leq^C$. We consider embedding

$$f : A \to B, f(a) = b_1,$$
$$g : A \to C, g(a) = c_1.$$  

Suppose there is $D = (D, \leq^D, (I^D_i)_{i=1}^n, \leq^D)$ and there are embeddings $f' : B \to D$ and $g' : C \to D$ such that $f' \circ f = g' \circ g$. Then we have

$$g'(c_2) \prec^D g'(c_1) = g' \circ g(a) = f' \circ f(a) = f'(b_1) \prec^D f'(b_2)$$
$$\& \ I^D_1(g'(c_2)) \ & I^D_1(f'(b_2)),$$

and $g'(c_2), f'(b_2)$ and $g' \circ g(a) = f' \circ f(a)$ are incomparable with respect to $\leq^D$. Since the set $\{d \in D : I^D_2(d)\}$ is linearly ordered by $\leq^D$, we have a contradiction.

(iii) By a similar argument, we show that these classes do not satisfy AP. It is a similar argument for these classes to show that they do not satisfy JEP, so we will demonstrate on the example of $\mathcal{COB}_n$. So let $\mathcal{A} = (A, \leq^A, (I^A_i)_{i=1}^n, \leq^A)$ and $\mathcal{B} = (B, \leq^B, (I^B_i)_{i=1}^n, \leq^B)$ be structures from $\mathcal{COB}_n$ such that

$$A = \{a_1, a_2\} \& B = \{b_1, b_2\}.$$  

The linear orderings are such that

$$a_1 \prec^A a_2 \ & b_2 \prec^B b_1.$$  

With respect to the partial orderings $\leq^A$ and $\leq^B$, respectively, the sets $A$ and $B$ are antichains. The unary relations are such that

$$I^A_1(a_1), I^A_2(a_2), I^B_1(b_1), I^B_2(b_2).$$

Suppose there is $\mathcal{C} = (C, \leq^C, (I^C_i)_{i=1}^n, \leq^C) \in \mathcal{COB}_n$ such that $\mathcal{A} \hookrightarrow \mathcal{C}$ and $\mathcal{B} \hookrightarrow \mathcal{C}$ and consider the sets

$$C_i = \{c \in C : I^C_i(c)\}$$

for $i = 1, 2$. The sets $C_1$ and $C_2$ form intervals with respect to $\leq^C$. Note that $\mathcal{A} \hookrightarrow \mathcal{C}$ implies $C_1 \prec^C C_2$, and $\mathcal{B} \hookrightarrow \mathcal{C}$ implies $C_2 \prec^C C_1$, so we have a contradiction.

By the previous lemma and Theorem 2, we have the following Fraïssé structures:

$$\mathcal{B}_n = F\text{lim}(\mathcal{B}_n), \mathcal{C}_n = F\text{lim}(\mathcal{C}_n), \mathcal{OB}_n = F\text{lim}(\mathcal{OB}_n), \mathcal{EB}_n = F\text{lim}(\mathcal{EB}_n).$$

From the proof of the previous lemma we may notice that JEP and AP fail because the order of intervals given by the unary relations is not fixed in advance. Therefore, we consider a linear ordering $\subseteq_n$ of the set $\{1, ..., n\}$ and classes

$$\mathcal{COB}_\subseteq_n = \{(A, \leq^A, (I^A_i)_{i=1}^n, \leq^A) \in \mathcal{COB}_n : (\forall i, j)(\forall x, y)[I^A_i(x), I^A_j(y), i \subseteq_n j \Rightarrow x \prec^A y]\},$$
$$\mathcal{CEB}_\subseteq_n = \mathcal{COB}_\subseteq_n \cap \mathcal{EB}_n,$$
$$\mathcal{COC}_\subseteq_n = \{(A, \leq^A, (I^A_i)_{i=1}^n, \leq^A) \in \mathcal{COC}_n : (\forall i, j)(\forall x, y)[I^A_i(x), I^A_j(y), i \subseteq_n j \Rightarrow x \prec^A y]\},$$
$$\mathcal{CEC}_\subseteq_n = \mathcal{COC}_\subseteq_n \cap \mathcal{CEC}_n.$$  

Now it is simple exercise to see the following.

**Lemma 2.** Let $\subseteq_n \in lo\{1, ..., n\}$. Then $\mathcal{COB}_\subseteq_n, \mathcal{CEB}_\subseteq_n, \mathcal{COC}_\subseteq_n$ and $\mathcal{CEC}_\subseteq_n$ are Fraïssé classes.

We have the following Fraïssé structures

$$\mathcal{COB}_\subseteq_n = F\text{lim}(\mathcal{COB}_\subseteq_n), \mathcal{CEB}_\subseteq_n = F\text{lim}(\mathcal{CEB}_\subseteq_n),$$
$$\mathcal{COC}_\subseteq_n = F\text{lim}(\mathcal{COC}_\subseteq_n), \mathcal{CEC}_\subseteq_n = F\text{lim}(\mathcal{CEC}_\subseteq_n).$$  

We have considered only classes with finitely many unary relations. We introduce classes with countably many unary relations.
Lemma 3. (i) $\mathcal{B}_\omega = \bigcup_{n=1}^{\infty} \mathcal{B}_n$, $\mathcal{C}_\omega = \bigcup_{n=1}^{\infty} \mathcal{C}_n$, $\mathcal{OB}_\omega = \bigcup_{n=1}^{\infty} \mathcal{OB}_n$ and $\mathcal{EB}_\omega = \bigcup_{n=1}^{\infty} \mathcal{EB}_n$ are Fraïssé classes with limits $\mathcal{B}_\omega$, $\mathcal{C}_\omega$, $\mathcal{OB}_\omega$ and $\mathcal{EB}_\omega$, respectively.
(ii) $\mathcal{EC}_\omega = \bigcup_{n=1}^{\infty} \mathcal{EC}_n$ and $\mathcal{OC}_\omega = \bigcup_{n=1}^{\infty} \mathcal{OC}_n$ satisfy HP and JEP, but they do not satisfy AP.
(iii) $\mathcal{COB}_\omega = \bigcup_{n=1}^{\infty} \mathcal{COB}_n$, $\mathcal{CC}_\omega = \bigcup_{n=1}^{\infty} \mathcal{CC}_n$, $\mathcal{CEB}_\omega = \bigcup_{n=1}^{\infty} \mathcal{CEB}_n$ and $\mathcal{CEC}_\omega = \bigcup_{n=1}^{\infty} \mathcal{CEC}_n$ satisfy HP but they do not satisfy JEP and AP.

Proof. Similar to Lemma 1. □

If $\sqsubseteq_\omega$ is a linear ordering of the set $\{1, 2, \ldots\}$, then we consider classes

$$\mathcal{COB}_{\sqsubseteq_\omega} = \{(A, \leq A, (I_i^A)_{i=1}^{\infty}, \gamma^A) \in \mathcal{COB}_\omega : (\forall i)(\forall x, y) \gamma[I_i^A(x), I_i^A(y), i \sqsubseteq_\omega j \Rightarrow x \nleq A y]\},$$

$$\mathcal{CEB}_{\sqsubseteq_\omega} = \mathcal{COB}_{\sqsubseteq_\omega} \cap \mathcal{CEB}_\omega,$$

$$\mathcal{COC}_{\sqsubseteq_\omega} = \{(A, \leq A, (I_i^A)_{i=1}^{\infty}, \gamma^A) \in \mathcal{COC}_\omega : (\forall i)(\forall x, y) \gamma[I_i^A(x), I_i^A(y), i \sqsubseteq_\omega j \Rightarrow x \nleq A y]\},$$

$$\mathcal{CEC}_{\sqsubseteq_\omega} = \mathcal{COC}_{\sqsubseteq_\omega} \cap \mathcal{CEC}_\omega.$$

By the same arguments as above we have the following.

Lemma 4. Let $\sqsubseteq_\omega \in lo(\{1, 2, \ldots\})$. Then, $\mathcal{COB}_{\sqsubseteq_\omega}, \mathcal{CEB}_{\sqsubseteq_\omega}, \mathcal{COC}_{\sqsubseteq_\omega}, \mathcal{CEC}_{\sqsubseteq_\omega}$ are Fraïssé classes with limits $\mathcal{COB}_{\sqsubseteq_\omega}, \mathcal{CEB}_{\sqsubseteq_\omega}, \mathcal{COC}_{\sqsubseteq_\omega}$ and $\mathcal{CEC}_{\sqsubseteq_\omega}$ respectively.

3. RP for linear extensions

The cardinality of a given set $S$ is denoted by $|S|$. Let $X$ be a nonempty set, and let $k, l, m, r$ be natural numbers. Then $[X]^k = \binom{X}{k} = \{S \subseteq X : |S| = k\}$. If for every set $C$ with $|C| = m$, and every coloring $c : \binom{C}{k} \to \{1, \ldots, r\}$ there is a $B \subseteq C$ with $|B| = l$ such that $c \upharpoonright \binom{B}{k} = \text{const}$, then we write

$$m \to (l, r)^k.$$

The following is the well-known classical Ramsey theorem.

Theorem 3. [6] For all natural numbers $r, k, l$ there is a natural number $m_0$ such that for all $m \geq m_0$ we have $m \to (l, r)^k$.

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a sequence of nonempty finite sets. A triple $\mathcal{X} = (X, f^X, \preceq^X)$ is called an $\alpha$-colored set if $\preceq^X \in lo(X)$, and if $f^X$ is a function $f^X : \bigcup_{i=1}^{k} [X]^i \to \bigcup_{i=1}^{k} \alpha_i$ such that for all $1 \leq i \leq k$, all $x \in [X]^i$ we have $f^X(x) \in \alpha_i$. If $\mathcal{Y} = (Y, f^Y, \preceq^Y)$ is also $\alpha$-colored set, then the map $F : X \to Y$ an embedding if it is $1 - 1$; and for all $x, x' \in X$ we have

$$x \preceq X x' \iff F(x) \preceq Y F(x');$$

and for all $1 \leq i \leq k$, all $z \in [X]^i$ we have

$$f^X(z) = f^Y(F(z)).$$

If there is an embedding from $X$ into $Y$, then we write $X \hookrightarrow Y$, and if the embedding is realized by identity the map, then we say that $X$ is a substructure of $Y$, or $X \preceq Y$. An embedding which is a bijection is called an isomorphism; we write $X \cong Y$. The class of finite $\alpha$-colored sets with the notion of embedding as defined above we denote by $\mathcal{K}(\alpha)$. Our proofs will use the following result.

Theorem 4. [1] For any finite sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ of finite nonempty sets, the class $\mathcal{K}(\alpha)$ satisfies RP.

The following result is useful for disproving RP.

Lemma 5. [8] If $\mathcal{K}$ is a class of finite rigid structures which satisfies HP, JEP, and RP, then $\mathcal{K}$ satisfies AP.

Theorem 5. For any natural number $n \geq 2$, $\mathcal{EB}_n$ satisfies RP, but $\mathcal{EC}_n$ does not satisfy RP.

Proof. (RP for $\mathcal{EB}_n$) We consider a sequence $\alpha = (\alpha_1)$ such that $\alpha_1 = \{1, \ldots, n\}$. There is a bijection:

$$\Delta : \mathcal{B}_n \to \mathcal{K}(\alpha),$$

$$\mathcal{E} = (E, \preceq_E, (I_i^E)_{i=1}^{n}, \preceq_E) \mapsto \Delta(\mathcal{E}) = (E, f^E, \preceq^E).$$
Note that the linear orderings on \( \mathbb{B} \) and \( \Delta(\mathbb{B}) \) are the same. For \( x, y \in E \), we define:
\[
f^E(\{x\}) = i \iff I^E_i(x).
\]
It is straightforward to see that if \( F \) is an embedding from \( \mathbb{A} \) into \( \mathbb{B} \), then \( F \) is an embedding from \( \Delta(\mathbb{A}) \) into \( \Delta(\mathbb{B}) \), and vice versa. Consequently the question of RP for \( \mathcal{EB}_n \) is translated into a question about RP for \( \mathcal{K}(\alpha) \). By Theorem 4, \( \mathcal{K}(\alpha) \) satisfies RP, so \( \mathcal{EB}_n \) also satisfies RP.

(no RP for \( \mathcal{EC}_n \)). Using Lemma 5 we disprove RP by giving a counterexample for AP. Let
\[
A = (A, \leq^A, (I^A_i)_{i=1}^n, \geq^A), \mathbb{B} = (B, \leq^B, (I^B_i)_{i=1}^n, \geq^B), \mathbb{C} = (C, \leq^C, (I^C_i)_{i=1}^n, \geq^C)
\]
be structures from \( \mathcal{EC}_n \). The partial orderings \( \leq^B \) and \( \leq^C \) are antichains of size \( n \), and \( \leq^A \) is also an antichain of size 1. Let
\[
A = \{a\}, B = \{b_1, \ldots, b_n\}, C = \{c_1, \ldots, c_n\}
\]
where the linear orderings are given by
\[
b_1 \leq^B \cdots \leq^B b_n \text{ and } c_1 \leq^C \cdots \leq^C c_n,
\]
and define unary relations such that for all \( 1 \leq i \leq n \) we have \( I^B_i(b_i) \) and \( I^C_i(c_i) \). There are maps
\[
f : A \to B, f(a) = b_n,
g : A \to C, g(a) = c_n,
\]
which are embeddings of \( \mathbb{A} \) into \( \mathbb{B} \) and \( \mathbb{C} \). Suppose there is \( \mathbb{D} = (D, \leq^D, (I^D_i)_{i=1}^n, \leq^D) \in \mathcal{EC}_n \) and embeddings from \( \mathbb{D} \) and \( \mathbb{C} \) into \( \mathbb{B} \) given by functions \( f' \) and \( g' \) such that \( f' \circ f = g' \circ g \). We consider the following elements from \( D \):
\[
d' = f' \circ f(a) = g' \circ g(a), b'_1 = f'(b_1), c'_1 = g'(c_1).
\]
Then, we have:
\[
I^D_1(b'_1), I^D_1(c'_1), b'_1 \leq^D a' \leq^D c'_1,
\]
and we know that the elements \( b'_1, a' \) and \( c'_1 \) are incomparable with respect to \( \leq^D \). This is a contradiction with the fact that maximal antichain with respect to \( \leq^D \) can have at most one element indicated by \( I^D_i \).

In similar way as in the previous Lemma we have the following.

**Corollary 1.** \( \mathcal{EB}_\omega \) satisfies RP, but \( \mathcal{EC}_\omega \) does not satisfy RP.

**Theorem 6.** For any natural number \( n \geq 2 \), \( \mathcal{CCEB}_n \) and \( \mathcal{CCEC}_n \) satisfy RP.

**Proof.** (RP for \( \mathcal{CCEB}_n \)) Let \( A = (A, \leq^A, (I^A_i)_{i=1}^n, \geq^A) \) and \( B = (B, \leq^B, (I^B_i)_{i=1}^n, \geq^B) \) be structures from \( \mathcal{CCEB}_n \) such that \( r(A) \neq \emptyset \), and let \( r \) be a natural number. There are decompositions of sets
\[
A = \bigcup_{i=1}^k A_i \text{ and } B = \bigcup_{i=1}^m B_i
\]
into non-empty maximal chains with respect to \( \leq^A \) and \( \leq^B \). The linear orderings \( \leq^A \) and \( \leq^B \) induce linear ordering of maximal chains, and without loss of generality we may assume to have:
\[
A_1 \leq^A \cdots \leq^A A_k \text{ and } B_1 \leq^B \cdots \leq^B B_m.
\]
It must be that \( k \leq m \leq n \). Let \( (a_i)_{i=1}^k \) be a sequence defined such that for all \( 1 \leq i \leq k \) we have \( a_i = |A_i| \), and let
\[
b = \max\{|B_1|, \ldots, |B_k|\}.
\]
By Theorem 9 there is \( c \) such that:
\[
c \rightarrow (b)^{(a_1, \ldots, a_k)}.
\]
Define \( C = (C, \leq^C, (I^C_i)_{i=1}^n, \geq^C) \in \mathcal{CCEB}_\omega \) such that \( C = \bigcup_{i=1}^m C_i \) is a decomposition into maximal chains with respect to \( \leq^C \), and \( I^C_i(x) \) for all \( 1 \leq i \leq m \) and all \( x \in C_i \). The linear ordering \( \leq^C \) is defined such that each \( C_i \) is an interval with respect to \( \leq^C \) and the induced linear ordering on chains is given by
\[
C_1 \leq^C \cdots \leq^C C_m.
\]
The partial ordering $\leq^C$ is defined such that for $i \neq j$, the elements $x$ and $y$ are incomparable if $x \in C_i$, $y \in C_j$, and $\leq^C$ is restriction of $\leq^C$ on each $C_i$. For $1 \leq i \leq k$ we have $|C_i| = c$, and for $k < i \leq m$ we have $|C_i| = |B_i|$.

We claim that $C \rightarrow (B)^A_r$. So, let

$$p : \left( \begin{array}{c} C \\ \mathcal{A} \end{array} \right) \rightarrow \{1, \ldots, r\}$$

be a given coloring. Then there is an induced coloring

$$\bar{p} : \left( \begin{array}{c} C_1 \times \cdots \times C_k \\ a_1 \times \cdots \times a_k \end{array} \right) \rightarrow \{1, \ldots, r\},$$

$$\bar{p}(F_1, \ldots, F_k) = p(\mathcal{F}),$$

where $\mathcal{F} \in \left( \begin{array}{c} C \\ \mathcal{A} \end{array} \right)$ is given as a substructure of $C$ on the underlying set $F_1 \cup \cdots \cup F_k$. Note that every $\mathcal{G} \in \left( \begin{array}{c} C \\ \mathcal{A} \end{array} \right)$ is given by an unique sequence from $\left( \begin{array}{c} C_1 \\ a_1 \end{array} \right) \times \cdots \times \left( \begin{array}{c} C_k \\ a_k \end{array} \right)$. Therefore, we have $(E_i)_{i=1}^k$ where $E_i \in \left( \begin{array}{c} C_i \\ a_i \end{array} \right)$ for $1 \leq i \leq k$ such that

$$\bar{p} \downarrow \left( \begin{array}{c} E_1 \times \cdots \times E_k \\ a_1 \times \cdots \times a_k \end{array} \right) = \text{const}.$$ 

Clearly, the structure $\mathcal{L} \in \left( \begin{array}{c} C \\ \mathcal{B} \end{array} \right)$ given by sequences $(E_i)_{i=1}^k$ and $(C_i)_{i=k+1}^m$ satisfies $p \downarrow \left( \begin{array}{c} C_r \end{array} \right) = \text{const}$, so RP is verified for $\mathcal{CEB}_n$.

(RP for $\mathcal{CEC}_n$) Let $\mathcal{B} = (A, \leq^A, (I_i^B)_{i=1}^n, \leq^A)$ and $\mathcal{B} = (B, \leq^B, (I_i^B)_{i=1}^n, \leq^B)$ be structures from $\mathcal{CEC}_n$ such that $\left( \begin{array}{c} C_r \end{array} \right) \neq \emptyset$, and let $r$ be a natural number. Without loss of generality we may assume that $x \leq^B y$ whenever $I_i^B(x), I_j^B(y)$ and $i < j$. Sets $A$ and $B$ are decomposed as

$$A = \bigcup_{i=1}^a A_i \quad \text{and} \quad B = \bigcup_{i=1}^b B_i$$

into maximal antichains with respect to $\leq^A$ and $\leq^B$ respectively. In addition to this we may assume that $\leq^A$ and $\leq^B$ induce linear orderings

$$A_1 \leq^A \cdots \leq^A A_a \quad \text{and} \quad B_1 \leq^B \cdots \leq^B B_b.$$ 

The structure $\mathcal{B}$ can be seen as a substructure of some $\mathcal{C} = (C, \leq^C, (I_i^C)_{i=1}^n, \leq^C)$ from $\mathcal{CEC}_n$ such that $C$ is decomposed as $C = \bigcup_{i=1}^b C_i$ and $|C_i| = n$ for all $1 \leq i \leq b$. Also we may assume that $x \leq^C y$ whenever $I_i^C(x), I_j^C(y)$ and $i < j$. Note that for $\mathcal{D} \in \mathcal{CEC}_n$ with $\mathcal{D} \rightarrow \left( \begin{array}{c} C \end{array} \right)^A_r$, we have $\mathcal{D} \rightarrow \left( \begin{array}{c} C \end{array} \right)^A_r$. By Theorem 3, there is a natural number $c$ such that

$$c \rightarrow (b)^a_c.$$

We define $\mathcal{D} = (D, \leq^D, (I_i^D)_{i=1}^n, \leq^D) \in \mathcal{CEC}_n$ such that we have a decomposition

$$D = \bigcup_{i=1}^c D_i$$

into maximal antichains with respect to $\leq^D$ such that

$$|D_1| = \cdots = |D_a| = n,$$

the induced linear ordering is $D_1 \leq^D \cdots \leq^D D_c$, and $x \leq^D y$ whenever $I_i^D(x), I_j^D(y)$ and $i < j$. Note that every $\mathcal{F} \in \left( \begin{array}{c} C \end{array} \right)$ is completely given by a set $H \in \left( \begin{array}{c} C \end{array} \right)$ such that the underlying set of $\mathcal{F}$ is subset of $\bigcup_{j \in H} D_j$. Clearly, we have $\mathcal{D} \rightarrow \left( \begin{array}{c} C \end{array} \right)^A_r$. \hfill $\square$

The Ramsey property of classes $\mathcal{CEB}_1 = \mathcal{EB}_1$ and $\mathcal{CEB}_1 = \mathcal{CE}_1$ follows directly from Theorem 3. Using similar argument as in the previous Theorem we get the following.

**Corollary 2.** (i) For $\mathcal{B} \in \mathcal{CEB}_{\leq n}$ and $\mathcal{CEC}_{\leq n}$ satisfy RP, $n \geq 2$.

(ii) $\mathcal{CEB}_{\omega}$ and $\mathcal{CEC}_{\omega}$ satisfy RP, and for $\omega \in \mathcal{CEB}_{\leq \omega}$ and $\mathcal{CEC}_{\leq \omega}$ satisfy RP

**Remark 1.** In many examples, classes that satisfy RP also satisfy AP, but classes $\mathcal{CEC}_n$ for $n \geq 2$ and $\mathcal{CEC}_{\omega}$ are examples of classes which satisfy HP but they do not satisfy JEP and AP.
4. \( T \)

Let \( \mathcal{L}_2 \) be the class of finite structures of the form \((A, \leq^A, \preceq^A)\), where \( \preceq^A \) and \( \leq^A \) are linear orderings on the set \( A \). Let \( \mathcal{A} = (A, \leq^A, \preceq^A) \) and \( \mathcal{B} = (B, \leq^B, \preceq^B) \) be structures from \( \mathcal{L}_2 \). An embedding from \( \mathcal{A} \) into \( \mathcal{B} \) is a map \( f : A \to B \) such that for all \( a_1, a_2 \in A \) we have

\[
a_1 \leq^A a_2 \leftrightarrow f(a_1) \leq^B f(a_2) \quad \text{and} \quad a_1 \preceq^A a_2 \leftrightarrow f(a_1) \preceq^B f(a_2).
\]

**Theorem 7.** \([17]\) \( \mathcal{L}_2 \) is a Ramsey class.

We consider a class \( T \) of finite structures \( \mathcal{A} = (A, I^A, \leq^A, \preceq^A) \) where:

1. \( I^A \) is a unary relation on \( A \),
2. \( \preceq^A \) is a linear relation on \( \{ a \in A : I^A(a) \} \).

Let \( \mathcal{A} = (A, I^A, \leq^A, \preceq^A) \) and \( \mathcal{B} = (B, I^B, \leq^B, \preceq^B) \) be two structures from the class \( T \). An embedding from \( \mathcal{A} \) into \( \mathcal{B} \) is an injection \( f : A \to B \) such that for all \( a, b \in A \) we have:

1. \( I^A(a) \leftrightarrow I^B(f(a)) \),
2. \( a \leq^A b \leftrightarrow f(a) \leq^B f(b) \),
3. \( a \preceq^A b \leftrightarrow f(a) \preceq^B f(b) \) whenever \( I^A(a) \) and \( I^A(b) \).

If there is an embedding from \( \mathcal{A} \) into \( \mathcal{B} \), we write \( \mathcal{A} \hookrightarrow \mathcal{B} \). If the embedding is given by the identity map we say that \( \mathcal{A} \) is a substructure of \( \mathcal{B} \), and we write \( \mathcal{A} \subseteq \mathcal{B} \). If there is an embedding which is a bijection we say that \( \mathcal{A} \) is isomorphic to \( \mathcal{B} \), \( \mathcal{A} \cong \mathcal{B} \).

We need the following result.

**Theorem 8.** \([19]\) Let \( (A_i)_{i=1}^l \) be a sequence of Ramsey classes of finite structures and let \( r \) be a natural number. Let \( (A_i)_{i=1}^l \) and \( (B_i)_{i=1}^l \) be sequences of finite structures such that \( A_i \in A_i, B_i \in A_i, \) and \( (A_i)_{i=1}^l \neq \emptyset \) for \( 1 \leq i \leq l \). Then, there is a sequence \( (C_i)_{i=1}^l \) such that \( C_i \in A_i \) for all \( 1 \leq i \leq l \) and such that for every coloring

\[
p : \left( \frac{C_1}{A_1} \right) \times \cdots \times \left( \frac{C_l}{A_l} \right) \to \{1, \ldots, r\},
\]

there is a sequence of structures \( \left( \frac{E_i}{A_i} \right)_{i=1}^l \) where \( E_i \in \left( \frac{C_i}{A_i} \right) \) for \( 1 \leq i \leq l \) and such that

\[
p \upharpoonright \left( \frac{E_1}{A_1} \right) \times \cdots \times \left( \frac{E_l}{A_l} \right) = \text{const}.
\]

For structures that satisfy the statement of the previous theorem we use arrow notation

\[
(C_1, \ldots, C_l) \rightarrow (B_1, \ldots, B_l)^{(A_1, \ldots, A_l)}.
\]

In particular, if we take in the previous Theorem \( A_i = \cdots = A_i \) to be the class of finite sets, then we get the product Ramsey theorem.

**Theorem 9.** \([6]\) Let \( (a_i)_{i=1}^l \) and \( (b_i)_{i=1}^l \) be sequences of non zero natural numbers, and let \( r \) be a natural number. Then there is a natural number \( c \) such that for every sequence of sets \( (C_i)_{i=1}^l \) with cardinality \( |C_1| = \cdots = |C_l| = c \) and every coloring

\[
p : \left( \frac{C_1}{a_1} \right) \times \cdots \times \left( \frac{C_l}{a_l} \right) \to \{1, \ldots, r\},
\]

there is a sequence of sets \( (E_i)_{i=1}^l \) where \( E_i \in \left( \frac{C_i}{b_i} \right) \) for \( 1 \leq i \leq l \) and such that

\[
p \upharpoonright \left( \frac{E_1}{a_1} \right) \times \cdots \times \left( \frac{E_l}{a_l} \right) = \text{const}.
\]

We use arrow notation to denote the statement of the previous Theorem as

\[
c \rightarrow (b_1, \ldots, b_l)^{(a_1, \ldots, a_l)}.
\]

**Proposition 1.** The class \( T \) satisfies RP.
Proof. Let \( r \) be a natural number. Let \( \mathbb{A} = (A, I^A, \leq^A, \preceq^A) \) and \( \mathbb{B} = (B, I^B, \leq^B, \preceq^B) \) be structures from \( T \) such that \( \left( \mathbb{B}^r \right) \neq \emptyset \).

First, we consider a class \( \mathcal{K}(\alpha) \) of \( \alpha \)-colored sets where
\[
\alpha = (a_1), a_1 = \{0, 1\}.
\]
To each \( F = (F, I^F, \leq^F, \preceq^F) \) from \( T \) we assign \( \bar{\Delta}_1(F) = (F, f^F, \leq^F) \) from \( \mathcal{K}(\alpha) \) such that the linear ordering on the set \( F \) is the same, and \( f^F \) is defined according to the unary relation \( I^F \):
\[
f^F(x) = 1 \iff I^F(x), \text{ for } x \in F.
\]
In particular, \( \Delta_1(\mathbb{A}) = (A, f^A, \leq^A) \) and \( \Delta_2(\mathbb{B}) = (B, f^B, \leq^B) \) are from \( \mathcal{K}(\alpha) \). Also, to each \( F = (F, I^F, \leq^F, \preceq^F) \) from \( T \) we assign the set \( \sigma(F) = \{ x \in F : I^F(x) \} \) and the structure \( \Delta_2(F) = (\sigma(F), \leq^F \cap \sigma(F), \preceq^F) \) from \( L_2 \). In particular we have structures \( \Delta_2(\mathbb{A}) = (\sigma(\mathbb{A}), \leq^{A \setminus B}, \preceq^{A \setminus B}) \) and \( \Delta_2(\mathbb{B}) = (\sigma(\mathbb{B}), \leq^B, \preceq^B) \) from \( L_2 \). By Theorem 4, \( \mathcal{K}(\alpha) \) is a Ramsey class and by Theorem 7, \( L_2 \) is a Ramsey class. Then by Theorem 8 there are \( C_1 = (C_1, f^{C_1}, \leq^{C_1}) \in \mathcal{K}(\alpha) \) and \( C_2 = (C_2, \leq^{C_2}, \preceq^{C_2}) \in L_2 \) such that
\[
(C_1, C_2) \to (\Delta_1(\mathbb{B}), \Delta_2(\mathbb{B}))_{\gamma(\Delta_1(\mathbb{A}), \Delta_2(\mathbb{A}))}.
\]

We consider a partition of the set \( C_1 = C_{10} \cup C_{11} \) given by the unary relation \( I^{C_1} \) where \( C_{11} = \{ c \in C_1 : I^{C_1}(c) \} \). Then, we define a set \( C \) as the disjoint union:
\[
C = C_{10} \cup (C_{11} \times C_2)
\]
and define the structure \( \mathbb{C} = (C, I^C, \leq^C, \preceq^C) \in T \) in the following way. Let \( c \in C \). We define projections
\[
\pi_1 : C \to C_1
\]
\[
\pi_1(c) = \begin{cases} 
    c : c \in C_{10}, \\
    c_1 : c = (c_1, c_2) \in C_{11} \times C_2
\end{cases}
\]
and
\[
\pi_2 : C_{11} \times C_2 \to C_2
\]
\[
\pi_2(c) = c_2 \text{ where } c = (c_1, c_2) \in C_{11} \times C_2.
\]
Let \( c \in C \). Then, we define the unary relation \( I^C \) such that:
\[
I^C(c) \iff c \in C_{11} \times C_2.
\]
The linear ordering \( \leq^C \) is given such that:
\[
c \leq^C c' \iff ((\pi_1(c) \leq^C \pi_1(c')) \text{ or } (I^C(c) \& I^C(c') \& \pi_1(c) = \pi_1(c') \& \pi_2(c) \leq^C \pi_2(c'))).
\]
The linear ordering \( \leq^C \) is defined only on the set \( \{ c \in C : I^C(c) \} \), so if \( I^C(c) \) and \( I^C(c') \) then we have:
\[
c \leq^C c' \iff ((\pi_2(c) \leq^C \pi_2(c')) \text{ or } (\pi_2(c) = \pi_2(c') \& \pi_1(c) \leq^C \pi_1(c')).
\]
We claim that \( C \to (B)^r \). So, let
\[
p : \left( \begin{array}{c} C \\ \mathcal{K}(\alpha) \end{array} \right) \to \{1, ..., r\}
\]
be a given coloring. For \( \mathbb{K} = (K, f^K, \leq^K) \), a substructure of \( C_1 \), and \( \mathbb{L} = (L, \leq^L, \preceq^L) \), a substructure of \( C_2 \), with the property that \(|\{ k \in K : f^K(k) = 1 \}| = |L| = a \) we assign a substructure \( \varphi(\mathbb{K}, \mathbb{L}) \) of \( \mathbb{C} \) as follows. Denote \( |K| = b \). List elements of \( K \) and \( L \) in increasing order with respect to \( \leq^K \) and \( \leq^L \)
\[
K : k_1 \leq^K \ldots \leq^K k_b \quad \text{and} \quad L : l_1 \leq^L \ldots \leq^L l_a,
\]
and \( k \in K \) such that \( f^K(K) = 1 \) are given by the increasing subsequence
\[
k_{s_1} \leq^K \ldots \leq^K k_{s_a}.
\]
Now, we consider the set of points \( M = \{ m_1, ..., m_b \} \) where
\[
m_i = \begin{cases} 
    (k_{s_i}, t_{s_i}) : & \text{for } 1 \leq i \leq a \\
    k_i : & \text{otherwise}
\end{cases}
\]
Then \( \varphi(\mathbb{K}, L) \) is a substructure of \( \mathbb{C} \) with underlying set \( M \). Then, there is an induced coloring:
\[
\hat{p} : \left( \Delta_1(\mathbb{A}) \times \Delta_2(\mathbb{A}) \right) \to \{1, \ldots, r\},
\]
\[
\hat{p}(\mathbb{K}, L) = p(\varphi(\mathbb{K}, L)).
\]
Clearly, this is well defined, and there are structures \( \mathbb{B}_1 = (B_1, f^{B_1}, \leq^{B_1}) \in (\Delta_1(\mathbb{B})) \) and \( \mathbb{B}_2 = (B_2, f^{B_2}, \leq^{B_2}) \in (\Delta_2(\mathbb{B})) \) such that
\[
\hat{p} \mid \left( \mathbb{B}_1 \right) \Delta_1(\mathbb{A}) \times \Delta_2(\mathbb{A}) = \text{const}.
\]
Note that \( \varphi(\mathbb{B}_1, \mathbb{B}_2) \cong \mathbb{B} \) and that every \( M \in (\mathbb{A})_i \) is of the form \( \varphi(U, V) \) for some \( U \in (\Delta_1(\mathbb{A})) \) and \( V \in (\Delta_2(\mathbb{A})) \). In particular we have
\[
p \mid \left( \frac{\mathbb{B}}{\mathbb{A}} \right) = \text{const},
\]
so RP for \( T \) is verified. \( \square \)

5. RP for \( \text{COB}_n \)

For \( n = 1, \text{COB}_1 = \mathcal{OB}_1 \) and the Ramsey property for this class is established in [17], so in the proof of the following Theorem we will concentrate on the case \( n > 1 \).

**Theorem 10.** For any natural number \( n \), \( \text{COB}_n \) is a Ramsey class.

**Proof.** Let \( r \) be a natural number. Let \( \mathbb{A} = (A, \leq^{A}, (I^n_i)_i=(1,n), \leq^{A}) \) and \( \mathbb{B} = (B, \leq^{B}, (I^n_i)_i=(1,n), \leq^{B}) \) be structures from \( \text{COB}_n \) such that \( \mathbb{B} \neq \emptyset \). For \( 1 \leq i \leq n \), we consider two sequences of sets \( (A_i)_{i=1}^n \) and \( (B_i)_{i=1}^n \) such that
\[A_i = \{ a \in A : I_i(a) \} \quad \text{and} \quad B_i = \{ b \in B : I_i(b) \} .\]

Without loss of generality, we may assume that \( A_i \neq \emptyset \) for \( 1 \leq i \leq k \) and \( A_i = \emptyset \) for \( i > k \), for some \( k \leq n \). Similarly, we may assume that \( B_i \neq \emptyset \) for \( 1 \leq i \leq l \) and \( A_i = \emptyset \) for \( i > l \), with some \( l \leq n \). Note that \( k + l \leq n \). Also we may assume that the linear orderings \( \leq^{A} \) and \( \leq^{B} \) induce linear orderings on the sets \( (A_i)_{i=1}^k \) and \( (B_i)_{i=1}^l \) as follows:
\[A_1 \prec^A A_2 \prec^A \cdots \prec^A A_k \quad \text{and} \quad B_1 \prec^B B_2 \prec^B \cdots \prec^B B_l.\]

There are sequences \( (A_i = (A_i, \leq^{A} | A_i, \leq^{A})_{i=1}^k \) and \( (B_i = (B_i, \leq^{B} | B_i, \leq^{B})_{i=1}^l \) of structures from \( \mathcal{L}_2 \). Note that every structure from \( \text{COB}_n \) is completely given by a sequence from \( \mathcal{L}_2 \) and by a linear ordering of its incomparable chains.

By Theorem 7 and Theorem 8 there is a sequence \( (\mathbb{C}_i)_{i=1}^k \) of structures from \( \mathcal{L}_2 \) such that
\[(C_1, \ldots, C_k) \to (B_1, \ldots, B_k)_{i=k+1}(A_1, \ldots, A_k).
\]
We may assume that the sets \( (C_i)_{i=1}^k \) and \( (B_i)_{i=k+1} \) are disjoint, and we define
\[C = (\bigcup_{i=1}^k C_i) \cup (\bigcup_{i=k+1}^l B_i).
\]
Now, we use \( (\mathbb{C}_i)_{i=1}^k \) and \( (B_i)_{i=k+1}^l \) in order to define \( C = (C, \leq^C, (I^n_i)_{i=1}^n, \leq^C) \in \text{COB}_n \). Let \( c, c' \in C \). Unary relation \( I^C_i \) is defined such that
\[I^C_i(c) \iff ((i \leq k \text{ and } c \in C_i) \text{ or } (i > k \text{ and } c \in B_i)).\]
The partial ordering \( \leq^C \) is defined with
\[\leq^C = \bigcup_{i=1}^k \leq^{C_i} \cup \bigcup_{i=k+1}^l \leq^{B_i} \cup B_i.
\]
The linear ordering \( \preceq^C \) is defined such that it induces the following linear ordering
\[C_1 \prec^C \cdots \prec^C C_k \prec^C B_{k+1} \prec^C \cdots \prec^C B_l.
\]
and such that we have
\[ \preceq^C | C_i \preceq^C | A, 1 \leq i \leq k, \]
\[ \preceq^C | B_i \preceq^B | B, k + 1 \leq i \leq l. \]

We claim that \( C \rightarrow (\mathbb{B})^k_A \). So let
\[ p : \left( \frac{C}{A} \right) \rightarrow \{1, \ldots, r\} \]
be a given coloring. Then, there is an induced coloring
\[ \bar{p} : \left( \frac{C_1}{A_1} \right) \times \cdots \times \left( \frac{C_k}{A_k} \right) \rightarrow \{1, \ldots, r\}, \]
\[ \bar{p}(F_1, \ldots, F_k) = p(F), \]
where \( F \in \binom{C}{r} \) is such that its restriction to \( C_1, \ldots, C_k \) gives \( F_1, \ldots, F_k \). Then we have \( (\mathbb{B})^k_i \) where \( \mathbb{B}_i \in \binom{C_i}{r} \) for \( 1 \leq i \leq k \) and such that
\[ p \upharpoonright \left( \frac{\mathbb{B}_1}{A_1} \right) \times \cdots \times \left( \frac{\mathbb{B}_k}{A_k} \right) = \text{const}. \]
The sequence \( (\mathbb{B}_i)^k_{i=1} \) with \( (\mathbb{B}_i)^k_{i=k+1} \) gives a structure \( K \in \binom{C}{r} \) such that \( p \upharpoonright \left( \frac{K}{A} \right) = \text{const} \), so RP is verified for the class \( \text{COB}_n \).

By the same approach as above we have the following.

**Corollary 3.** (i) For \( \subseteq n \in \text{lo}(\{1, \ldots, n\}) \), \( \text{COB}_{\subseteq n} \) satisfies RP, \( n \geq 2 \).
(ii) \( \text{COB}_{\omega} \) satisfy RP, and for \( \subseteq \omega \in \text{lo}(\{1, 2, \ldots\}) \), \( \text{COB}_{\subseteq \omega} \) satisfies RP.

### 6. RP for \( \text{COC}_n \)

Since \( \text{COC}_1 = L_2 \), then by Theorem 7 we get the Ramsey property for \( \text{COC}_1 \), so the proof of the following Theorem is focused on \( n \geq 2 \).

**Theorem 11.** For any natural number \( n \), \( \text{COC}_n \) is a Ramsey class.

**Proof.** Let \( A = (A, \preceq^A, (I^A_i)_{i=1}^n, \preceq^A) \) be a structure from \( \text{COC}_n \). Then the set \( A \) is decomposed into maximal antichains with respect to the partial ordering \( \preceq^A \) such that
\[ A = A_1 \cup \cdots \cup A_n, \]
and without loss of generality we may assume that \( \preceq^A \) induces a linear ordering on the sets \( \{A_1, \ldots, A_n\} \) as follows
\[ A_1 \preceq^A \cdots \preceq^A A_n. \]

To a structure \( A \) we assign a sequence \( (\Delta^A_i(A) = (\{1, \ldots, a\}, I^i, \preceq^i, \preceq^i))_{i=1}^n \) of structures from \( T \).

The linear ordering \( \preceq^i \) is given by \( 1 <^i 2 <^i \cdots <^i a \). The unary relation \( I^i \) is given such that
\[ I^i(x) \Leftrightarrow (\exists y \in A)[I^i_1(y) \land y \in A_x]. \]

For \( x, x' \in \{1, \ldots, a\} \) with \( I^i(x) \) and \( I^i(x') \) we define a linear ordering \( \preceq^i \) such that
\[ x <^i x' \Leftrightarrow (\exists y, y' \in A)[y \in A_x \land y' \in A_{x'} \land y <^A y']. \]

Note that any sequence \( (\Delta^A_i(A))_{i=1}^n \) of structures from \( T \) of the same length gives the unique structure from \( \text{COC}_n \).

Let \( r \) be a natural number and let \( A = (A, \preceq^A, (I^A_i)_{i=1}^n, \preceq^A) \) and \( B = (B, \preceq^B, (I^B_i)_{i=1}^n, \preceq^B) \) be structures from \( \text{COC}_n \) such that \( \binom{A}{r} \neq \emptyset \). We consider the sequences \( (\Delta^A_i(A))_{i=1}^n \) and \( (\Delta^B_i(B))_{i=1}^n \) of structures from \( T \).

Note that \( (\Delta^A_i(A)) \neq \emptyset \) for all \( 1 \leq i \leq n \). By Theorem 1 and Theorem 8 there is a sequence \( (C_i = (C_i, I^{C_i}, \preceq^{C_i}, \preceq^{C_i}))_{i=1}^n \) of structures from \( T \) such that
\[ (C_1, \ldots, C_n) \rightarrow (\Delta^A(B), \ldots, \Delta^B(B))_{(\Delta^A_1(A), \ldots, \Delta^A_n(A))}. \]

Now, we define the set
\[ C_0 = C_1 \times \cdots \times C_n \]
and \(d = |C_0|\). On the set \(C_0\) we define a linear ordering \(\leq_{C_0}\) such that for distinct \(c = (c_1, ..., c_n)\) and \(c' = (c'_1, ..., c'_n)\) from \(C_0\) we have
\[
eq_{C_0} c' \iff (c_i < c'_i & i = \min\{j: c_j \neq c'_j\}).
\]
We define the structure \(C = (C, \leq^C, (I_i^n)_{i=1}^n, \leq^C)\) from \(\text{COC}_n\). Set \(C\) is decomposed into the disjoint union
\[C = \bigcup_{c \in C_0} C_c\]
where each \(C_c\) is a maximal antichain with respect to \(\leq^C\) and the linear ordering \(\leq^C\) induces a linear ordering of the set \(\{C_c\}_{c \in C_0}\) such that
\[C_c \leq^C C_{c'} \iff c \leq^C c'
\]
for all \(c, c' \in C_0\). Now, we have to define the unary relations \((I_i^n)_{i=1}^n\). First, let \(c = (c_1, ..., c_n)\) be a point from \(C_0\). If \(I_i^n(c_i)\) then the set \(C_c\) contains exactly one point indicated by \(I_i^n\), otherwise set \(C_c\) does not contain any point indicated by \(I_i^n\). Clearly, this defines unary relations \((I_i^n)_{i=1}^n\). For \(1 \leq i \leq n\) we define the linear ordering \(\leq^C\) by defining its restriction to each \(x \in C : I_i^n(x)\) such that for distinct \(x, y \in C\) with \(I_i^n(x), I_i^n(y)\) and \(x \in C_c, y \in C_{c'}\), \(c = (c_1, ..., c_n)\) and \(c' = (c'_1, ..., c'_n)\) we have
\[x \leq^C y \iff c_i \leq^C c'_i.
\]
It remains to show that \(C_c \neq \emptyset\) for all \(c \in C_0\). If this happens we have just to take the restriction of \(C\) to the set of nonempty antichains \(C_c\). So without loss of generality we assume that \(C\) is well defined.
We claim that \(C \rightarrow (B)_{r}^A\). So let
\[p : (C_A^n) \rightarrow \{1, ..., r\}\]
be a given coloring. Then we have the induced coloring
\[\tilde{p} : (C_A^n) \times \cdots \times (C_A^n) \rightarrow \{1, ..., r\},\]
\[\tilde{p}(\mathcal{F}_1, ..., \mathcal{F}_n) = p(\mathcal{F}),\]
where \(\mathcal{F} \in (A)^n_A\) is the unique structure such that \((\Delta_1(\mathcal{F}), ..., \Delta_n(\mathcal{F})) = (\mathcal{F}_1, ..., \mathcal{F}_n)\). Then there is a \((A_i)_{i=1}^n\) where \(A_i \in (C_i^n)_{A_i}\) for \(1 \leq i \leq n\) and such that
\[\tilde{p} | (A_1)_{A_1} \times \cdots \times (A_n)_{A_n} = \text{const}.
\]
Clearly, the sequence \((A_i)_{i=1}^n\) gives us a structure \(G \in (C)^n_A\) such that \((E_i)_{i=1}^n = (\Delta_i(G))_{i=1}^n\) and \(p | (C)_{A} = \text{const}\), so \(\text{RP}\) is verified.

Note that \(\text{COC}_n\) for \(n \geq 2\) is also an example of the class which satisfies \(\text{HP}\) and \(\text{RP}\) but it does not satisfy \(\text{JEP}\) and \(\text{AP}\).

**Corollary 4.** (i) For \(\sqsubseteq_n \epsilon \text{lo}(\{1, ..., n\})\), \(\text{COC}_{\sqsubseteq_n}\) satisfy \(\text{RP}\), \(n \geq 2\).
(ii) \(\text{COC}_\omega\) satisfy \(\text{RP}\), and for \(\sqsubseteq_\omega \epsilon \text{lo}(\{1, 2, ..., \})\), \(\text{COC}_{\sqsubseteq_\omega}\) satisfy \(\text{RP}\).

### 7. RP for \(\text{OB}_n\) and \(\text{OC}_n\)

We will use a similar idea as in the proof of the Ramsey property for the class \(\text{COC}_n\). Since \(\text{OB}_1 = \text{OC}_1 = \mathcal{L}_2\) is a Ramsey class, proofs in this section will focus on \(n > 1\).

**Theorem 12.** For any natural number \(n\), \(\text{OB}_n\) is a Ramsey class.

**Proof.** Let \(A = (A, \leq^A, (I_i^n)_{i=1}^n, \leq^A)\) be a structure from \(\text{OB}_n\). Then the set \(A\) is decomposed into maximal chains with respect to the partial ordering \(\leq^A\) such that
\[A = A_1 \cup \cdots \cup A_k,
\]
\[A_i = \{a \in A : I_i^n(a)\} \text{ for } 1 \leq k \leq n.
\]
To the structure $\mathcal{A}$ we assign a sequence $\{(\Delta_i(\mathcal{A}))^n_{i=0}\}$ of finite structures. The structure $\Delta_0(\mathcal{A}) = (A, f^A, \leq^A) \in \mathcal{K}(\alpha)$ for $\alpha = (\alpha_1)$, $\alpha_1 = \{1, \ldots, n\}$, is defined such that

$$f^A(a) = i \iff a \in A_i.$$  

For $1 \leq i \leq n$, the structures $\Delta_i(\mathcal{A}) = (A_i, \leq_i, \leq_i)$ are from $L_2$ such that

$$\leq_i = \leq^A | A_i \land \leq_i = \leq^A | A_i.$$  

So the structure $\Delta_0(\mathcal{A})$ carries information about the distribution of indicators, and structures $\Delta_i(\mathcal{A})$ carries information how we put the partial ordering on each $A_i$ with respect to $\leq^A$.

Let $\mathcal{A} = (A, \leq^A, (I_i^A)^n_{i=1}, \leq^A)$ and $\mathcal{B} = (B, \leq^B, (I_i^B)^n_{i=1}, \leq^B)$ be structures from $\mathcal{OB}_n$ such that $\mathcal{B} \neq \emptyset$, and let $r$ be a natural number. Then we have assigned sequences of structures

$$(\Delta_i(\mathcal{A}))^n_{i=0}, \quad \Delta_i(\mathcal{B})^n_{i=0},$$  

where

$$A_i = \{a \in A : I_i^A(a) \land B_i = \{b \in B : I_i^B(b)\}$$  

for $1 \leq i \leq n$. According to Theorem 4, Theorem 7, and Theorem 8 there is a sequence of structures $(C_i)^n_{i=0}$ such that $C_0 = (C_0, f^{C_0}, \leq^{C_0}) \in \mathcal{K}(\alpha)$, $C_i = (C_i, \leq_i^{C_i}, \leq_i^{C_i}) \in L_2$ for $1 \leq i \leq n$ and

$$(C_0,\ldots,C_n) \rightarrow (\Delta_0(\mathcal{B}),\ldots,\Delta_n(\mathcal{B}))(\Delta_0(\mathcal{A}),\ldots,\Delta_n(\mathcal{A})).$$

We have a partition of the set $C_0$ given by the function $f^{C_0}$ as follows:

$$C_0 = \bigcup^n_{i=1} C_{0,i},$$

$$C_{0,i} = \{c \in C_0 : f^{C_0}(c) = 1\}, 1 \leq i \leq n.$$  

Now, we define $D = (D, \leq^D, (I_i^D)^n_{i=1}, \leq^D) \in \mathcal{OB}_n$ using the sequence $(C_i)^n_{i=0}$. The underlying set of the structure $D$ is given by the disjoint union

$$D = \bigcup^n_{i=1} D_i,$$

$$D_i = C_{0,i} \times C_i.$$  

The unary relations $(I_i^D)^n_{i=1}$ are defined such that $D_i = \{d \in D : I_i^D(d)\}$. In order to define the linear ordering $\leq^D$ and the partial ordering $\leq^D$ let $d = (d_0, d_1)$ and $d' = (d_0', d_1')$ be from $D$. First, we define the linear ordering

$$d \leq^D d' \iff (d_0 \leq^{C_0} d_0') \lor (d_0 = d_0' \land d_1 \leq^{C_i} d_1').$$

The partial ordering $\leq^D$ is defined such that $d$ and $d'$ which belong to distinct $D_i$ are incomparable, and if $d$ and $d'$ belong to the same $D_i$, then the restriction of $\leq^D$ to $D_i$ is a linear ordering defined as follows

$$d \leq^D d' \iff (d_1 \leq^{C_i} d_1') \lor (d_1 = d_1' \land d_0 \leq^{C_0} d_0').$$

Let $(\mathbb{H})^n_{i=0}$ be a sequence of structures such that $\mathbb{H}_i \leq C_i$ for $0 \leq i \leq n$. Let $\mathbb{H}_0 = (H_0, f^{H_0}, \leq^{H_0})$ and let

$$H_{0,i} = \{h \in H_0 : f^{H_0}(h) = i\}.$$  

Also, we have $H_i = (H_i, \leq^{H_i}, \leq^{H_i})$ for $1 \leq i \leq n$. Suppose that we have $|H_{0,i}| = |H_i| = s_i$ for all $1 \leq i \leq n$. For each $i$ we list points from $H_{0,i}$ in increasing order with respect to $\leq^{H_0}$ as

$$h_{0,i,1} \prec^{H_0} \cdots \prec^{H_0} h_{0,i,s_i},$$

and we list points of $H_i$ in increasing order with respect to $\leq^{H_i}$ as

$$h_{i,1} \prec^{H_i} \cdots \prec^{H_i} h_{i,s_i}.$$  

Then we consider a set $G_i \subset H_{0,i} \times H_i \subset D_i$ such that

$$G_i = \{(h_{0,i,j}, h_{i,j}) : 1 \leq j \leq s_i\}.$$
Finally we have a set
\[ G = \bigcup_{i=1}^{n} G_i \subset D \]
and we define the structure \( \varphi(\mathbb{H}_0, ..., \mathbb{H}_n) \) to be a substructure of \( D \) defined on the set \( G \), i.e.
\[ \varphi(\mathbb{H}_0, ..., \mathbb{H}_n) = D \upharpoonright G. \]

We claim that \( D \to (\mathbb{B})^k_\varphi \). So, let us fix a coloring
\[ p : (\mathbb{B})_{\varphi} \rightarrow \{1, ..., r\}. \]
Then there is an induced coloring
\[ \bar{p} : \left( \frac{\mathbb{C}_0}{\Delta_0(\mathbb{A})} \times \cdots \times \frac{\mathbb{C}_n}{\Delta_n(\mathbb{A})} \right) \rightarrow \{1, ..., r\}, \]
\[ \bar{p}(\mathbb{E}_0, ..., \mathbb{E}_n) = p(\varphi(\mathbb{E}_0, ..., \mathbb{E}_n)). \]
So, there is a sequence \( (\mathbb{E}_i)_{i=0} \) where \( \mathbb{E}_i \in \left( \frac{\mathbb{C}_i}{\Delta_i(\mathbb{B})} \right) \) for \( 0 \leq i \leq n \) and such that
\[ \bar{p} \upharpoonright \left( \frac{\mathbb{E}_0}{\Delta_0(\mathbb{A})} \times \cdots \times \frac{\mathbb{E}_n}{\Delta_n(\mathbb{A})} \right) = \text{const}. \]
Note that \( \varphi(\mathbb{E}_0, ..., \mathbb{E}_n) \cong \mathbb{B} \) and that any \( \mathbb{K} \in (\mathbb{B}_{\varphi}) \) is of the form \( \varphi(\mathbb{K}_0, ..., \mathbb{K}_n) \) for some \( \mathbb{K}_i \in \left( \frac{\mathbb{E}_i}{\Delta_i(\mathbb{A})} \right) \) for \( 0 \leq i \leq n \). Therefore, we have
\[ p \upharpoonright \left( \varphi(\mathbb{E}_0, ..., \mathbb{E}_n) \right) = \text{const}, \]
so RP is verified for the class \( \text{OC}_n \).

\[ \text{Lemma 6. OC}_n \text{ is not a Ramsey class for } n > 1. \]

\[ \text{Proof. This follows from the fact that OC}_n \text{ is a class of rigid structures which satisfy HP and JEP and by Lemma 5.} \]

\[ \text{Corollary 5. OB}_\omega \text{ satisfies RP, but OC}_\omega \text{ does not satisfy RP.} \]

\[ \text{Remark 2. It remains an open problem to find the Ramsey degree of objects in the classes OC}_n \text{ and OC}_\omega. \]

8. Final remarks

Let \( L \) and \( L' \) be signatures such that \( L' = L \cup \{ \leq \} \) and \( \leq \not\in L \) where \( \leq \) is a binary relational symbol. Let \( \mathcal{K}' \) be a class of finite structures such that \( \leq \) is interpreted in all structures from \( \mathcal{K}' \) as a linear ordering and let \( \mathcal{K} = \mathcal{K}' | L \). Then we say that \( \mathcal{K}' \) is an ordered class. If for all structures \( \mathbb{A} \) and \( \mathbb{B} \) from \( \mathcal{K} \), which are defined on the sets \( A \) and \( B \) respectively, every embedding \( f : A \to B \), and every \( \leq_A \in \text{lo}(A) \) such that \( (\mathbb{A}, \leq_A) \in \mathcal{K}' \) there is \( \leq_B \in \text{lo}(B) \) such that \( (\mathbb{B}, \leq_B) \in \mathcal{K}' \) and \( f \) is embedding from \( (\mathbb{A}, \leq_A) \) into \( (\mathbb{B}, \leq_B) \) then we say that \( \mathcal{K}' \) is a reasonable expansion of \( \mathcal{K} \). If for every \( \mathbb{A} \in \mathcal{K} \) there is \( \mathbb{B} \in \mathcal{K} \) such that for every \( \leq_A \in \text{lo}(A) \) and every \( \leq_B \in \text{lo}(B) \) with \( (\mathbb{A}, \leq_A) \in \mathcal{K}' \) and \( (\mathbb{B}, \leq_B) \in \mathcal{K}' \) we have \( (\mathbb{A}, \leq_A) \leftrightarrow (\mathbb{B}, \leq_B) \), then we say that \( \mathcal{K}' \) satisfies the ordering property (OP) with respect to \( \mathcal{K} \). In almost all cases it is straightforward to check if we have a reasonable expansion, but OP may require more work.

\[ \text{Proposition 2. Let } n > 1 \text{ be a natural number or } n = \omega. \text{ For any natural number } n, \text{ let } \subseteq_n \in \text{lo}(\{1, ..., n\}). \text{ For } n = \omega, \text{ let } \subseteq_\omega \in \text{lo}(\{1, 2, ...\}). \]

(i) \( \text{OB}_n, \mathcal{E}B_n, \mathcal{C}OB_n, \mathcal{C}EB_n, \text{ and } \mathcal{C}OB_{\subseteq_n} \) do not satisfy OP with respect to \( \mathcal{B}_n \), but \( \mathcal{C}EB_{\subseteq_n} \) satisfies OP with respect to \( \mathcal{B}_n \).

(ii) \( \text{OC}_n, \mathcal{E}C_n, \mathcal{C}OC_n, \mathcal{C}EC_n, \text{ and } \mathcal{C}OC_{\subseteq_n} \) do not satisfy OP with respect to \( \mathcal{C}_n \), but \( \mathcal{C}EC_{\subseteq_n} \) satisfies OP with respect to \( \mathcal{C}_n \).
Proof. (i) Let \( \mathcal{A} = (A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A) \) be a structure from \( \mathcal{OB}_n \) such that
\[
A = \{a_1, a_2\}, I_1^A(a_1), I_2^A(a_2), a_1 \preceq^A a_2,
\]
and \( a_1 \) and \( a_2 \) are incomparable with respect to \( \preceq^A \). Suppose, there is \( (B, \leq^B, (I_i^B)_{i=1}^n) \in \mathcal{B}_n \) such that
\[
(A, \leq^A, (I_i^A)_{i=1}^n) \rightarrow (B, \leq^B, (I_i^B)_{i=1}^n),
\]
and let \( B_i = \{b \in B : I_i^B(b)\} \) for \( i = 1, 2 \). Then there is \( \preceq^B \) such that \( \mathcal{B} = (B, \preceq^B, (I_i^B)_{i=1}^n, \preceq^B) \in \mathcal{OB}_n \) and \( x \preceq^B y \) for all \( x \in B_2 \) and all \( y \in B_1 \). Clearly it is impossible to have \( \mathcal{A} \rightarrow \mathcal{B} \), so \( \mathcal{OB}_n \) does not satisfy OP with respect to \( \mathcal{B}_n \). In the same way, we show that \( \mathcal{EB}_n, \mathcal{COB}_n, \mathcal{CEB}_n \) do not satisfy OP with respect to \( \mathcal{B}_n \).

In order to show that \( \mathcal{COB}_n \) does not satisfy OP with respect to \( \mathcal{B}_n \), we consider the structure \( \mathcal{A} = (A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A) \) from \( \mathcal{COB}_n \) such that
\[
A = \{a_1, a_2\}, I_1^A(a_1), I_2^A(a_2), a_1 \preceq^A a_2, a_1 \leq^A a_2.
\]
Suppose, there is \( (B, \leq^B, (I_i^B)_{i=1}^n) \in \mathcal{B}_n \) such that
\[
(A, \leq^A, (I_i^A)_{i=1}^n) \rightarrow (B, \leq^B, (I_i^B)_{i=1}^n),
\]
and let \( B_1 = \{b \in B : I_1^B(b)\} \). Then there is \( \preceq^B \) such that for all \( x, y \in B_1 \) we have
\[
x \preceq^B y \Leftrightarrow y \leq^B x
\]
and \( \mathcal{B} = (B, \preceq^B, (I_i^B)_{i=1}^n, \preceq^B) \in \mathcal{COB}_n \). Clearly, it is impossible to have \( \mathcal{A} \rightarrow \mathcal{B} \), so we have contradiction.

Since for \( \mathcal{A} = (A, \leq^A, (I_i^A)_{i=1}^n) \in \mathcal{B}_n \) there is only one \( \preceq^A \in \text{lo}(A) \) such that
\[
(A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A) \in \mathcal{CEB}_n,
\]
the structure \( \mathcal{A} \) will verify for itself the condition of OP. Therefore, \( \mathcal{CEB}_n \) satisfies OP with respect to \( \mathcal{B}_n \).

(ii) This is done in the similar way as in (i). \( \square \)

We say that a topological group \( G \) is extremely amenable iff for every continuous action of \( G \) on a compact space there is a fixed point; see [2] and [8].

**Theorem 13.** [8] Let \( \mathcal{K} \) be an ordered Fraïssé class in a signature \( L \) with limit \( \mathbb{K} = F \lim(\mathcal{K}) \). Then \( \text{Aut}(\mathbb{K}) \) with the pointwise topology is extremely amenable iff \( \mathcal{K} \) is a Ramsey class.

From Section 2, we have a list of ordered Fraïssé classes, so by the previous Theorem we have the following.

**Theorem 14.** Let \( n > 1 \) be a natural number or \( n = \omega \). For any natural number \( n \), let \( \mathbb{C}_n \in \text{lo}(\{1, \ldots, n \}) \). For \( n = \omega \), let \( \mathbb{C}_\omega \in \text{lo}(\{1, 2, \ldots \}) \). Then the following groups are extremely amenable:
\[
\text{Aut}(\mathcal{OB}_n), \text{Aut}(\mathcal{EB}_n), \text{Aut}(\mathcal{COB}_n), \text{Aut}(\mathcal{CEB}_n), \text{Aut}(\mathcal{COC}_n), \text{Aut}(\mathcal{CEC}_n).
\]

We make a note about universal minimal flow and for more details we refer the reader to [8]. Using the previous Theorem together with Proposition 2 and the technique from [8], we may calculate the universal minimal flow of \( \text{Aut}(\mathcal{B}_n) \) and \( \text{Aut}(\mathcal{C}_n) \). Since \( \text{Aut}(\mathcal{B}_n) \cong (\text{Aut}(\mathbb{Q}, <))^n \) and \( \text{Aut}(\mathcal{C}_n) \cong (\text{Aut}(\mathbb{Q}, <))^\omega \), \( \text{Aut}(\mathcal{B}_n) \) and \( \text{Aut}(\mathcal{C}_n) \) are extremely amenable (product of extremely amenable groups is extremely amenable).

**References**


DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA CA 91125
E-mail address: msokic@caltech.edu