

Problem Set #5

Total points: 100. Each question weighted equally.

1. Consider $X_1, \dots, X_n \sim i.i.d. U[0, \theta]$. The unknown parameter is θ . Derive:
- (a) a Method of Moments estimator
 - (b) a Maximum Likelihood estimator (tricky!)
 - (c) the Bayesian posterior mean estimator, for prior $\theta \sim U[0, 1]$. (Be careful about the bounds of integration!)

Note: the answer for each part may not be unique. For estimators (a) and (b), answer the following:

- (i) Derive the expectation $E(W_n)$. Is the estimator unbiased?
- (ii) Derive the variance $Var(W_n)$.
- (iii) Derive the Cramer-Rao lower-bound (if it exists). Is the estimator UMVUE?
- (iv) Derive the mean-squared error.
- (v) Is it consistent?

2. Consider $X_1, \dots, X_n \sim i.i.d. N(1, \sigma^2)$. The unknown parameter is σ^2 . Derive:

- (a) a Method of Moments estimator
- (b) a Maximum Likelihood estimator
- (c) the Bayesian posterior mean estimator, for the flat (or uniform) prior $U[0, 20]$.

Note: the answer for each part may not be unique. For estimators (a) and (b), answer the following:

- (i) Derive the expectation $E(W_n)$. Is the estimator unbiased?
- (ii) Derive the Cramer-Rao lower-bound (if it exists). Is the estimator UMVUE? (Hint: try to apply Corollary 7.3.15 rather than deriving $Var(W_n)$.)
- (iii) Is it consistent?
- (iv) For estimator (b), derive the asymptotic variance. Make any assumption that are necessary.

3. **(Labor supply models: simple examples)** Consider the structural relation

$$Y_i^* = X_i\beta + \epsilon_i$$

where Y_i^* is person i 's log wage offer, X_i denotes person i 's years of schooling, and ϵ_i denotes her unobserved ability. Assume that $\epsilon_i \sim N(0, 1)$ and is *i.i.d.* for all persons i .

- (a) We observe data $\{Y_i^*, X_i\}$, $i = 1, \dots, n$.
 - (i) Derive the likelihood function $L(\beta|Y_1, \dots, Y_n; X_1, \dots, X_n)$. (Hint: you need to derive the density of Y_i^* .)
 - (ii) Derive a method of moments estimator for β .
- (b) Assume that you only observe an indicator for whether each person works, and we know that people work only when their log wage offer exceeds zero (i.e., only when their wage exceeds one).

In other words, our data consists of:

$$Y_i = \begin{cases} 0 & \text{if } Y_i^* < 0 \\ 1 & \text{if } Y_i^* \geq 0 \end{cases}, \quad i = 1, \dots, n.$$

We observe data $\{Y_i, X_i\}$, $i = 1, \dots, n$.

- (i) Derive the likelihood function $L(\beta|Y_1, \dots, Y_n; X_1, \dots, X_n)$.
- (ii) For this part only, assume a simpler model where $X_i = 1$, for all i (ie. is a constant). Derive the asymptotic variance for the MLE of β in this case.
- (c) Now assume that you observe a person's wages as long as they work. In other words, our data consists of:

$$Z_i = \begin{cases} 0 & \text{if } Y_i^* < 0 \\ Y_i^* & \text{if } Y_i^* \geq 0 \end{cases}, \quad i = 1, \dots, n.$$

We observe data $\{Z_i, X_i\}$, $i = 1, \dots, n$.

- (i) Derive the likelihood function $L(\beta|Z_1, \dots, Z_n; X_1, \dots, X_n)$.