Problem Set #4: Miscellaneous, and Large Sample Theory

1. **(Probability limits)** Consider a sequence of random variables $\{X_i\}$, $i = 1, 2, \ldots$ which are *i.i.d.* (i.e., independently and identically distributed) with $EX_i = \mu$ and $\text{Var}X_i = \sigma^2$, for all i.

   Let $\bar{X}_n \equiv \frac{1}{n}\sum_i X_i$. For each of the following random sequences, derive the probability limit (if it exists). Be rigorous and explicit about the theorems used in each step of your argument.

   (a) $\bar{X}_n^2$
   (b) $\frac{1}{\bar{X}_n^2}$
   (c) $\exp\left(\bar{X}_n\right)$
   (d) $\frac{1}{\bar{X}_n - \mu}$

2. **(Limit distribution)** Consider a sequence of random variables $\{X_i\}$, $i = 1, 2, \ldots$ which are *i.i.d.* (i.e., independently and identically distributed) with $EX_i = \mu$ and $\text{Var}X_i = \sigma^2$, for all i.

   Let $\bar{X}_n \equiv \frac{1}{n}\sum_i X_i$. For each of the following random sequences, derive the limit distribution (if it exists). Be rigorous and explicit about the theorems used in each step of your argument.

   (a) $\bar{X}_n^2$
   (b) $\frac{1}{\bar{X}_n^2}$
   (c) $\exp\left(\bar{X}_n\right)$
   (d) $\frac{1}{\bar{X}_n - \mu}$

3. **(Asymptotic behavior of binomial experiments)** Consider two sequences of independent and identical Bernoulli experiments. In the first sequence of experiments, the outcomes are

   $$Y_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p. \end{cases}, \quad i = 1, 2, 3, \ldots$$

   In the second sequence, the outcomes are

   $$Z_i = \begin{cases} 1 & \text{with probability } q \\ 0 & \text{with probability } 1-q. \end{cases}, \quad i = 1, 2, 3, \ldots$$

   For each of the following, derive the probability limit and limit distribution (if they exist). Be rigorous and explicit about the theorems used in each step of your argument.

   (a) $\frac{1}{n}\sum_{i=1}^n Y_i$
   (b) $\frac{1}{n}\sum_{i=1}^n Y_i + Z_i$
   (c) $\frac{1}{n}\sum_{i=1}^n (1 - Y_i)$
   (d) $\frac{1}{n}\sum_{i=1}^n Y_i + (1 - Z_i)$
   (e) $\frac{1}{n}\sum_{i=1}^n Y_i + X_i$, where $X_i \equiv 1 - Y_i$.

4. **(Almost sure convergence)** Assume that $X \sim U[0,1]$. Define the random sequences

$$S_i = \begin{cases} X & \text{if } X \in \{1, 1/2, 1/4, 1/8, 1/16 \dots\} \\ \frac{X}{i} & \text{if } X \notin \{1, 1/2, 1/4, 1/8, 1/16 \dots\} \end{cases}, \; i = 1, 2, 3, \dots .$$

Does the random sequence S_i converge almost surely? If so, to what?