



Identification of first-price auctions with non-separable unobserved heterogeneity

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ABSTRACT

We propose a novel methodology for identification of first-price auctions, when bidders' private valuations are independent conditional on one-dimensional unobserved heterogeneity. We extend the existing literature (Li and Vuong, 1998; Krasnokutskaya, 2011) by allowing the unobserved heterogeneity to be non-separable from bidders' valuations. Our central identifying assumption is that the distribution of bidder values is increasing in the state. When the state-space is finite, such monotonicity implies the full-rank condition needed for identification. Further, we extend our approach to the conditionally independent private values model of Li et al. (2000), as well as to unobserved heterogeneity settings in which the implicit reserve price or the cost of bidding varies across auctions.

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1. Introduction

This paper considers the problem of identification in first-price auctions in which bidders have independent private values conditional on an unobserved one-dimensional state Y . Given the joint distribution of bids, (B_1, \dots, B_n) , what can be inferred about the joint distribution of bidder values and the state, (V_1, \dots, V_n, Y) ? In many empirical applications, such a state variable Y captures an auction-specific characteristic commonly observed by the bidders but unobserved by the econometrician. The resulting model is one of *Independent Private Values with Unobserved Heterogeneity (the UH model)*.

The existing literature has mainly focused on the convolution case, in which the unobserved heterogeneity has either an additive or multiplicative effect on bidder values, e.g., $V_i = S_i Y$, with independent signals S_i that are independent of Y (Li and Vuong, 1998; Krasnokutskaya, 2011). Our approach to identify (V_1, \dots, V_n, Y) differs in that we rely instead on the weaker assumption that the distribution of bidder values is *monotone* in the state, in the sense of first-order stochastic dominance (FOSD).

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This additional generality in the relationship between V and Y comes at some cost. Our identification approach relies on recent results in the econometric literature on nonlinear measurement error, namely Hu (2008). At least *three* bidders per auction are required to apply these results, whereas previous approaches have required only two bidders per auction. In addition, for our results to hold, the distribution of equilibrium bids conditional on the state must satisfy a full-rank condition. When the state-space is finite, however, this full-rank condition follows immediately from our monotonicity assumption on the underlying distribution of bidder values.

Most of the paper focuses on a setting in which the state can be interpreted as product quality that is observed by the bidders but not the econometrician. In particular, higher states correspond to FOSD-higher conditional distributions of bidder values. While the distribution of equilibrium bids need not be FOSD-increasing in the state, the *maximum of the equilibrium bid support* is increasing in the state. Such “monotonicity of the maximum” is enough to satisfy the monotonicity condition required by our approach. Further, when the state-space is finite, monotonicity of the maximum is also enough to satisfy the full-rank condition.

In Section 3, we consider the *Conditionally Independent Private Value (CIPV)* model, the identification and estimation of which has been analyzed in Li et al. (2000, 2002). This model is behaviorally distinct from but statistically similar to the UH model. The key

difference between the CIPV model and the UH model is that, in the CIPV model, bidders also do not observe the state.

To illustrate the breadth of applicability of our identification approach, Section 4 considers two settings in which the state induces unobserved heterogeneity in the distribution of bids *without* affecting the underlying distribution of bidder values. First, suppose that the seller’s (implicit) reserve price is known to the bidders but unobserved by the econometrician. The maximum of the equilibrium bid support is higher when bidders are faced with a higher reserve. Thus, the central monotonicity assumption of our approach is satisfied, and we identify the joint distribution of bidder values and the reserve. Second, and similarly, suppose that bidding is costly, but the cost of bidding is not observed by the econometrician. The maximum of the equilibrium bid support is higher when bidders are faced with a lower cost of bidding, again satisfying our monotonicity requirement when the space of possible costs is endowed with the reverse order. In this case, we identify the joint distribution of bidder values and the cost of bidding.

Related literature. Our paper builds on the pioneering work of Li and Vuong (1998), Li et al. (2000, 2002), and Krasnokutskaya (2011), who applied results from the literature on classical measurement error to identify CIPV models and UH models in which the state has a separable effect on bidder values.^{1,2} We relax this separability assumption by applying non-classical measurement error results in Hu (2008), though at the cost of requiring three bids per auction rather than two.

Another closely related paper is An et al. (2010), in which unobserved heterogeneity takes the form of an *unobserved number of potential bidders* N^* . An et al. (2010)’s identification approach exploits the fact that the number of observed bidders is always less than or equal to N^* . By contrast, our approach exploits the fact that some location of the distribution of equilibrium bids is increasing in the unobserved state. Thus, while their approach is specific to their particular application, our approach can be adapted to a wide variety of auction settings that exhibit monotonicity of bids in the state. Indeed, our approach can even be adapted to identify An et al. (2010)’s model, since the distribution of equilibrium bids is increasing in N^* .³ Further, An, Hu, and Shum focuses exclusively on the UH model, while we also extend our analysis to the CIPV model.

Several other papers in the recent literature address identification when there is an unobserved state. d’Haultfoeuille and Février (2008) identify a common value model with conditionally independent signals, assuming that the support of bidders’ signals is strictly increasing in the underlying common value. d’Haultfoeuille and Février (2010) extends this approach to a broader class of mixture models with at least three conditionally independent observations, again under a moving-support assumption. Roberts (2009) identifies a model with independent private values and unobserved heterogeneity given two bidders plus a reserve price that serves as an instrument for the unobserved heterogeneity. Aradillas-Lopez et al. (forthcoming) partially identify a model with unobserved heterogeneity and correlated private values, given data generated in an ascending auction.

The rest of the paper is organized as follows. Section 2 introduces and analyzes our main model of unobserved heterogeneity

in first-price auctions with independent private values (UH model). Sufficient conditions for the monotonicity and full-rank conditions are discussed in detail in Sections 2.1–2.2. Section 3 extends our approach to a setting with conditionally independent private values (CIPV model). Two other UH applications, to settings with an unobserved implicit reserve price or an unobserved cost of bidding, are considered in Section 4. Section 5 offers concluding remarks, followed by a technical appendix.

2. Independent private values with unobserved heterogeneity

Model. $n \geq 3$ symmetric risk-neutral bidders participate in a first-price auction with zero reserve price. Bidders’ private values (V_1, \dots, V_n) are independent conditional on an auction-specific “state” $Y \in \{1, \dots, K\}$ and bounded with support $[0, \bar{v}]$. The conditional probability distribution function (pdf) $f(v_i|Y = y)$ is continuously differentiable in v_i and bounded away from both zero and infinity on its support. Let $f(v_i, y)$ denote the joint density of (V_i, Y) .⁴

These assumptions are sufficient to imply existence of a unique bidding equilibrium, which is in symmetric strategies. Let $b(v_i|y)$ denote the equilibrium bid of a bidder having value v_i , conditional on realized state y . Let $G(b_i|y)$ and $g(b_i|y)$ be the cumulative distribution function (cdf) and pdf of each bidder’s equilibrium bid conditional on state $Y = y$, and $g(b_i, y)$ the joint density of (B_i, Y) . Note that, since bidder values are i.i.d. conditional on Y , so are bids.

The state Y is common knowledge among the bidders prior to the bidding, but the econometrician does not observe Y and does not even know the distribution from which it is drawn. (However, for convenience, we assume that the number of points of support K is known.) Y constitutes a source of unobserved heterogeneity, since equilibrium bidding strategies vary with the state.

Our central identifying assumption is a monotonicity property on model primitives, that the distribution of bidder values is stochastically increasing in the state.

Monotonicity assumption. $V_i|Y = y$ is strictly increasing in y with respect to first-order stochastic dominance (FOSD-increasing). That is, $y' > y$ implies that $F(v_i|y') \leq F(v_i|y)$ for all v_i , with strict inequality at some v_i .⁵

Proposition 1 (Monotonicity of the Maximum). *Suppose that the monotonicity assumption is satisfied. Then $\bar{b}(y) = b(\bar{v}|y)$ is strictly increasing in y .*

Proof. By the envelope theorem, a bidder with value v_i earns interim expected payoff equal to $\int_0^{v_i} F(v|y)^{n-1} dv$, where $F(v|y)^{n-1}$ is each bidder’s conditional probability of winning given value v . Since a bidder with the highest possible value \bar{v} always wins, this means that such a bidder’s equilibrium bid equals

$$\bar{b}(y) = b(\bar{v}|y) = \bar{v} - \int_0^{\bar{v}} F(v|y)^{n-1} dv.$$

Suppose that $y' > y$. By the monotonicity assumption, $F(v|y') \leq F(v|y)$ for all v , with strict inequality for a positive measure of values. Thus, $\bar{b}(y') > \bar{b}(y)$. \square

Identification: assumptions and argument. When there is unobserved heterogeneity, the unconditional distribution of bids observed by the econometrician differs from the conditional distribution that bidders use when formulating their bids. Thus, correct

¹ These papers also developed important results on estimation. Our focus is on identification.

² Krasnokutskaya (2012) has recently extended her approach to a setting in which a *multi-dimensional* state has a separable effect on bidder values.

³ An et al. (2010) assume that the distribution of bidder values does not depend on N^* , and that $N^* \geq 3$ with positive probability. As long as bidder identities are observed, one can then apply our identification results to the joint distribution of three bidders’ bids—including when they each fail to bid.

⁴ Random variables are capitalized while realizations of random variables are in lower case.

⁵ Since $F(v_i|y)$ is continuous in v_i , $F(\hat{v}_i|y') < F(\hat{v}_i|y)$ implies that $F(v_i|y') < F(v_i|y)$ for a positive measure of values v_i .

inference about bidder values requires identification of the joint distribution of bids and the unobserved heterogeneity. We achieve such identification by applying a result from Hu (2008), for which we need two conditions to hold on the distribution of equilibrium bids: the UH monotonicity condition and the UH full-rank condition.

UH monotonicity condition: There exists a known functional M such that $M[g(\cdot|y)]$ is strictly increasing in y .

Definition 1 (*Discretization of Bids*). A “discretization of bids” is any monotone onto mapping $\mathcal{D} : \mathbb{R}_+ \rightarrow \{1, \dots, K\}$. Each such mapping is equivalent to a partition of the bid-space \mathbb{R}_+ into K intervals. Let $D_k = \mathcal{D}(B_k)$ be shorthand for the interval to which bidder k 's bid belongs and $d_k = \mathcal{D}(b_k)$ be the realization of D_k .

UH full-rank condition: There exists a discretization of bids such that the $K \times K$ matrix $L_{D_i, D_k} = [\Pr \{D_i = i', D_k = k', \}]_{i', k' \in \{1, \dots, K\}}$ has rank K .

Theorem 1 (*UH Identification*). Suppose that $n \geq 3$. If the UH monotonicity condition and the UH full-rank condition hold, then the joint distribution of (V_1, \dots, V_n, Y) is identified from the joint distribution of bids (B_1, \dots, B_n) .

Proof. The proof has two steps. First, under the assumptions of Theorem 1, the joint distribution of (B_1, \dots, B_n, Y) is identified from that of (B_1, \dots, B_n) . For this, we apply Theorem 1 of Hu (2008), using three bids (B_i, B_j, B_k) as multiple conditionally independent measurements of Y . Then, by established methods in the literature (see, e.g., Guerre et al., 2000), the joint distribution of values (V_1, \dots, V_n) conditional on Y is identified from the distribution of bids (B_1, \dots, B_n) conditional on Y by the first-order conditions of equilibrium bidding as in Guerre et al. (2000). □

The key step of our identification argument applies Theorem 1 of Hu (2008) to identify (B_1, \dots, B_n, Y) from (B_1, \dots, B_n) . For completeness, and to shed light on why we require three bids and impose the UH monotonicity and UH full-rank conditions, we provide a self-contained proof of this key step in the Appendix.

Lemma 1 (*Corollary to Hu, 2008, Theorem 1*). Suppose that $n \geq 3$. If the UH monotonicity and UH full-rank conditions hold, then (B_1, \dots, B_n, Y) is identified from the joint distribution of bids (B_1, \dots, B_n) .

Proof. See the Appendix. □

Discussion. Three bidders. If there are only two bidders, our analysis still applies if an appropriate alternative instrument can be found that satisfies the full-rank condition. Loosely speaking, such an instrument should be correlated with the bids but independent of the bids conditional on Y . For example, consider a timber auction in which Y denotes the quality of the timber for sale. An instrument in this context might be average rainfall or soil quality, which is related to timber quality but does not directly affect bidders' valuations.

Symmetry. The analysis can be easily generalized to allow for asymmetric bidders, as long as one is willing to assume or able to prove (as in Proposition 1) that the maximum of each bidder's equilibrium bid support, or some other location of the distribution of bids such as the mean, is higher in higher states.

2.1. Sufficient conditions for the UH monotonicity condition

UH monotonicity is a non-primitive condition on the distribution of equilibrium bids. However, this monotonicity property of bids follows directly from our maintained monotonicity assumption on values. (Proposition 2 is an immediate corollary of Proposition 1.)

Proposition 2 (*UH Monotonicity*). When the monotonicity assumption on bidder values is satisfied, the UH monotonicity condition is satisfied with respect to the operator corresponding to the maximum of the equilibrium bid support, $M[g(\cdot|y)] = \max \text{supp}(B_i|y)$.

2.2. Sufficient conditions for the UH full-rank condition

The UH full-rank condition is also a non-primitive condition on the distribution of equilibrium bids. However, in the finite state-space case considered here, this full-rank condition follows immediately from our maintained monotonicity assumption on values. Thus, no further assumptions are required.

Proposition 3 (*UH Full-Rank*). When the monotonicity assumption on bidder values is satisfied, the UH full-rank condition is satisfied.

Proof. To establish that the UH full-rank condition is satisfied, it suffices to show that there exists a discretization such that

$$\text{Rank}(L_{D_i, D_k}) = K, \tag{1}$$

where $L_{D_i, D_k} = [\Pr \{D_i = i', D_k = k', \}]_{i', k' \in \{1, \dots, K\}}$. Consider the following discretization: $\mathcal{D}(b) = 1$ if $b \leq \bar{b}(1)$; $\mathcal{D}(b) = k$ if $b \in (\bar{b}(k-1), \bar{b}(k)]$ for all $k = 2, \dots, K-1$; and $\mathcal{D}(b) = K$ if $b > \bar{b}(K-1)$. (Recall that $\bar{b}(K) > \bar{b}(K-1) > \dots > \bar{b}(1) > 0$ by Proposition 1.) In each state $y = k$, bids are at most $\bar{b}(k)$. Thus, for each bidder i , $L_{D_i|Y} = [\Pr \{D_i = i' | Y = y, \}]_{i' \in \{1, \dots, K\}}$ is a triangular matrix of rank K . Finally, note that $L_{D_i, D_k} = L_{D_i|Y} \times D_Y \times L_{D_k|Y}^T$, where $D_Y = \text{diag} \{ \Pr \{Y = y\}_{y \in \{1, \dots, K\}} \}$ also has rank K . Thus, L_{D_i, D_k} has rank K . □

Discussion: The proof of Proposition 3 leverages the fact that bidder values are bounded, but this is not essential. Suppose that bidder values are unbounded but, for simplicity, that the state-space has exactly two elements, “low” and “high”. This special case is useful from an expositional point of view, since the full-rank condition (1) reduces to a simple correlation condition, that there exists some threshold $b^* > 0$ such that

$$\Pr(B_j > b^* | B_i > b^*) \neq \Pr(B_j > b^* | B_i < b^*). \tag{2}$$

Let $b(v_i|low)$, $b(v_i|high)$ denote equilibrium bidding strategies in each state. Lebrun (1998) shows that equilibrium bids are increasing in the state, i.e., $b(v_i|high) > b(v_i|low)$ for all v_i , whenever the distribution of bidder values has the monotonicity property that $d \left[\frac{F(v_i|y')}{F(v_i|y)} \right] / dv > 0$ for all v and all $y' > y$. (This monotonicity property is stronger than our monotonicity assumption. See Lebrun, 1998 for details.) Thus, $G(b|Y = high) < G(b|Y = low)$ for all $b > 0$, and hence $\Pr(B_j > b | B_i > b) > \Pr(B_j > b | B_i < b)$ for all $b > 0$. The correlation condition (2) is therefore satisfied with respect to any $b^* > 0$.

2.3. Estimation

Our identification strategy is constructive, and one may follow the identification procedure to estimate the model. For a nonparametric approach, one may follow the estimator in An et al. (2010) to estimate the bid distribution $g_{B_j|Y}$. For a semiparametric approach, one may use the estimator in Hu (2008).

Here, we provide a brief description of the nonparametric approach, focusing on how to recover $g_{B_j|Y}$ from a random sample $\{D_{kt}, B_{jt}, D_{it}\}_{t=1, \dots, T}$. From the discussion above, we have

$$g_{D_k, B_j, D_i}(d_k, b_j, d_i) = \sum_{y=1}^K g_{D_k|Y}(d_k|y) g_{B_j|Y}(b_j|y) g_{Y, D_i}(y, d_i). \tag{3}$$

Averaging over B_j leads to

$$\begin{aligned} & \int b g_{D_k, B_j, D_i}(d_k, b, d_i) db \\ &= \sum_{y=1}^K g_{D_k|Y}(d_k|y) \left(\int b g_{B_j|Y}(b|y) db \right) g_{Y, D_i}(y, d_i) \\ &= \sum_{y=1}^K g_{D_k|Y}(d_k|y) E(B_j|y) g_{Y, D_i}(y, d_i). \end{aligned}$$

Following An et al. (2010), we estimate $g_{B_j|Y}$ in two steps: first, we use the equation above to obtain $g_{D_k|Y}$; thereafter, we estimate $g_{B_j|Y}$ from the observed g_{B_j, D_k} . In the first step, we define

$$L_{D_k, EB_j, D_i} = \left[\int b g_{D_k, B_j, D_i}(k', b, i') db \right]_{k', i' = 1, 2, \dots, K}$$

$$L_{D_i, D_k} = [g_{D_i, D_k}(i', k')]_{i', k' = 1, 2, \dots, K}$$

$$L_{D_k|Y} = [g_{D_k|Y}(k'|y)]_{k', y = 1, 2, \dots, K}$$

$$L_{Y, D_i} = [g_{Y, D_i}(y, i')]_{y, i' = 1, 2, \dots, K}$$

and

$$D_{EB_j|Y} = \text{diag} \left\{ [E(B_j|y)]_{y=1, 2, \dots, K} \right\}.$$

We then have

$$L_{D_k, EB_j, D_i} = L_{D_k|Y} D_{EB_j|Y} L_{Y, D_i}$$

and similarly

$$L_{D_i, D_k}^T = L_{D_k|Y} L_{Y, D_i}.$$

The invertibility of L_{D_i, D_k} implies that

$$L_{D_k, EB_j, D_i} (L_{D_i, D_k}^T)^{-1} = L_{D_k|Y} D_{EB_j|Y} L_{D_k|Y}^{-1}. \tag{4}$$

Next, we apply the UH monotonicity condition, and use the conditional expectation of the bid as the functional which satisfies this condition, i.e., $M[g(\cdot|y)] = \int b g(b|y) db$. Therefore, we may order the eigenvalues $E(B_j|y)$ to obtain a unique decomposition. Further,

$$L_{D_k|Y} = \psi \left(L_{D_k, EB_j, D_i} (L_{D_i, D_k}^T)^{-1} \right),$$

where $\psi(\cdot)$ denotes the mapping from the square matrix on the left-hand side (LHS) of Eq. (4) to its eigenvector matrix following the identification procedure. We may estimate $L_{D_k|Y}$ as follows:

$$\widehat{L}_{D_k|Y} = \psi \left(\widehat{L}_{D_k, EB_j, D_i} (\widehat{L}_{D_i, D_k}^T)^{-1} \right), \tag{5}$$

where $\widehat{L}_{D_k, EB_j, D_i}$ and \widehat{L}_{D_i, D_k} are defined as

$$\widehat{L}_{D_k, EB_j, D_i} = \left[\frac{1}{T} \sum_{t=1}^T B_{jt} \mathbf{1}(D_{kt} = k', D_{it} = i') \right]_{k', i' = 1, 2, \dots, K} \tag{6}$$

$$\widehat{L}_{D_i, D_k} = \left[\frac{1}{T} \sum_{t=1}^T \mathbf{1}(D_{it} = i', D_{kt} = k') \right]_{i', k' = 1, 2, \dots, K}. \tag{7}$$

In the second step, we estimate $g_{B_j|Y}(b_j|y)$ from⁶

$$g_{B_j, D_k}(b_j, d_k) = \sum_{y=1}^K g_{D_k|Y}(d_k|y) g_{B_j, Y}(b_j, y),$$

which is equivalent to

$$\vec{g}(b_j, d_k) = L_{D_k|Y} \times \vec{g}(b_j, y),$$

with the vector of densities $\vec{g}(b_j, d) \equiv [g(b_j, d = 1), g(b_j, d = 2), \dots, g(b_j, d = K)]^T$.

Define $e_y = (0, \dots, 0, 1, 0, \dots, 0)^T$, where 1 is at the y th position in the vector. Our corresponding estimator is

$$\widehat{g}(b_j, y) = e_y^T (\widehat{L}_{D_k|Y})^{-1} \vec{g}(b_j, d_k),$$

where $\widehat{L}_{D_k|Y}$ is from the first step, $\vec{g}(b_j, d_k) = [\widehat{g}(b_j, d_k = 1), \widehat{g}(b_j, d_k = 2), \dots, \widehat{g}(b_j, d_k = K)]^T$, and

$$\widehat{g}(b_j, d_k = k') = \left[\frac{1}{T} \sum_{t=1}^T \frac{1}{h} \kappa \left(\frac{b_j - B_{jt}}{h} \right) \mathbf{1}(D_{kt} = k') \right], \tag{8}$$

with a kernel function κ and a bandwidth h .

Similarly, we may estimate the marginal distribution of Y as follows:

$$\widehat{g}(y) = e_y^T (\widehat{L}_{D_k|Y})^{-1} \vec{g}(d_k),$$

where $\vec{g}(d_k) \equiv [\frac{1}{T} \sum_t \mathbf{1}_{D_{kt}=1}, \dots, \frac{1}{T} \sum_t \mathbf{1}_{D_{kt}=K}]$. Therefore, the conditional bid densities $g_{B_j|Y}(b_j|y)$ may be estimated as

$$\widehat{g}_{B_j|Y}(b_j|y) = \frac{e_y^T (\widehat{L}_{D_k|Y})^{-1} \vec{g}(b_j, d_k)}{e_y^T (\widehat{L}_{D_k|Y})^{-1} \vec{g}(d_k)}. \tag{9}$$

The empirical conditional cdf for the bids is

$$\widehat{F}_{B_j|Y}(b_j|y) = \frac{e_y^T (\widehat{L}_{D_k|Y})^{-1} \vec{F}(b_j, d_k)}{e_y^T (\widehat{L}_{D_k|Y})^{-1} \vec{g}(d_k)}, \tag{10}$$

where $\vec{F}(b_j, d_k)$ denotes the vector of the following elements:

$$\begin{aligned} \widehat{F}(b_j, d_k = k') &= \frac{1}{T} \sum_{t=1}^T \mathbf{1}(B_{jt} < b_j, D_{kt} = k'), \\ k' &= 1, \dots, K, \end{aligned} \tag{11}$$

which can be recovered from the sample.

A complete discussion of the asymptotic theory for this procedure is provided in the Appendix of An et al. (2010), and we summarize the results here. Since the first step only involves sample averages in different subsamples, our estimator $\widehat{L}_{D_k|Y}$ converges at a \sqrt{T} -rate. The convergence rate of $\widehat{g}_{B_j|Y}(b_j|y)$ is determined by the convergence properties of the kernel estimator $\widehat{g}(b_j, d_k = k')$ in Eq. (8); for a fixed b_j , $(Th)^{1/2} [\widehat{g}_{B_j|Y}(b_j|y) - g_{B_j|Y}(b_j|y)]$ converges to a normal distribution. For the empirical distribution $\widehat{F}_{B_j|Y}(b_j|y)$, we have that $T^{1/2} [\widehat{F}_{B_j|Y}(b_j|y) - F_{B_j|Y}(b_j|y)]$ converges to a normal distribution with mean zero. Complete details can be found in Hu (2008) and An et al. (2010).

3. Extension: conditionally independent private values

Our results also apply to a setting in which bidders have conditionally independent private values (CIPV model). The main behavioral distinction between the UH and CIPV models is that, in the UH model, bidders observe Y before they choose their bid while, in the CIPV model, bidders do not observe Y .

As before, we achieve identification by applying a result from Hu (2008). The main difference is that our monotonicity and full-rank conditions are now primitive conditions on bidder values, rather than non-primitive conditions on equilibrium bids.

CIPV monotonicity condition: There exists a known functional M such that $M[f(\cdot|y)]$ is strictly increasing in y .

⁶ We omit the subscripts when it does not cause any confusion.

Definition 2 (*Discretization of Values*). A “discretization of values” is any monotone onto mapping $\mathcal{D} : \mathbb{R} \rightarrow \{1, \dots, K\}$. Each such mapping is equivalent to a partition of the value-space $[0, \bar{v}]$ into K intervals. Let $D_k = \mathcal{D}(V_k)$ be shorthand for the interval to which bidder k 's value belongs and $d_k = \mathcal{D}(v_k)$ be the realization of D_k .

CIPV full-rank condition: There exists a discretization of values such that the $K \times K$ matrix $L_{D_i, D_k} = [\Pr\{D_i = i', D_k = k', \}]_{i', k' \in \{1, \dots, K\}}$ has rank K .

Theorem 2 (*CIPV Theorem*). Suppose that $n \geq 3$. If the CIPV monotonicity and CIPV full-rank conditions hold, then the joint distribution of (V_1, \dots, V_n, Y) is identified from the joint distribution of bids (B_1, \dots, B_n) .

Proof. The proof has two steps. First, by established methods in the literature (see, e.g., Li et al., 2000), the joint distribution of values (V_1, \dots, V_n) is identified from the distribution of bids (B_1, \dots, B_n) by the first-order conditions of equilibrium bidding in the CIPV model.⁷ The second step is to show that, under the assumptions of the CIPV theorem, the joint distribution of values and the state (V_1, \dots, V_n, Y) is identified from (V_1, \dots, V_n) . For this step, we apply Theorem 1 of Hu (2008), where three values (V_i, V_j, V_k) serve as multiple conditionally independent measurements of Y . □

Sufficient condition for the CIPV monotonicity condition. As in the UH model, it suffices for the distribution of bidder values to be FOSD-increasing in Y .

Proposition 4 (*CIPV Monotonicity*). Suppose that the monotonicity assumption is satisfied. The CIPV monotonicity condition is then satisfied with respect to the mean operator, $M[g(\cdot|Y = y)] = E[B_i|Y = y]$.

Proof. In the CIPV model, bidders' bids depend on their realized values but not the state. Thus, the distribution of equilibrium bids is just a monotone transformation of the distribution of bidder values, where this transformation does not vary with the state. Since the distribution of bidder values is FOSD-increasing in the state by the monotonicity assumption, the distribution of equilibrium bids must also be FOSD-increasing in the state. In particular, the mean equilibrium bid is strictly increasing in the state. □

Sufficient conditions for the CIPV full-rank condition. The CIPV full-rank condition is identical to the UH full-rank condition in all but one respect: the CIPV full-rank condition applies to the primitive joint distribution of the state and bidder values while the UH full-rank condition applies to the non-primitive joint distribution of the state and equilibrium bids. Thus, our results on full rank in the UH model translate directly to the CIPV model, once interpreted as applying to values rather than bids. In particular, the CIPV full-rank condition holds whenever the joint distribution of two bidders' (discretized) values satisfies a full-rank condition analogous to that of Eq. (1).

4. Additional extensions

This section aims to illustrate the breadth of application of our identification approach, by examining two settings *outside*

⁷ The CIPV model has a unique equilibrium when bidders are symmetric (McAdams, 2007). Without uniqueness, the equilibrium selected by bidders would be a source of unobserved heterogeneity. Further, the approach developed here cannot accommodate this sort of unobserved heterogeneity, since equilibrium bids across multiple equilibria need not be ordered so as to satisfy our monotonicity condition.

of the model of Section 2 with independent private values and unobserved heterogeneity. In Section 4.1, we consider a situation in which the seller's (implicit) reserve price is known to the bidders but not observed by the econometrician. The reserve price affects the distribution of equilibrium bids and therefore constitutes a source of unobserved heterogeneity, even if it is uncorrelated with the distribution of bidder values. Given the joint distribution of at least three bids, we identify the distribution of the reserve and the distribution of bidder values. In Section 4.2, we consider a situation in which bidding is costly, but the econometrician does not observe the cost of bidding. Given the joint distribution of at least three bids, we identify the distribution of the cost of bidding and the distribution of bidder values.

4.1. Implicit reserve price as unobserved heterogeneity

Auction datasets often include the *explicit* reserve price, the minimal bid permitted by the auction rules. However, real-world sellers sometimes refuse to sell to the highest bidder, unless the highest bid exceeds an even higher *implicit* reserve price.⁸ This observation has motivated an important empirical literature on auctions with a “random reserve price”; see, e.g., Li and Perrigne (2003). Papers in this literature presume that bidders do not know the seller's implicit reserve price when bidding, only the distribution from which it is drawn. Yet sellers can have an incentive to reveal their implicit reserve price to bidders prior to the bidding.⁹ If bidders observe the implicit reserve price, the distribution of equilibrium bids will vary with the implicit reserve, making this a potentially important source of unobserved heterogeneity.

We consider here the simplest case in which (i) the realized implicit reserve price $R = r$ is common knowledge among the bidders prior to the bidding, (ii) bidder values are independent of R , and (iii) the econometrician knows nothing about the implicit reserve, not even the distribution from which it is drawn. Such econometrician ignorance could arise naturally, if the seller's implicit reserve depends on his/her own cost but the econometrician has no data on seller cost.

For simplicity of the exposition, we shall henceforth suppress the distinction between explicit and implicit reserve prices, treating the reserve price as if it is an explicit reserve that is unobserved by the econometrician.¹⁰ Also, we will assume that the reserve has finite support, $\text{supp}(R) = \{r_1, \dots, r_K\}$.

Monotonicity. Equilibrium bids $b(v|r)$ vary monotonically with the reserve price r .

Proposition 5 (*Monotonicity in the Reserve Price*). Given any reserve prices $r' > r$, $b(v|r') > b(v|r)$ for all values $v > r'$.

Proof. Given reserve price r , each bidder earns zero surplus given any value $v_i \leq r$. By the envelope theorem, each bidder's expected surplus given value $v_i > r$ takes the form $\int_r^{v_i} F(v)^{n-1} dv$, where

⁸ If the seller is bound to accept the highest bid whenever it exceeds the explicit reserve, the implicit reserve equals the explicit reserve and there is no problem. Otherwise, the “true” reserve typically exceeds the explicit reserve, and bidders' beliefs about the reserve price affect the distribution of equilibrium bids.

⁹ For example, suppose that the seller's cost is random and unobserved by the bidders. Since the optimal reserve price varies with the seller's cost, the seller cannot implement the optimal auction unless he/she credibly reveals the reserve price to bidders. Consequently, any seller who can commit to an optimal implicit reserve price will always choose to reveal it to bidders. See also Brisset and Naegelen (2006) for another context in which the seller chooses to reveal the reserve price.

¹⁰ When R is an implicit reserve price, the data will include *permissible but unacceptable* bids between the explicit and implicit reserves. R is the minimal acceptable bid, rather than the minimal permissible bid.

$F(v)^{n-1}$ is the probability of winning given value v . Equilibrium bids $b(v_i|r)$ must therefore satisfy the condition

$$(v_i - b(v_i|r)) F(v_i)^{n-1} = \int_r^{v_i} F(v)^{n-1} dv. \tag{12}$$

The right-hand side (RHS) of (12) is decreasing in r for all $v_i > r$. Thus, $b(v_i|r)$ must be increasing in r for all $v_i > r$. \square

Proposition 5 implies that the maximum of the equilibrium bid support, $b(\bar{v}|r)$, is strictly increasing in r . Thus, the UH monotonicity condition is satisfied with respect to the operator corresponding to this maximum.

Full rank. Since the maximum of the equilibrium bid support is increasing in the reserve, the UH full-rank condition is automatically satisfied by the argument of **Proposition 3**.

Identification. As noted above, the UH monotonicity condition and the UH full-rank condition are automatically satisfied. Thus, we may apply **Theorem 1** to identify the joint distribution of the bids and the unobserved state. More precisely, we may identify the joint distribution of $(B_1, \dots, B_n, \gamma(R))$, where $\gamma : \{r_1, \dots, r_K\} \rightarrow \{1, \dots, K\}$ is a normalization.

Identifying the *unnormalized* support of the unobserved heterogeneity requires extra work which, in this case, is trivial and immediate. Namely, since the realized reserve price r_k is the minimum of the support of submitted bids conditional on $\gamma(R) = k$, we may infer r_k directly from the distribution of bids conditional on $\gamma(R) = k$. Thus, in fact, the joint distribution of (B_1, \dots, B_n, R) is identified.

Identifying bidder values is now straightforward. In particular, the conditional distribution of $(V_1, \dots, V_n)|R$ is identified as usual from that of $(B_1, \dots, B_n)|R$, by the first-order conditions of equilibrium bidding as in **Guerre et al. (2000)**. (More precisely, for each realized reserve price $R = r_k$, we may identify the distribution of values above r_k .)

4.2. Bidding cost as unobserved heterogeneity

Samuelson (1985) noted that “competing firms must bear significant bid-preparation and documentation costs” in order to bid in an auction, spawning a large literature on auctions with costly bidding. Suppose that, as in **Samuelson (1985)**, each bidder costlessly learns his/her private value and then simultaneously decides whether to pay $C \geq 0$ to submit a bid in a first-price auction with zero reserve price. The distribution of equilibrium bids varies with the cost of bidding, making this a potentially important source of unobserved heterogeneity if the cost of bidding varies across auctions but is not observed by the econometrician. (Such a case of unobserved heterogeneity is also discussed in **Li and Zheng (2009)**.) Also, we will assume that the cost of bidding is drawn from finite support, $\text{supp}(C) = \{c_1, \dots, c_K\}$.

Monotonicity. Equilibrium bids $b(v|c)$ vary monotonically with the bidding cost c .

Proposition 6 (Monotonicity in the Bidding Cost). *Given any bidding costs $c' < c$, $b(v|c') > b(v|c)$ for all values $v > \underline{v}(c)$, where $\underline{v}(c)$ is the equilibrium participation threshold implicitly defined by the indifference condition $c = \underline{v}(c)F(\underline{v}(c))^{n-1}$.*

Proof. Each bidder earns zero surplus given any value $v_i \leq \underline{v}(c)$. By the envelope theorem, each bidder's expected surplus given value $v_i > \underline{v}(c)$ takes the form $\int_{\underline{v}(c)}^{v_i} F(v)^{n-1} dv$. Since this is *net* expected surplus, equilibrium bids $b(v_i|c)$ must therefore satisfy the condition

$$\begin{aligned} (v_i - b(v_i|c)) F(v_i)^{n-1} &= c + \int_{\underline{v}(c)}^{v_i} F(v)^{n-1} dv \\ &= \int_0^{v_i} F(\max\{v, \underline{v}(c)\})^{n-1} dv, \end{aligned} \tag{13}$$

where the last equality comes from substituting $c = \underline{v}(c)F(\underline{v}(c))^{n-1}$. The threshold $\underline{v}(c)$ is strictly increasing in c . Thus, the RHS of (13) is strictly increasing in c for all $v > \underline{v}(c)$. Thus, $b(v_i|c)$ must be strictly decreasing in c for all $v_i > \underline{v}(c)$. This completes the proof. \square

Proposition 6 implies that the maximum of the equilibrium bid support, $b(\bar{v}|c)$, is strictly decreasing in c . Thus, the UH monotonicity condition is satisfied with respect to the operator corresponding to this maximum when the state-space of possible bidding costs is endowed with the reverse order.

Full rank. Since the maximum of the equilibrium bid support is monotone in the bidding cost, the UH full-rank condition is automatically satisfied by the argument of **Proposition 3**.

Identification. As noted above, the UH monotonicity condition and the UH full-rank conditions are automatically satisfied. Thus, we may apply **Theorem 1** to identify the joint distribution of the bids and the unobserved state. That is, we may identify the joint distribution of $(B_1, \dots, B_n, \gamma(C))$, where $\gamma : \{c_1, \dots, c_K\} \rightarrow \{1, \dots, K\}$ is a normalization.

Identifying the *unnormalized* support of the unobserved heterogeneity requires extra work which, in this case, is not as immediate as in the reserve price example of Section 4.1. First, for every realization of the normalized cost of bidding $k = 1, \dots, K$, the conditional distribution of values $(V_1, \dots, V_n)|(\gamma(C) = k)$ is identified from the conditional distribution of bids $(B_1, \dots, B_n)|(\gamma(C) = k)$, by the first-order conditions of equilibrium bidding as in **Guerre et al. (2000)**. More precisely, the distribution of values is identified above the minimal value $\underline{v}(c_k)$ given which each bidder submits a bid, conditional on bidding cost $C = c_k$.

The threshold $\underline{v}(c_k)$ is determined by the indifference condition

$$c_k = \underline{v}(c_k)F(\underline{v}(c_k))^{n-1} \quad \text{for all } k = 1, \dots, K. \tag{14}$$

(A bidder having value $v_i = \underline{v}(c_k)$ bids zero in equilibrium, wins with probability $F(\underline{v}(c_k))^{n-1}$, and is indifferent between bidding or not.) Both the probability of non-bidding $F(\underline{v}(c_k))$ and the bidding threshold $\underline{v}(c_k)$, conditional on $C = c_k$, are identified from the distribution of bidder values conditional on $\gamma(C) = k$. Eq. (14) therefore allows us to identify c_k from the distribution of bidder values conditional on $\gamma(C) = k$. Thus, in fact, the joint distribution of (V_1, \dots, V_n, C) is identified.

5. Concluding remarks

This paper has developed a novel approach to identify first-price auction models with independent private values in the face of one-dimensional unobserved heterogeneity.¹¹ Our key identifying assumption is that the distribution of *bidder values* is increasing in the state, in the sense of first-order stochastic dominance. This monotonicity assumption suffices to imply both the monotonicity and full-rank conditions on the distribution of *equilibrium bids* that are necessary for our identification approach.

Our identification approach can be adapted to a wide variety of auction environments, in which some location of the distribution of equilibrium bids is increasing in the unobserved state. We consider three such applications: (i) bidders also do not observe the underlying state, so the model is one of conditionally independent private values (Section 3); (ii) the seller's implicit reserve price is known to the bidders but unobserved by the econometrician (Section 4.1); and (iii) the bidders' cost of preparing a bid is known to the bidders but unobserved by the econometrician (Section 4.2).

¹¹ We focused on the simplest case with a finite state-space. See the working-paper version **Hu et al. (2011)** for some additional results in the more challenging case with a continuous state-space.

We see three important directions for future work building upon this paper. First, in many applications it is likely that there are multiple potential sources of unobserved heterogeneity. We are currently exploring an extension of this paper’s analysis to a setting with multi-dimensional unobserved heterogeneity. Second, the proof of our main result relies heavily on the assumption of *independent* private values. We are working to extend our results to settings in which bidders have affiliated private values conditional on the state. Finally, we are considering how to extend our results to “endogenous participation” models, roughly defined as models of entry in auctions. Unlike the “bidding cost” example of Section 4.2, bidders’ entry decisions in such models depend on unobserved auction characteristics which also affect their valuations; see, e.g., Haile et al. (2003) and Li and Zheng (2009).

Appendix. Proof of Lemma 1

Proof. Since bidders are symmetric and equilibrium bids are conditionally independent, the joint distribution of (B_1, \dots, B_n, Y) is identified from the distributions of Y and $B_k|Y$, for any $k = 1, \dots, n$. Fix any three bidders i, j, k . (The following proof can be repeated for any triplet of bidders.) We will show that the distributions of Y and $B_k|Y$ are identified from the joint distribution of (D_i, B_j, D_k) with $D_k = \mathcal{D}(B_k)$.

The following matrices will be useful in the proof. (In what follows, variables with bars ($\bar{\cdot}$) denote fixed values.)

$$L_{D_k|Y} = [g_{D_k|Y}(k'|y)]_{k',y=1,2,\dots,K}$$

$$L_{D_i,D_k} = [g_{D_i,D_k}(i',k')]_{i',k'=1,2,\dots,K}$$

$$L_{D_k,\bar{b}_j,D_i} = [g_{D_k,B_j,D_i}(k',\bar{b}_j,i')]_{k',i'=1,2,\dots,K}$$

$$D_{\bar{b}_j|Y} = \text{diag} \left\{ [g_{B_j|Y}(\bar{b}_j|y)]_{y=1,2,\dots,K} \right\}$$

$$L_{Y,D_i} = [g_{Y,D_i}(y,i')]_{y,i'=1,2,\dots,K}$$

Invertibility of $L_{D_k|Y}$.

Note that

$$E[h(D_k)|d_i] = \sum_{k'=1}^K h(k') g_{D_k|D_i}(k'|d_i)$$

$$= \frac{1}{g_{D_i}(d_i)} \sum_{k'=1}^K g_{D_i,D_k}(d_i,k') h(k').$$

By the UH full-rank condition, $E[h(D_k)|d_i] = 0$ for all $d_i \in \{1, \dots, K\}$ implies that $h(k') = 0$ for all k' . Thus, the matrix L_{D_i,D_k} is invertible. The conditional independence between D_i and D_k , i.e., $g_{D_k,B_j,D_i}(d_k,d_i) = \sum_{y=1}^K g_{D_k|Y}(d_k|y)g_{Y,D_i}(y|d_i)$, implies that

$$L_{D_i,D_k}^T = L_{D_k|Y}L_{Y,D_i}, \tag{15}$$

where L_{D_i,D_k}^T is the transpose of L_{D_i,D_k} . Therefore, the invertibility of L_{D_i,D_k} implies that $L_{D_k|Y}$ is invertible.

Eigenvalue/eigenvector decomposition.

Since bidder values are assumed to be conditionally independent, bids are also independent conditional on Y :

$$g_{D_k,B_j,D_i}(d_k,b_j,d_i) = \sum_{y=1}^K g_{D_k|Y}(d_k|y)g_{B_j|Y}(b_j|y)g_{Y,D_i}(y,d_i). \tag{16}$$

In particular, this equation is equivalent to the matrix equation as follows:

$$L_{D_k,\bar{b}_j,D_i} = L_{D_k|Y}D_{\bar{b}_j|Y}L_{Y,D_i}. \tag{17}$$

Since L_{D_i,D_k} and $L_{D_k|Y}$ are invertible, as shown before, Eq. (15) implies that

$$L_{Y,D_i} = L_{D_k|Y}L_{D_i,D_k}^T. \tag{18}$$

Substituting this expression in (17), for any fixed \bar{b}_j , yields

$$L_{D_k,\bar{b}_j,D_i} = L_{D_k|Y}D_{\bar{b}_j|Y}L_{D_k|Y}^{-1}L_{D_i,D_k}^T. \tag{19}$$

By the invertibility of L_{D_i,D_k} , finally,

$$L_{D_k,\bar{b}_j,D_i} (L_{D_i,D_k}^T)^{-1} = L_{D_k|Y}D_{\bar{b}_j|Y}L_{D_k|Y}^{-1}. \tag{20}$$

This equation implies that the observed LHS has an eigenvalue and eigenvector decomposition. The eigenvalues are $g_{B_j|Y}(\bar{b}_j|y)$ in the diagonal matrix $D_{\bar{b}_j|Y}$ and the eigenvectors are $g_{D_k|Y}(\cdot|y)$ in the matrix $L_{D_k|Y}$. Because $L_{D_k|Y}$ is a conditional probability matrix, the fact that its column sums are all equal to 1 provides a natural normalization for the eigenvectors. The realization of unobserved heterogeneity y is the index for the eigenvalues and eigenvector. The UH monotonicity condition implies that for any two possible values y and \tilde{y} of Y there exists a nonzero-measure set of \bar{b}_j such that the corresponding two eigenvalues $g_{B_j|Y}(\bar{b}_j|y)$ and $g_{B_j|Y}(\bar{b}_j|\tilde{y})$ are distinctive. In other words, for any two eigenvectors corresponding to two indices y and \tilde{y} , there must exist a \bar{b}_j such that the two corresponding eigenvalues are different. The eigendecompositions corresponding to different \bar{b}_j may share the same set of K eigenvectors because the eigenvectors $g_{D_k|Y}(\cdot|y)$ do not depend on \bar{b}_j . This means that the K eigenvectors are uniquely determined. However, the ordering of the K eigenvectors is still arbitrary in Eq. (20). This can be seen from the following equation:

$$L_{D_k,\bar{b}_j,D_i} (L_{D_i,D_k}^T)^{-1} = L_{D_k|Y}D_{\bar{b}_j|Y}L_{D_k|Y}^{-1}$$

$$= (L_{D_k|Y}Q) (Q^{-1}D_{\bar{b}_j|Y}Q) (L_{D_k|Y}Q)^{-1}, \tag{21}$$

where Q is an elementary matrix generated by interchanging columns of the identity matrix. Let $(L_{D_k|Y}Q)$ be an eigenvector matrix with a fixed ordering of the eigenvectors arbitrarily set by the econometrician. Notice that the corresponding unknown matrix Q does not change with different \bar{b}_j . We then have

$$(L_{D_k|Y}Q)^{-1} [L_{D_k,\bar{b}_j,D_i} (L_{D_i,D_k}^T)^{-1}] (L_{D_k|Y}Q) = (Q^{-1}D_{\bar{b}_j|Y}Q). \tag{22}$$

The RHS is a diagonal matrix, whose only difference from $D_{\bar{b}_j|Y}$ is an unknown permutation of the diagonal entries implied by Q . We may then apply the functional M in the UH monotonicity condition to each diagonal entry for different values of \bar{b}_j . The UH monotonicity condition directly implies an ordering of the diagonal entries, which uniquely determines the elementary matrix Q . Therefore, the eigenvector matrix $L_{D_k|Y}$ is identified from $(L_{D_k|Y}Q)$ and the eigenvalue matrix $D_{\bar{b}_j|Y}$ is uniquely determined from $(Q^{-1}D_{\bar{b}_j|Y}Q)$ for any given \bar{b}_j . In addition, both $g_{Y,D_i}(y,d_i)$ and $g_Y(y) = \sum_{d_i=1}^K g_{Y,D_i}(y,d_i)$ are identified from (18). This completes the proof. \square

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