Problem Set #1: Probability Theory

1. (Discrete random variable) Consider the statistical experiment where a coin is tossed *twice*. Derive the probability space for the following random variables.

   a. X=#heads

   b. X=#tails

   c. X=#heads/2

   d. X=2*(#heads)+5*(#tails)

   e. X=log (2*(#heads)+5*(#tails))

2. (σ-fields) Let the sample space $\Omega \subset \mathbb{R}$, and consider the the following collection of events $\mathcal{C}$:

$$\mathcal{C} = \{A \subset \mathbb{R} : \ A \text{ is countable or } A^c \text{ is countable}\}.$$

   Is $\mathcal{C}$ a σ-field?

3. (σ-fields) Let $\Omega = (0, 1]$. Let the collection $\mathcal{C}$ of events consists of $\emptyset$ and all finite unions of disjoint intervals of the form $(c, c'] \subset \Omega$. That is, a typical element in $\mathcal{C}$ takes the form $\cup_{i=1}^{n}(l_i, u_i]$ where the intervals are disjoint. Is $\mathcal{C}$ a σ-field?

4. (Definition of cumulative distribution functions) Prove that the following functions are CDF's:
   (a) $\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x), \ x \in (-\infty, +\infty)$
   (b) $e^{-e^{-x}}, \ x \in (-\infty, +\infty)$
   (c) $F_Y(y) = \begin{cases} \frac{1-\epsilon}{1+e^{-y}} & \text{if } y < 0 \\ \epsilon + \frac{(1-\epsilon)}{1+e^{-y}} & \text{if } y \geq 0 \end{cases}$ for a given ϵ such that $0 < \epsilon < 1$.

5. (Truncated random variables) CB, 1.52.

   Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. For a fixed number x_0, define the function

$$g(x) = \begin{cases} f(x)/[1 - F(x_0)] & \text{if } x \geq x_0 \\ 0 & \text{if } x < x_0. \end{cases}$$

   Prove that $g(x)$ is a pdf. (Assume that $F(x_0) < 1$.)

   Derive, via integration, the CDF (cumulative distribution function) corresponding to the pdf $g(x)$.

6. (Transformations of random variables) Consider the random variable X which is uniformly distributed on the real unit interval [0,1]. In other words, the cdf is

$$F(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   a. Derive (by differentiation) the corresponding density function $f(x)$ (easy!)

b. Consider the random variable $Y = a + X(b - a)$ with $b > a$. Derive the cdf $G(y)$. (Hint: start from the definition of $G(y) = \text{prob}\,(Y \leq y)$.) Via differentiation, derive the corresponding density function $g(y)$.

c. Consider the random variable $Y = \log(X + 1)$. Derive the cdf $H(y)$. Via differentiation, derive the corresponding density function $h(y)$.

d. Consider the random variable $Y = F(X)$ (that is, Y is the random variable derived by evaluating $F(x)$, the cdf of X at X). Derive the cdf $K(y)$. Via differentiation, derive the corresponding density function $k(y)$.

7. Provide an example (if one exists) where (X, Y) are independent, (Y, Z) are independent, but (X, Z) are not independent.

8. Go back to the Monty Hall example (Lecture #1, pp. 6,7). Instead of Monty opening door 2, consider the case where the contestant chooses to open door 2, and discovers that there is no prize behind it. Should the contestant switch in this case? Explain.