Problem Set #7

Total points: 100. Each question weighted equally.

1. This question reconsiders the problem you solved in the previous problem set (you can use your results from that problem).

   Consider $\mathbf{X}_n \equiv X_1, \ldots, X_n \sim$ *i.i.d.* $U[0, \theta]$. Let $\lambda(\mathbf{X}_n; \theta_0)$ denote the likelihood ratio statistic for testing

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_1 &: \theta \neq \theta_0 \end{aligned} \tag{1}$$

   where $0 < \theta_0 \leq 5$.

   Consider the likelihood ratio test of the form: $\mathbf{1}(\lambda(\mathbf{X}_n) < c)$, for some $c \in [0, 1]$. Recall from the previous problem set that, for a size α test, you would choose $c = \alpha$.

   (a) Is $\lambda(\mathbf{X}_n; \theta_0)$ a pivotal statistic?

   (b) Derive the $(1 - \alpha)$ confidence set for θ_0 by inverting the likelihood ratio test.

2. For the above problem, construct an *asymptotic* 95% confidence interval for θ, based on

   - the method of moments estimator $\hat{\theta}_n \equiv 2\frac{1}{n}\sum_{i=1}^{n} X_i$; and
   - the T-statistic $\frac{\sqrt{n}(\hat{\theta}_n - \theta)}{\sqrt{\hat{V}}}$. $\hat{V}$ denotes an estimate of the asymptotic variance, equal to $\frac{\hat{\theta}_n}{3}$.

3. Consider three *i.i.d.* draws X_1, X_2, X_3 from a Bernoulli experiment

$$X_i = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{with prob } 1 - p. \end{cases}$$

   Assume that $X_1 = 1$, $X_2 = 0$, and $X_3 = 1$, and let $\bar{X}_3 \equiv \frac{1}{3}(X_1 + X_2 + X_3)$ denote an estimator for p.

   (a) Derive the exact distribution of $\bar{X}_3$.

   (b) Consider the interval estimator $p \in [\bar{X}_3 - 0.1, \bar{X}_3 + 0.1]$. Derive the coverage probability of this interval for each $p \in [0, 1]$, and derive the confidence coefficient.

   (c) Use asymptotic theory (law of large numbers and central limit theorem) to derive the asymptotic distribution of $\bar{X}_3$. Derive a symmetric asymptotic 50% confidence interval for p, as a function of $\bar{X}_3$, of the sort $p \in [\bar{X}_3 - t, \bar{X}_3 + t]$, for some $t > 0$.

   (d) Analytically derive the bootstrap approximation to the distribution of $\bar{X}_3$. (Please use a spreadsheet program to do this!)

   Also construct the bootstrap approximation to the distribution of $\exp(\bar{X}_3)$. Compare your result to the one obtained from asymptotic approximation and the Delta method.

   (e) Perform a subsampling-based test (using subsamples of size $M = 2$) for the null hypothesis of $H_0 : p = 0.5$ vs. $H_1 : p \neq 0.5$.