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Econometric models of asymmetric ascending auctions

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Abstract

We develop econometric models of ascending (English) auctions which allow for both bidder asymmetries as well as common and/or private value components in bidders' underlying valuations. We show that the equilibrium inverse bid functions in each round of the auction are implicitly defined (pointwise) by a system of nonlinear equations, so that conditions for the existence and uniqueness of an increasing-strategy equilibrium are essentially identical to those which ensure a unique and increasing solution to the system of equations. We exploit the computational tractability of this characterization in order to develop an econometric model, thus extending the literature on structural estimation of auction models. Finally, an empirical example illustrates how equilibrium learning affects bidding during the course of the auction.

JEL classification: C51; D44; D82; L96

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1. Introduction

We develop a framework for estimating structural models of asymmetric ascending (English) auctions. In these auctions, the bidding process is modeled as a multi-stage game in which bidders obtain more and more information during the course of the auctions as rivals drop out of the bidding. Equilibrium learning is a feature of these dynamic games, in contrast to static (first- or second-price) sealed-bid auctions which offer participants no opportunity to gain information during the course of the auction. In a common-value setting, information revelation during the auction reduces the effects of the winner's curse, thereby encouraging participants to bid more aggressively and

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raising expected seller revenue relative to a sealed-bid auction. Many real-world auction mechanisms—from art and collectible auctions to the Japanese “button” auction cited by Milgrom and Weber (1982, p. 1104)—resemble the auctions we study, and perhaps these mechanisms arose to allow for the possibility of information revelation.

The theoretical literature on ascending auctions (including the paradigmatic model presented in Milgrom and Weber (1982)) has focused primarily on symmetric models, in which the bidders’ signals about the value of the object are assumed to be generated from identical distributions. However, recent applied research in auctions (e.g. work by Hendricks and Porter (1988) on offshore gas auctions, and by Klemperer (1998) on the PCS spectrum auctions) suggests that symmetry may not be a realistic assumption for many real-world situations. For these reasons, we develop an econometric framework for asymmetric ascending auctions which can be used in applied analyses.

We begin with a brief characterization of Bayesian–Nash equilibrium bidding behavior in asymmetric ascending auctions. This complements recent work (Maskin and Riley, 2000; Bulow et al., 1999; Bajari, 1998; Campo et al., 1998; Froeb et al., 2000) on asymmetric first-price auctions, and by Wilson (1998) and Maskin and Riley (2000) on asymmetric ascending auctions. We find that the increasing-strategy equilibrium bid functions in each round of an ascending auction exhibit an attractive analytic property: specifically, the *inverse bid functions* are implicitly defined by a system of nonlinear equations, pointwise in the bids. Therefore, conditions for the existence of an increasing-strategy equilibrium are essentially identical to those which ensure an increasing solution to the system of equations, given primitive model assumptions about the joint distribution of the bidders’ underlying valuations and private signals.

This attractive analytic property also facilitates numerical calculation of the equilibrium bidding strategies, which makes the econometric implementation of these models feasible. This was recognized by Wilson (1998), who analytically derives the equilibrium bid functions for a log-additive log-normal asymmetric ascending auction model given a diffuse prior assumption on the distribution of the common value component. Following this cue, we develop an econometric model of the asymmetric ascending auction for this log-additive case which differs from Wilson’s model in that we do not assume a diffuse prior for the common value distribution.[1] This extends the scope of the literature on the structural estimation of auction models (e.g. Paarsch, 1992; Laffont et al., 1995; Li et al., 2000) to asymmetric ascending auctions within the CV paradigm. Perhaps the closest antecedents of our work are papers by Donald et al. (1997) on bidding in simultaneous ascending auctions within the symmetric independent private values paradigm, and by Bajari and Hortacsu (1999) on bidding in symmetric common-value ascending auctions.

We provide an empirical illustration of this model by estimating it using data from the PCS spectrum auctions run by the U.S. Federal Communications Commission (FCC). While our model accommodates the multiple-round aspect of these auctions, it does not include other essential details, such as the simultaneous selling of multiple licenses, and the flexible eligibility rules. Therefore, we view the main purpose of this

[1] Furthermore, we show that the log-additive log-normal information structure satisfies a *diagonal dominance* condition which ensures the existence of an equilibrium in monotonic bidding strategies.

example as illustrating the econometric model and suggesting solutions to problems which arise in implementing the estimation method in practice, rather than providing robust empirical findings concerning the FCC auctions.[2] We present estimated bid functions which illustrate how equilibrium learning affects bidding behavior during the course of an ascending auction.

We start, in Section 2, by a brief description of equilibrium bidding behavior in the asymmetric ascending auction. In Section 3 we develop an econometric model based on a log-normal specification of the auction model, and discuss estimation issues in Section 4. Section 5 contains the empirical example, and Section 6 concludes.

2. Asymmetric ascending auctions

Consider an auction in which N bidders compete for the possession of a single object. The ascending auction proceeds in "rounds", with a new round beginning whenever another bidder drops out. At this point, we introduce the indexing convention we will follow in this paper. With N bidders participating in the auction, there will be $N-1$ rounds, indexed $k=0,\ldots,N-2$. In round 0, all N bidders are active, and in round k, only $N-k$ bidders are active: each round ends when a bidder drops out. Bidders are indexed by $i=1,\ldots,N$ where, without loss of generality, the ordering $1,\ldots,N$ indicates the order of dropout, so that bidder N drops out in round 0, and bidder 1 wins the auction. The dropout prices are indexed by rounds, i.e., $P_0,\ldots,P_{N-2}$. To sum up, bidder $N-k$ drops out at the end of round k, at the price P_k.

Each bidder i values the object at V_i, but does not observe his valuation directly. Rather, before the auction begins, each bidder i observes a private and noisy signal X_i of his valuation V_i. The auction format which we focus on in this paper is an asymmetric version of the "irrevocable dropout" auction described in Milgrom and Weber (1982, p. 1104). In this auction bidders drop out one by one irrevocably as the auctioneer raises the price. By observing the dropout prices in previous rounds, remaining bidders can infer the private information possessed by the bidders who have dropped out. In a common value setting, bidder j's signal X_j is useful to bidder i in estimating V_i, his valuation of the object.[3] This equilibrium learning (i.e., losing bidders revealing their private information to remaining bidders in equilibrium) is a distinctive feature of irrevocable dropout common value English auctions.

2.1. Equilibrium bidding in the ascending auction

A Bayesian–Nash equilibrium in the ascending auction game consists of bid functions $\beta_i^k(X_i;\Omega_k)$ for each bidder i, and for each round k, $k=0,\ldots,N-2$, i.e., $\{\beta_i^0(X_i;\Omega_0),\ldots,$

[2] Recent empirical work on these auctions has been done by, among others, Cramton (1997), Ausubel et al. (1997), Moreton and Spiller (1998). The focus in most of these papers has been on detecting the presence of cross-license complementarities.

[3] In the *private values* (PV) paradigm, in contrast, where each bidder has a private value for the object (which he knows), the (undominated) equilibrium bidding strategy is independent of others' valuations: a bidder will bid (up to, in the ascending case) his private valuation of the object.

$\beta_i^{N-2}(X_i;\Omega_{N-2})\}$, where X_i denotes bidder i's private signal and Ω_k the public information set at the beginning of round k. The contents of Ω_k will be described later, but in what follows we sometimes suppress the dependence of the bid functions $\beta_i^k(\cdots)$ on Ω_k, for notational simplicity. $\beta_i^k(X_i)$ tells bidder i at which price he should drop out during round k. The collections of bid functions $\beta_i^0(\cdot),\ldots,\beta_i^{N-2}(\cdot)$ for bidders $i=1,\ldots,N$ are common knowledge.

Up to the beginning of round k, bidders $N-k+1,\ldots,N$ have already dropped out, at prices $P_{k-1},\ldots,P_0$, respectively. Since the equilibrium bid functions are common knowledge, an active bidder i can use this information to infer the private signals $X_{N-k+1},\ldots,X_N$ observed by these bidders by inverting their bid functions: i.e., $X_j=(\beta_j^{N-j})^{-1}(P_{N-j})$, for $j=N-k+1,\ldots,N$.

In what follows, we focus on equilibria in *increasing* bidding strategies (i.e., $\beta_i^k(X_i)$ is increasing in X_i, for $k=0,\ldots,N-2$).[4] The structure of the equilibrium strategies extends the construction of the symmetric equilibrium strategies described in Milgrom and Weber (1982, p. 1104ff), to the asymmetric case.[5]

Next we state three assumptions which are sufficient to ensure the existence of an equilibrium in monotonic bidding strategies. As before, let $i=1,\ldots,N$ denote the dropout order.[6] For any round k ($0\leqslant k\leqslant N-2$) fix the realizations of $X_N,\ldots,X_{N-k+1}$ (the private signals corresponding to the bidders who have already dropped out prior to round k). The $N-k$ conditional expectations for the $N-k$ bidders active in round k constitute a system of $N-k$ equations with $N-k$ unknowns

$$\begin{aligned} &\mathrm{E}[V_1|X_1,\ldots,X_{N-k};X_{N-k+1},\ldots,X_N]=P,\\ &\vdots\\ &\mathrm{E}[V_{N-k}|X_1,\ldots,X_{N-k};X_{N-k+1},\ldots,X_N]=P, \end{aligned}\tag{1}$$

where $X_1,\ldots,X_{N-k}$ are the unknown variables and P is taken as a parameter.

(A1) The conditional expectation $\mathrm{E}[V_i\,|\,X_1,\ldots,X_N]$ is *strictly increasing* in X_i, for each bidder $i=1,\ldots,N$.

(A2) (Monotonic solution) The solution of the $N-k$ unknown variables in Eqs. (1) are unique and strictly increasing in P, for all possible realizations of $X_N,\ldots,X_{N-k+1}$.

[4] This rules out implausible Nash equilibria involving bidding rules such as "stay in no matter what", in which case $\beta_i(X_i)=+\infty$ regardless of the value of X_i.

[5] Bikhchandani et al. (2000) point out that, in fact, a continuum of symmetric equilibria exist in these symmetric auctions. In this paper, we focus on one equilibrium for asymmetric auctions which is very similar in structure to the equilibrium described in Milgrom and Weber's paper for symmetric auctions.

[6] Strictly speaking, the assumptions in this section apply to any permutation $(i_1,\ldots,i_N)$ of the bidder indices $(1,\ldots,N)$. This is because, although we observe only one particular dropout order in the dataset, the equilibrium bidding strategies are constructed ex ante from the information structure of the bidding game, before the signals are realized and the realized dropout order is known. However, for notational clarity, we state the equilibrium conditions only for the observed (inverse) dropout order $(1,\ldots,N)$.

Assumption A1 is implied by strict affiliation but may also hold in the absence of affiliation. This assumption rules out "garbling"[7] scenarios where (for example) $\mathrm{E}[V_1|X_1,\dots,X_N] = \mathrm{E}[V_1|X_2,\dots,X_N]$, in which case bidder 1 is less informed than the other bidders. Assumption A2 relates the existence of a monotonic equilibrium to the existence of a monotonic (in the parameter P) solution to the nonlinear system of equations (1).

These conditions lead to an equilibrium proposition for the English auction:

Proposition 1. *Given assumptions* A1 *and* A2, *there exists an increasing-strategy Bayesian–Nash Equilibrium of the asymmetric English auction for which the strategies are defined recursively. In round* k:

$$\beta_i^k(X_i) = \mathrm{E}[V_i|X_i; X_j = (\beta_j^k)^{-1}(\beta_i^k(X_i)),\ j = 1,\dots,N-k, j \neq i; \Omega_k] \tag{2}$$

for the bidders $i=1,\dots,N-k$ *remaining in round* k, *and where* Ω_k *denotes the public information set consisting of the signals observed by the bidders* $N-k+1,\dots,N$ *who have dropped out prior to round* k, *i.e.*,

$$\Omega_k = \{X_j = (\beta_j^{N-j})^{-1}(P_{N-j}), j = N-k+1,\dots,N\}.$$

In other words, at each round k, *we can solve for the set of inverse bid function for all remaining bidders pointwise in* P *by solving the* $(N-k)$*-dimensional system of equations*

$$P = \mathrm{E}[V_i\,|\,X_i = (\beta_i^k)^{-1}(P); X_j = (\beta_j^k)^{-1}(P), j = 1,\dots,N-k, j \neq i; \Omega_k]. \tag{3}$$

for the $N-k$ *unknowns* $(\beta_i^k)^{-1}(P),\ i = 1,\dots,N-k$.

Proof. In the appendix.

3. Log-normal asymmetric ascending auction model

A structural econometric model of the ascending auction would use the equilibrium mapping between unobserved signals and bids (2) as the basis for obtaining estimates of the underlying joint distribution of unobserved valuations and signals $F(V_1,\dots,V_N,X_1,\dots,X_N)$. In this paper, we take a *parametric* approach by restricting attention to a family of joint distribution $F(V_1,\dots,V_N,X_1,\dots,X_N;\theta)$ parameterized by a finite-dimensional vector θ, and use the equilibrium mapping (2) to derive the likelihood function for the observed dropout prices, which can subsequently be maximized with respect to θ to obtain parameter estimates.

Difficulties arise in doing this because the updating process in the common-value ascending auction introduces a large amount of recursivity into the definition of the bid function. For example, assume four bidders (A,B,C,D) and assume the first three drop out in rounds 0,1, and 2, respectively. After bidder A drops out, the remaining bidders (B,C,D) invert the equilibrium bid function for bidder A, in order to obtain his

[7] See Milgrom and Weber (1982, Theorem 7), Engelbrechet-Wiggans et al. (1983), and Hendricks and Porter (1988) for additional discussion and applications of these scenarios.

private signal X_A. In round two, bidder C and D must invert bidder B's bid function during round one, which is her expected value of the object in round one, conditional not only on her private signal, but also on X_A which she inferred by inverting bidder A's conditional expectation function from round zero. The recursive structure which results (involving conditional expectations functions which have as arguments inversions of other conditional expectation functions which are themselves inversions of other conditional expectation functions) quickly becomes intractable if the conditional expectations derived during the updating process do not have analytic solutions.

Therefore, the feasibility of structural estimation lies in choosing a parametric family of joint distributions $F(V_1,\ldots,V_N,X_1,\ldots,X_N;\theta)$ for the latent valuations and signals such that the resulting conditional expectation functions have closed-form expressions which are easy to invert. Among the limited choice of parameterizations which satisfy this criterion, we assume that the bidders' valuations are log-normally distributed. Previously, Wilson (1998) has derived closed-form equilibrium bid functions for a log-additive log-normal information structure, but in this paper we differ from Wilson in not assuming a diffuse prior for the common value distribution. [8] In the rest of this section, we discuss the derivation of the likelihood function for the sequence of dropout prices observed in an ascending auction, under a log-additive log-normal information structure. [9]

V_i, the value of the object to bidder i is assumed to take a multiplicative form $V_i = A_i \times V$, where A_i is a bidder-specific private value for bidder i, and V is a common value component unknown to all bidders. In other words, V_i is the product of a common value part and a private value part.

We assume that V and the A_i's are independently log normally distributed. Letting $v \equiv \ln V$, and $a_i \equiv \ln A_i$: [10]

$$v = m + \varepsilon_v \sim \mathrm{N}(m,\ r_0^2),$$

$$a_i = \bar{a}_i + \varepsilon_{a_i} \sim \mathrm{N}(\bar{a}_i,\ t_i^2).$$

Each bidder is assumed to have a single noisy signal of the value of the object, X_i, which has the form $X_i = A_i \times \mathrm{E}_i$ where $\mathrm{E}_i = V\exp\{s_i\xi_i\}$ and ξ_i is an (unobserved) error term that has a normal distribution with mean 0 and variance 1. If we let $v_i \equiv \ln V_i$ and $x_i \equiv \ln X_i$, then conditional on v_i, $x_i = v_i + s_i\xi_i \sim \mathrm{N}(v_i,\ s_i^2)$. Note that bidder i observes X_i which, in equilibrium, is revealed to other bidders after bidder i drops out. Finally, define $r_i \equiv \sqrt{t_i^2 + s_i^2}$ and denote the variance for ε_v by r_0^2.

[8] The diffuse prior assumption was needed by Wilson (1998) because he allowed each bidder i to observe *two* distinct signals: his private component A_i as well as his noisy estimate of the common component E_i. In contrast, we only allow bidder i to observe a single signal $X_i \equiv A_i \times \mathrm{E}_i$. Under a diffuse prior assumption, observing X_i and observing E_i and A_i separately are informationally equivalent; this is not true without the diffuse prior assumption.

[9] As pointed out by a referee, the assumptions of log-normality help to avoid the high dimensional integration problem of computing bidders' expected valuations conditional on other bidder's drop out prices. However, this approach does not generalize easily to other functional forms. The log-normality assumption plays an important simplifying role, but also represents a limitation of the analysis.

[10] Here r_0^2 represents the variance in bidders' prior distributions on v. Wilson (1998) makes the diffuse prior assumption that $r_0^2 = \infty$.

The joint distribution of $(V_i, X_i, i=1,\ldots,N)=\exp(v_i, x_i, i=1,\ldots,N)$ is fully characterized by $\{m, \bar{a}, t, s, r_0\}$ where $\bar{a}$ denotes the collection of $\bar{a}_i$'s, t denotes the collection of t_i's, and s denotes the collection of s_i's. These parameters are all common knowledge among the bidders.

3.1. Deriving the equilibrium bid functions

We show in this section that the log-normality assumption implies that the system of equations (3) defining the inverse bidding strategies in each round of the auction is log-linear in the signals, allowing us to derive the equilibrium bid functions for each round in closed form. We begin with the system of equations which, following Proposition 1, defines the equilibrium *inverse* bidding strategies for the $N-k$ bidder active in round k, for any value of the bid P:

$$\begin{aligned}
P &= \mathrm{E}[V_1 \,|\, X_1 = (\beta_1^k)^{-1}(P), X_2 = (\beta_2^k)^{-1}(P), \ldots, X_{N-k} = (\beta_{N-k}^k)^{-1}(P),\\
&\qquad X_{N-k+1}, \ldots, X_N],\\
P &= \mathrm{E}[V_2 \,|\, X_1 = (\beta_1^k)^{-1}(P), X_2 = (\beta_2^k)^{-1}(P), \ldots, X_{N-k} = (\beta_{N-k}^k)^{-1}(P),\\
&\qquad X_{N-k+1}, \ldots, X_N],\\
&\vdots\\
P &= \mathrm{E}[V_{N-k} \,|\, X_1 = (\beta_1^k)^{-1}(P), X_2 = (\beta_2^k)^{-1}(P), \ldots, X_{N-k} = (\beta_{N-k}^k)^{-1}(P),\\
&\qquad X_{N-k+1}, \ldots, X_N]. \qquad (4)
\end{aligned}$$

Given the log-normality assumption, the conditional expectation functions for V_i take the form:

$$\mathrm{E}[V_i \,|\, X_1, \ldots, X_N] = \exp(\mathrm{E}(v_i \,|\, x_1, \ldots, x_N) + \tfrac{1}{2} Var(v_i \,|\, x_1, \ldots, x_N)), \;\; i = 1, \ldots, N. \quad (5)$$

Furthermore, we denote the marginal mean-vector and variance–covariance matrix of $(v_i, x_1, \ldots, x_N)$ by $\mu_i \equiv (u_i, \mu^*)$ and $\Sigma_i \equiv \begin{pmatrix} \sigma_i^2 & \sigma_i^{*\prime} \\ \sigma_i^* & \Sigma^* \end{pmatrix}$ [11]. Then, using the conditional mean and variance of jointly normal random variables: [12]

$$\mathrm{E}(v_i \,|\, x \equiv (x_1, \ldots, x_N)') = (u_i - \mu^{*\prime} \Sigma^{*-1} \sigma_i^*) + x' \Sigma^{*-1} \sigma_i^*$$

and

$$V(v_i \,|\, x) = \sigma_i^2 - \sigma_i^{*\prime} \Sigma^{*-1} \sigma_i^*. \quad (6)$$

By plugging (6) into Eq. (5) above, and noting that the conditional variance expression is not a function of x, we see that the conditional expectation function in (5) are log-linear in x_i.

[11] Explicit formulas for the elements of the vector μ_i and the matrix Σ_i can be derived from the distributional assumptions made in the previous section.

[12] See, for example, Amemiya (1985, p. 3).

At round k, let $x_d^k \equiv (x_{N-k+1},\ldots,x_N)'$ denote the vector of k valuations for the bidders who have dropped out prior to round k, and $x_r^k \equiv (x_1,\ldots,x_{N-k})'$ denote the vector of $(N-k)$ valuations for the bidders who have not yet dropped out as of round k. Analogously, partition Σ^{*-1} into $(\Sigma_{k,1}^{*-1\prime},\Sigma_{k,2}^{*-1\prime})'$ where $\Sigma_{k,1}^{*-1}$ is a $((N-k)\times N)$ matrix and $\Sigma_{k,2}^{*-1}$ is a $(k\times N)$ matrix. Then the conditional mean function can be re-written as

$$\mathrm{E}(v_i\,|\,x)=(u_i-\mu^{*\prime}\Sigma^{*-1}\sigma_i^*)+x_r^{k\prime}\Sigma_{k,1}^{*-1}\sigma_i^*+x_d^{k\prime}\Sigma_{k,2}^{*-1}\sigma_i^*. \tag{7}$$

After substituting the conditional mean and variance formulas (7) and (6) into the equations in (5) and taking the log of both sides, we get the following set of $(N-k)$ linear equations for the $N-k$ bidders active in round k, for $p=\ln P$:

$$p=(u_i-\mu^{*\prime}\Sigma^{*-1}\sigma_i^*)+\sigma_i^{*\prime}\Sigma_{k,2}^{*-1\prime}x_d^k+\sigma_i^{*\prime}\Sigma_{k,1}^{*-1\prime}x_r^k+\tfrac{1}{2}(\sigma_i^2-\sigma_i^{*\prime}\Sigma^{*-1}\sigma_i^*) \tag{8}$$

for $i=1,2,\cdots,N-k$. This is analogous to the system of equations in (4) above for the log-normal distribution.

If we let l_k be the $(N-k)\times 1$ vector of 1's, $\mu_k=(u_1,\ldots,u_{N-k})'$, $\Gamma_k=(\sigma_1^2,\ldots,\sigma_{N-k}^2)'$, $\Lambda_k=(\sigma_1^*,\ldots,\sigma_{N-k}^*)'$, then we could rewrite the above system of linear equations (8) as

$$\begin{aligned}p\times l_k=&\tfrac{1}{2}(\Gamma_k-diag(\Lambda_k\Sigma^{*-1}\Lambda_k'))+\Lambda_k\Sigma_{k,2}^{*-1\prime}x_d^k+\mu_k\\&-\Lambda_k\Sigma^{*-1}\mu^*+\Lambda_k\Sigma_{k,1}^{*-1\prime}x_r^k.\end{aligned} \tag{9}$$

Next, let us define

$$\mathscr{A}^k\equiv(\Lambda_k\Sigma_{k,1}^{*-1\prime})^{-1}l_k,$$

$$\mathscr{C}^k\equiv\tfrac{1}{2}(\Lambda_k\Sigma_{k,1}^{*-1\prime})^{-1}(\Gamma_k-diag(\Lambda_k\Sigma^{*-1}\Lambda_k')+2\mu_k-2\Lambda_k\Sigma^{*-1}\mu^*),$$

$$\mathscr{D}^k\equiv(\Lambda_k\Sigma_{k,1}^{*-1\prime})^{-1}(\Lambda_k\Sigma_{k,2}^{*-1\prime}).$$

Solving out for the x_r^k, we obtain the set of $(N-k)$ log-inverse bid functions at round k:

$$x_r^k=\mathscr{A}^k\,p-\mathscr{D}^kx_d^k-\mathscr{C}^k \tag{10}$$

or, each equation singly:

$$x_{r^i}^k=\mathscr{A}_i^k\,p-\mathscr{D}_i^kx_d^k-\mathscr{C}_i^k \tag{11}$$

for $i=1,\ldots,N-k$, where $\mathscr{A}_i^k$ and $\mathscr{C}_i^k$ denote the ith elements of the vectors $\mathscr{A}^k$ and $\mathscr{C}^k$, and $\mathscr{D}_i^k$ denotes the ith row of $\mathscr{D}^k$. The system of equations (11) can be inverted to obtain the $(N-k)$-dimensional system of (log-)bidding strategies for the bidders active in round k, as a function of each bidder's signal and the public information set $\Omega_k\equiv x_d^k$:

$$b_i^k(x_i;x_d^k)\equiv\ln\beta_i^k(e^{x_i};\Omega_k)=\frac{1}{\mathscr{A}_i^k}(x_i+\mathscr{D}_i^kx_d^k+\mathscr{C}_i^k),\ \ i=1,\ldots,N-k. \tag{12}$$

Existence of monotonic equilibrium in log-additive model: We end this section by verifying assumptions A1 and A2 for the log-normal model. For the log-additive information structure, the joint distribution of $(V_1,\ldots,V_N,X_1,\ldots,X_N)$ is strictly affiliated, thereby satisfying assumption A1 using the same argument as in Milgrom and Weber (1982, Theorem 5). The following lemma directly verifies Assumptions A1 and A2.

Lemma 1. $A_i^k > 0\ \forall k \in [0, N-2],\ \forall i \in [1, N-k]$.

Proof. In the appendix.

By Proposition 1, therefore, an increasing-strategy equilibrium exists for the log-additive log-normal information structure.

3.2. *Deriving the likelihood function of the dropout price vector*

The system of equations (12) describes the monotonic mapping from bidders' unobserved signals to their equilibrium dropout prices in round k. However, in round k, we only observe the dropout price for bidder $N-k$, so that only the equation corresponding to this bidder will be used in constructing the likelihood function. Although likelihood-based estimation procedures are not used in the empirical illustration presented later in this paper, we derive the likelihood function in this section to understand the data generating process of the sequence of dropout prices.

Looping over all rounds $0 \leqslant k \leqslant N-2$, the equations relating the sequence of observed bids to the latent signals in a given auction are, similar to Eq. (12) above, given by

$$b_{N-k}^k(x_{N-k}; x_d^k) = \frac{1}{\mathscr{A}_{N-k}^k}(x_{N-k} + \mathscr{D}_{N-k}^k x_d^k + \mathscr{C}_{N-k}^k) \quad \forall k = 0, \ldots, N-2. \tag{13}$$

If we introduce more shorthand notation:

$$\mathscr{F} = \left(\frac{\mathscr{C}_N^0}{\mathscr{A}_N^0}, \ldots, \frac{\mathscr{C}_2^{N-2}}{\mathscr{A}_2^{N-2}} \right)', \qquad \mathscr{G}_i = \left(\underbrace{0, \ldots, 0}_{N-i-2}, \frac{1}{\mathscr{A}_{N-i}^i}, \frac{\mathscr{D}_{N-i}^i}{\mathscr{A}_{N-i}^i} \right).$$

Let $\mathscr{G} \equiv (\mathscr{G}_0', \ldots, \mathscr{G}_{N-2}')'$. Then the system of equations describing the sequence of observed dropout prices (13) can be very succinctly written as:

$$\mathscr{P} = \mathscr{G}(x_2, \ldots, x_N)' + \mathscr{F}. \tag{14}$$

This describes the mapping from the unobserved log-signals $x \equiv (x_2, \ldots, x_N)'$ to the observed log-bids $\mathscr{P} = (p_0, \ldots, p_{N-2})'$. We denote the model parameters by θ. Note that both $\mathscr{F}$ and $\mathscr{G}$ will be explicit functions of θ.

Conditioning on the observed dropout order: In each auction we observe (1) the vector $\mathscr{P}$ of dropout prices for bidders $2, \ldots, N$; and (2) the order in which the participating bidders drop out, and their identities. In deriving the likelihood function for the set of bids (equivalently, dropout prices) observed in an asymmetric ascending auction, the researcher must condition on the observed dropout order. More precisely, in order to specify the likelihood contribution of (say) the dropout price observed in round k in a manner consistent with the equilibrium bidding strategies (2), one must condition on both the order as well as the identities of the bidders who dropped out prior to round

k.[13] The observed dropout order restricts the support of the log-signals $x_1,\ldots,x_N$ to a region $\mathscr{T}_1(\theta) \subset R^N$, for a fixed parameter vector θ. Let $\Pr(\mathscr{T}_1(\theta);\theta)$ denote the probability that $x_1,\ldots,x_N \in \mathscr{T}_1(\theta)$.

Censoring of winning bid: Furthermore, for a given realization of $x_1,\ldots,x_N$, the researcher never observes p_{N-1}, the winning bidder's log-dropout price, since it is censored by p_{N-2}, the log-dropout price of bidder 2.[14] Therefore, all one knows about the winning bidder's log-signal x_1 is that it lies within some region of its support which is consistent with bidder 1's winning the auction, again fixing θ. We let $\mathscr{T}_2(x_2,\ldots,x_N;\theta) \subset R^1$ denote this region of the support of the winner's log-signal x_1. Both of these sets will be described in more detail below.

The likelihood function: Given the distributional assumption that the log-signals $x_2,\ldots,x_N$ are unconditionally multivariate normal, the mapping (14) implies that without conditioning on the event $\mathscr{T}_1(\theta) \subset R^N$, the distribution of a log-bid vector $\mathscr{P}$ is also multivariate normal via a standard change of variables formula, with (unconditional) mean and variance given by:

$$\mu_p(\theta) = \mathscr{F}(\theta) + \mathscr{G}(\theta)\,\mu_2^*(\theta),$$
$$\Sigma_p(\theta) = \mathscr{G}(\theta)\Sigma_2^*(\theta)\,\mathscr{G}(\theta)', \tag{15}$$

where μ_2^* is the $N-1$ subvector of μ^* and Σ_2^* is the $(N-1)(N-1)$ submatrix of Σ^* corresponding to bidders $2,\ldots,N$. Let $f(\cdot;\theta)$ denote the $(N-1)$-variate normal distribution with mean and variance given in (15) above:

$$f(\mathscr{P};\theta) \equiv (2\pi)^{-(N-1)/2}|\Sigma_p(\theta)|^{-1/2}\exp[-\tfrac{1}{2}(\mathscr{P}-\mu_p(\theta))'\Sigma_p(\theta)^{-1} \times(\mathscr{P}-\mu_p(\theta))]. \tag{16}$$

We can then write the likelihood function for a given auction as

$$L(\mathscr{P}\,|\,\theta) = \frac{f(\mathscr{P};\theta)\Pr(\mathscr{T}_2(\mathscr{G}^{-1}(\mathscr{P}-\mathscr{F});\theta);\theta)}{\Pr(\mathscr{T}_1(\theta);\theta)}, \tag{17}$$

where $\mathscr{G}^{-1}(\mathscr{P}-\mathscr{F})$ denotes the realization of $x_2,\ldots,x_N$ consistent with the observed dropout prices $\mathscr{P}$ and the observed dropout order, and $\Pr(\mathscr{T}_2(\mathscr{G}^{-1}(\mathscr{P}-\mathscr{F});\theta);\theta)$ is the probability of $x_1 \in \mathscr{T}_2$ conditional on $\mathscr{P}$. The likelihood function (17) resembles a truncated multivariate normal likelihood, where the numerator is the likelihood for the observed log-dropout prices $p_0,\ldots,p_{N-2}$ and the conditional probability associated with the censored winning log-bid p_{N-1}.[15] The denominator is the truncation probability, which is required since we are deriving the likelihood of the observed log-bids

[13] In an asymmetric model, we cannot derive the joint density of the bids without conditioning explicitly on the orders and identities of the dropout bidders, since each bidder employs nonidentical bidding strategies in equilibrium. For more details, see our discussion in Hong and Shum (1999, pp. 135–137). As we also point out there, conditioning on the observed dropout order is not required to derive the likelihood in symmetric models, or sealed-bid auctions.

[14] Censoring of the winning bid also occurs in empirical models of second-price sealed-bid auctions; see, for example, Paarsch (1997).

[15] For symmetric independent private value models, Eq. (17) reduces to the joint density of the $N-1$ lowest order statistics shown in Donald and Paarsch (1996).

conditional on the observed dropout order. In what follows, we refer to $\Pr(\mathscr{T}_1(\theta);\theta)$ as the truncation probability, and we completely characterize the regions $\mathscr{T}_1$ and $\mathscr{T}_2(\mathscr{G}^{-1}(\mathscr{P}-\mathscr{F});\theta)$ in the next section.

3.3. Truncation probability and equilibrium consistency conditions

3.3.1. Characterization of $\mathscr{T}_1(\theta)$

As discussed in the previous section, for a fixed value of the parameter vector θ, the observed dropout order restricts the log-signals $x_1,\dots,x_N$ to a region $\mathscr{T}_1(\theta)\subset R^N$ within which the log-signals imply, in equilibrium, a dropout order corresponding to the observed order. For a fixed value of θ, this region is defined by inequalities involving the log-signals $x_1,\dots,x_N$ which we refer to as *equilibrium consistency conditions*. These consistency considerations ensure that, in each round of the auction, given the parameters θ, the targeted dropout prices of the remaining bidders for that round are higher than the dropout price at that round.

More precisely, to ensure that the "correct" dropout order occurs, we need to impose that, at the given parameter values, all remaining bidders $i=1,\dots,N-k-1$ have expected valuations greater than $b^k_{N-k}(x_{N-k};x^k_d,\theta)$, the equilibrium dropout price for bidder $N-k$ in round k suppress

$$b^k_i(x_i;x^k_d,\theta)\geqslant b^k_{N-k}(x_{N-k};x^k_d,\theta)=p_k \tag{18}$$

for all rounds k and all $i=1,\dots,N-k-1$, the bidders who remain in the auction after round k. We can now define the truncation region:

$$\mathscr{T}_1(\theta)=\{x_1,\dots,x_N:(18)\text{ is satisfied; }\ \theta\}. \tag{19}$$

At first glance, (18) consists of $\frac{1}{2}N(N-1)$ inequalities; however, we will show that all of these inequalities are implied by the smaller set of $N-1$ inequalities:

$$b^k_{N-k-1}(x_{N-k-1};x^k_d,\theta)\geqslant b^k_{N-k}(x_{N-k};x^k_d,\theta),\quad k=0,\dots,N-2. \tag{20}$$

In order to show this, we first introduce the following important lemma, which holds in the context of the general model in Proposition 1.

Lemma 2. *Let $\phi^k_i(p;x^k_d,\theta)$ denote the inverse function of $b^k_i(x_i;x^k_d,\theta)$ with respect to the x_i argument. For all $j>0, j\leqslant N-2$, and for all $i\leqslant N-j$,* at $x^j_d\equiv(\phi^{j-1}_{N-j+1}(p_{j-1};x^{j-1}_d,\theta),x^{j-1}_d)$:

$$\phi^j_i(p_{j-1};x^j_d,\theta)=\phi^{j-1}_i(p_{j-1};x^{j-1}_d,\theta). \tag{21}$$

In other words, the log-bid functions for rounds j and $j-1$, for each bidder $i=1,\dots,N-j$, intersect at the point $(\phi^{j-1}_i(p_{j-1};x^{j-1}_d,\theta),p_{j-1})$, since an equivalent statement of the above lemma is

$$b^j_i(\phi^{j-1}_i(p_{j-1};x^{j-1}_d,\theta);x^j_d,\theta)=b^{j-1}_i(\phi^{j-1}_i(p_{j-1};x^{j-1}_d,\theta);x^{j-1}_d,\theta)=p_{j-1}.$$

Proof (sketch). Let $(\phi_i^{j-1}(p_{j-1}; x_d^{j-1}, \theta),\ i=1,\dots,N-j+1)$ denote the vector of signals which solves system (4) for round $j-1$ at p_{j-1}. Since $x_d^j = (\phi_{N-j+1}^{j-1}(p_{j-1}; x_d^{j-1}, \theta), x_d^{j-1})$, by careful inspection of (4) and (8), the first $N-j$ elements of the same vector

$$(\phi_i^{j-1}(p_{j-1}; x_d^{j-1}, \theta),\ i = 1, \dots, N-j)$$

also solves system (4) for round j at p_{j-1}. A detailed proof is given in the appendix.

Corollary 1. (20) ⇒ (18).

Proof. In the appendix.

3.3.2. Characterization of $\mathscr{T}_2(\mathscr{G}^{-1}(\mathscr{P}-\mathscr{F});\theta)$

Unlike $\mathscr{T}_1(\theta)$, the set $\mathscr{T}_2(\mathscr{G}^{-1}(\mathscr{P}-\mathscr{F});\theta)$ describes equilibrium restrictions on x_1, the log-signal for the winning bidder, as a function of the observed price vector $\mathscr{P}$ as well as the parameter vector θ.

Let e_i^N be the $(i-1)$th column of a $(N-1)$ $(N-1)$ identity matrix, and let $\mathscr{E}_k^N = (e_{N-k+1}^N, \dots, e_N^N)'$. Using this notation, the log-signals $x_2, \dots, x_N$ of the losing bidders can be denoted $\bar{x}_i = e_i^{N\prime}\mathscr{G}^{-1}(\mathscr{P}-\mathscr{F})$ and $\bar{x}_d^k = \mathscr{E}_k^{N\prime}\mathscr{G}^{-1}(\mathscr{P}-\mathscr{F})$, where the bars emphasize that these log-signals are explicitly functions of the observed prices $\mathscr{P}$ and θ.

Then the set $\mathscr{T}_2(\mathscr{G}^{-1}(\mathscr{P}-\mathscr{F});\theta)$ consists of the following conditions:

$$\{x_1 : b_1^l(x_1; \bar{x}_d^l, \theta) \geqslant p_l,\ l = 0, \dots, N-2\}. \tag{22}$$

At first look, (22) also involves $N-1$ inequality constraints. However, we now show that the only binding constraint will always be

$$b_1^{N-2}(x_1; \bar{x}_d^{N-2}, \theta) \geqslant p_{N-2}. \tag{23}$$

Before proving this, we introduce another preliminary lemma which summarizes some important restrictions on bidders' log-signals induced by the observed price vector $\mathscr{P}$.[16]

Lemma 3. *For any two rounds $k, l \in \{0, \dots, N-2\}$, $l < k$; and for all bidders $i \geqslant N-k$; for all nondecreasing sequences $p_0, \dots, p_{N-2}$ and the corresponding signals $(\bar{x}_2, \dots, \bar{x}_N)$ solved at θ,*

$$\phi_i^k(p_k; \bar{x}_d^k, \theta) \geqslant \phi_i^l(p_l; \bar{x}_d^l, \theta).$$

Proof. In the appendix.

The desired result is a direct corollary of the above lemma.

[16] This lemma is also interesting in its own right from a theoretical point of view, since it is related to a generalization of the "no regret" property in Milgrom (1981), for the symmetric ascending auction, to the asymmetric case covered in this paper. It states that, for *any dropout order*, the bidder dropping out in round k will never "regret" staying in the auction in any round l prior to round k and, analogously, will never "regret" having dropped out in any round subsequent to round k. It also ensures a monotonic equilibrium price path.

Corollary 2. (23) ⇒ (22).

Proof. Note that (23) is a special case of Lemma 3 for $k = N-2$ and $i = 1$, since

$$(23) \Leftrightarrow x_1 \geqslant \phi_1^{N-2}(p_{N-2}; \bar{x}_d^{N-2}, \theta) \geqslant \phi_1^l(p_l; \bar{x}_d^l, \theta)$$
$$\Rightarrow b_1^l(x_1; \bar{x}_d^l, \theta) \geqslant b_1^l(\phi_1^l(p_l; \bar{x}_d^l, \theta); \bar{x}_d^l, \theta) = p_l$$

where the last inequality in the first line uses Lemma 3.[17] □

3.3.3. The likelihood function: log-normal specification

For the log-normal information structure, the regions $\mathcal{T}_1$ and $\mathcal{T}_2$ can be characterized by sets of linear inequalities, using (12). Specifically, $\mathcal{T}_1(\theta)$ is described by this set of linear inequalities regarding (20), for all $k \in \{0, \ldots, N-2\}$

$$\frac{1}{\mathscr{A}_{N-k-1}^k}(x_{N-k-1} + \mathscr{D}_{N-k-1}^k x_d^k + \mathscr{C}_{N-k-1}^k) \geqslant \frac{1}{\mathscr{A}_{N-k}^k}(x_{N-k} + \mathscr{D}_{N-k}^k x_d^k + \mathscr{C}_{N-k}^k).$$

For $\mathcal{T}_2(\mathscr{G}^{-1}(\mathscr{P} - \mathscr{F}); \theta)$, condition (23) can be written as

$$x_1 \geqslant [A_1^{N-2} e_N^{N\prime} - D_1^{N-2} \mathscr{E}_{N-2}^{N\prime} \mathscr{G}^{-1}]\mathscr{P} - C_1^{N-2} + D_1^{N-2} \mathscr{E}_{N-2}^{N}{}' \mathscr{G}^{-1} \mathscr{F}. \tag{24}$$

Using (20) and (24), the likelihood function (17) can be written as

$$\mathscr{L}(\mathscr{P}; \theta) = \frac{f(\mathscr{P}; \theta)}{\Pr(\mathcal{T}_1(\theta); \theta)}$$
$$\times \Phi\left(\frac{m + \bar{a}_1 - [A_1^{N-2} e_N^{N\prime} - D_1^{N-2} \varepsilon_{N-2}^{N\prime} \mathscr{G}^{-1}]\mathscr{P} + C_1^{N-2} - D_1^{N-2} \varepsilon_{N-2}^{N\prime} \mathscr{G}^{-1} \mathscr{F}}{\sqrt{r_0^2 + t_1^2 + s_1^2}}\right).$$

In the next section, we discuss MLE, as well as alternative estimation methods, which may be preferable from a computational perspective.

4. Estimation issues

4.1. Maximum likelihood estimation

Since $\mathcal{T}_1(\theta)$, the support of the log-signals $x_1, \ldots, x_N$ consistent with the observed dropout order, depends explicitly on the parameter vector θ, one may be concerned

[17] Note that if we substituted in $p_{N-2} = b_2^{N-2}(x_2; \bar{x}_d^{N-2}, \theta)$ and $p_l = b_{N-l}^l(x_{N-l}; \bar{x}_d^l, \theta)$ into the right-hand sides of the inequalities in (23) and (22), respectively, then Corollary 2 states that the condition in (20) corresponding to round $k = N-2$ implies the conditions in (18) that pertain to bidder 1. While the corollaries are similar in this way, the statement of Corollary 2 is not explicitly implied by that of Corollary 1. Furthermore, the probability associated with the censored x_1 is conditional on the observed log-dropout prices $p_0, \ldots, p_{N-2}$, rather than the unobserved log-signals $x_2, \ldots, x_N$. We prove Corollary 1 using Lemma 3, which may be of independent interest. Alternatively, we could have proven Corollary 2 by retracing a subset of the arguments used in proving Corollary 1.

that the set of dropout prices generated from $\mathscr{T}_1(\theta)$, i.e.,

$$\mathscr{P}(\theta) \equiv \{b^k_{N-k}(x_{N-k}; x^k_d, \theta),\ k = 0, \ldots, N-2 : x_1, \ldots, x_N \in \mathscr{T}_1(\theta)\}$$

also depends explicitly on θ. Any dependence of $\mathscr{P}(\theta)$, the support of the dropout prices, on θ would violate regularity conditions which are required to derive the usual asymptotic normality for the MLE. However, an interesting corollary of Lemma 2 suggests that this will not be a problem. In what follows, sometimes we suppress the explicit dependence of $b^k_i(\cdot)$ and $\phi^k_i(\cdot)$ on $\bar{x}^k_d$ for notational convenience.

Corollary 3. *For every θ, and every increasing sequence $p_0 < p_1 < \cdots < p_{N-2}$ of log-dropout prices,*

$$[x_1, \phi^{N-2}_2(p_{N-2}; \bar{x}^{N-2}_d, \theta), \ldots, \phi^0_N(p_0; \theta)] \in \mathscr{T}_1(\theta)$$

for all $x_1 \in \mathscr{T}_2(\phi^{N-2}_2(p_{N-2}; \bar{x}^{N-2}_d, \theta), \ldots, \phi^0_N(p_0; \theta); \theta)$.

Proof. We must show that the vector of signals $[x_1, \phi^{N-2}_2(p_{N-2}; \theta), \ldots, \phi^0_N(p_0; \theta)]$, for all $x_1 \in \mathscr{T}_2(\cdots; \theta)$, satisfies conditions (20). Note that, for all rounds $k = 0, \ldots, N-3$,

$$b^k_{N-k-1}(\phi^{k+1}_{N-k-1}(p_{k+1}; \theta)) \geqslant b^k_{N-k-1}(\phi^{k+1}_{N-k-1}(p_k; \theta))$$
$$= p_k = b^k_{N-k}(\phi^k_{N-k}(p_k; \theta)),$$

thus satisfying (20), where the first equality follows from Lemma 2. Therefore, by Corollary 1, the statement holds. □

This corollary implies that, for every θ, every vector of nondecreasing dropout prices $\mathscr{P}$ has strictly positive likelihood: the support of $\mathscr{P}$ does not depend on θ. Alternatively, even though $\mathscr{T}_1(\theta)$ depends on θ, the set $\mathscr{P}(\theta)$ is just the set of nondecreasing dropout price vectors:

$$\forall\theta: \ \mathscr{P}(\theta) = \{\mathbf{p} \equiv (p_0, \ldots, p_{N-2})' \in R^{N-1} : p_0 < p_1 < \cdots < p_{N-2}\},$$

which is just a "rectangular" region in R^{N-1} which does not depend on θ. Therefore the standard asymptotics for MLE obtain. The derivation of the likelihood function for our model complements the results of Donald and Paarsch (1996) for independent private value models.

The major difficulty in implementing the likelihood function is calculating the multivariate integral $P(\mathscr{T}_1(\theta); \theta)$. These difficulties can be overcome using simulation techniques. Given the necessity of evaluating this integral, estimation methods based on simulated moments of the underlying distribution are also attractive alternatives to maximum likelihood estimation. We discuss these alternatives in the following sections.

4.2. Simulated nonlinear least-squares estimation

We consider next a simulated nonlinear least-squares (SNLS) estimator, based on the methodology of Laffont et al. (1995). This estimator minimizes the usual nonlinear least-squares (NLS) objective function

$$Q_T(\theta) = \frac{1}{T}\sum_{t=1}^{T}\sum_{k=0}^{N_t-2}(p_k^t - m_k^t(\theta))^2, \tag{25}$$

where p_k^t is the kth observed log dropout price for auction t, and $m_k^t(\theta)$ is its corresponding expectation conditional on the covariates z_t, taken with respect to its data generating process as given in Eq. (17). Note that, generally speaking, the data generating process depends not only on the parameters θ, but also on covariates z_t which describe auction- as well as bidder-specific characteristics. Therefore the expected bids $m_k^t(\theta)$ should also depend on z_t, but for notational clarity we usually suppress this dependence on z in what follows.[18]

Because $m_k^t(\theta)$, the mean of a multivariate truncated distribution, is difficult to compute analytically, we replace $m_k^t(\theta)$ in Eq. (25) by a simulation estimator $\tilde{m}_k^t(\theta)$ that is consistent as S, the number of simulation draws, goes to infinity. The ensuing SNLS objective function

$$Q_{S,T}(\theta) = \frac{1}{T}\sum_{t=1}^{T}\sum_{k=0}^{N_t-2}(p_k^t - \tilde{m}_k^t(\theta))^2 \tag{26}$$

yields a consistent estimate of θ when $S \to \infty$. In the rest of this section, we give complete details on the simulation of the expected value of each bid $\tilde{m}_k^t(\theta)$.

Simulating $m_k^t(\theta)$: To be specific, we can write the first moment $m_k^t(\theta)$ of the kth dropout price p_k^t, for $k = 0, \ldots, N_t - 2$, as

$$m_k^t(\theta) = \int_{\vec{x}} p_k^t(\vec{x};\theta)\mathbf{1}(\vec{x} \in \mathscr{T}_{1t}(\theta))\frac{f_t(\vec{x};\theta)}{\Pr(\mathscr{T}_{1t}(\theta))}\,\mathrm{d}\vec{x} \tag{27}$$

where $\vec{x} \equiv \{x_1, \ldots, x_{N_t}\}$ denotes the vector of the signals of bidders in the order of dropping out, $p_k^t(\vec{x};\theta)$ specifies the kth dropout price as a function of the parameters and realized vector of bidder signals in (12), $f_t(\vec{x};\theta)$ denotes the multivariate normal density of $\vec{x}$ parameterized by θ. $\mathscr{T}_{1t}(\theta)$ denotes the event that the observed order of dropping out is realized for the tth auction. The integration is over the N_t-dimensional vector of bidder signals.

An "acceptance/rejection" algorithm can be used to simulate $m_t^k(\theta)$. Using this algorithm, for each fixed value of the parameter vector, we draw a multivariate normal random vector of the bidders' private signals (the $\vec{x}$'s) for each auction, and calculate all the targeted dropout prices at all rounds.[19] Then we check all of the truncation

[18] In principle, efficiency considerations may lead to other weighted least squares or other method of moments-based estimators. In addition, one could also exploit other conditional moments of p_k^t in the nonlinear least-squares estimation, by adding summations of terms of the form $(\phi(p_k^t) - \bar{m}_k^{t,\phi}(\theta))^2$ to (25), where $\phi(\cdot)$ is some transformation of p_k^t and $\bar{m}_k^{t,\phi}(\theta)$ denotes the conditional expectation of $\phi(p_k^t)$ given z under (17).

[19] In practice, we draw a vector of i.i.d. N[0, 1] random variables (which are held fixed across all iterations of the estimation procedure), and transform them into the desired multivariate normal vector by premultiplying by the Cholesky factorization of the estimated variance–covariance matrix and adding the estimated mean of the log-signals.

inequalities in $\mathcal{T}_{1t}(\theta)$, and we average the targeted dropout prices over the subset of simulations for which the truncation conditions in $\mathcal{T}_{1t}(\theta)$ are all satisfied. In short, $m_k^t(\theta)$ can be simulated by

$$\frac{1}{S}\sum_{s=1}^{S}[p_k^t(\vec{x}_s;\theta)\mathbf{1}(\vec{x}_s \in \mathcal{T}_{1t}(\theta))] \Big/ \left[\frac{1}{S}\sum_{s=1}^{S}\mathbf{1}(\vec{x}_s \in \mathcal{T}_{1t}(\theta))\right], \tag{28}$$

where the denominator is a simulated approximation of the truncation probability $\Pr(\mathcal{T}_{1t}(\theta))$.

Bias correction in SNLS estimation: The SNLS procedure we have described so far requires the number of simulations $S \to \infty$ to obtain consistency, due to the bias introduced by simulating the denominator probability $\Pr(\mathcal{T}_{1t}(\theta))$. We could remove this denominator bias by multiplying each summand in (25) by the truncation probability $\Pr(\mathcal{T}_{1t}(\theta))$: [20]

$$\bar{Q}_T(\theta) = \frac{1}{T}\sum_{t=1}^{T}\sum_{k=0}^{N_t-2}[\Pr(\mathcal{T}_{1t}(\theta))\ (p_k^t - m_k^t(\theta))]^2. \tag{29}$$

A simulated version of this would be

$$\bar{Q}_{S,T}(\theta) = \frac{1}{T}\sum_{t=1}^{T}\sum_{k=0}^{N_t-2}(\bar{P}_{\mathcal{T}_{1t}}(\theta)p_k^t - \bar{\Pi}_t^k(\theta))^2, \tag{30}$$

where

$$\bar{P}_{\mathcal{T}_{1t}}(\theta) \equiv \frac{1}{S}\sum_{s=1}^{S}[\mathbf{1}(\vec{x}_s \in \mathcal{T}_{1t}(\theta))],$$

$$\bar{\Pi}_t^k(\theta) \equiv \frac{1}{S}\sum_{s=1}^{S}[p_k^t(\vec{x}_s;\theta)\mathbf{1}(\vec{x}_s \in \mathcal{T}_{1t}(\theta))] \tag{31}$$

are unbiased acceptance/rejection simulators for $\Pr(\mathcal{T}_{1t}(\theta))$ and $m_k^t(\theta)\Pr(\mathcal{T}_{1t}(\theta))$, respectively. As shown in Laffont et al. (1995), pg. 959, for every finite S, as $T \to \infty$,

$$\begin{aligned}\text{plim }\bar{Q}_{S,T}(\theta) &= \mathrm{E}\bar{Q}_T(\theta) + \text{plim }\frac{1}{T}\sum_{t=1}^{T}\sum_{k=0}^{N_t-2} Var_S(\bar{P}_{\mathcal{T}_{1t}}(\theta)p_k^t - \bar{\Pi}_t^k(\theta))\\ &\neq \mathrm{E}\bar{Q}_T(\theta).\end{aligned} \tag{32}$$

The second term in the probability limit of $\bar{Q}_{S,T}(\theta)$ is a bias term consisting of conditional variances (across simulation draws) of the simulated difference $\bar{P}_{\mathcal{T}_{1t}}(\theta)p_k^t - \bar{\Pi}_t^k(\theta)$ for the round k dropout price in auction t. The bias term in this probability limit can be corrected, however, using an unbiased estimate of $Var_S(\cdots)$, yielding a modified

[20] We are grateful to the associate editor for pointing this out to us.

NLS objective function

$$\tilde{Q}_{S,T}(\theta) \equiv \bar{Q}_{S,T}(\theta) - \frac{1}{T}\sum_{t=1}^{T}\sum_{k=0}^{N_t-2}\frac{1}{S(S-1)}\sum_{s=1}^{S}(\mathbf{1}(\vec{x}_s \in \mathscr{T}_{1t}(\theta))p_k^t$$

$$-[p_k^t(\vec{x}_s;\theta)\mathbf{1}(\vec{x}_s \in \mathscr{T}_{1t}(\theta))] - (\bar{P}_{\mathscr{T}_{1t}}(\theta)p_k^t - \bar{\Pi}_t^k(\theta)))^2. \tag{33}$$

Then for finite S, $\tilde{\theta} \equiv \operatorname{argmin} \tilde{Q}_{S,T}(\theta) \xrightarrow{p} \theta_0$, because $\tilde{Q}_{S,T}(\theta) \xrightarrow{p} \mathrm{E}\bar{Q}_T(\theta)$.[21]

In principle, therefore, minimization of the modified objective function (33) yields an estimate of the parameter vector θ which is consistent even when the number of simulation draws S remains fixed while the number of auctions $T \to \infty$. In practice, however, this modified objective function is ill-behaved due to the nonsmoothness in θ (for any fixed S) of the indicator functions in the simulators $\bar{P}_{\mathscr{T}_{1t}}(\theta)$ and $\bar{\Pi}_t^k(\theta)$. We overcome this problem by employing an *independent probit kernel-smoother*[22] for these indicator functions. In particular, we estimate $m_k^t(\theta)$ by

$$\tilde{m}_k^t(\theta) = \frac{\left[\frac{1}{S}\sum_{s=1}^{S} p_k^t(\vec{x}_s;\theta)\prod_{k'=0}^{N_t-2}\prod_{j=1}^{N_t-k'-1}\Phi\left(\frac{p_{k',j}^t(\vec{x}_s;\theta) - p_{k'}^t(\vec{x}_s;\theta)}{h}\right)\right]}{\left[\frac{1}{S}\sum_{s=1}^{S}\prod_{k'=0}^{N_t-2}\prod_{j=1}^{N_t-k'-1}\Phi\left(\frac{p_{k',j}^t(\vec{x}_s;\theta) - p_{k',N_t-k'}^t(\vec{x}_s;\theta)}{h}\right)\right]} \tag{34}$$

where $p_{k,j}^t$ is the targeted dropout price for bidder j in round k, for auction t, as a function of $\vec{x}_s$ and θ, $\Phi(\cdot)$ is the standard normal CDF, and h is a bandwidth. Therefore in the empirical illustration we use the following SNLS objective function:

$$\tilde{\tilde{Q}}_{S,T}(\theta) \equiv \frac{1}{T}\sum_{t=1}^{T}\sum_{k=0}^{N_t-2}[\tilde{P}_{\mathscr{T}_{1t}}(\theta)p_k^t - \tilde{\Pi}_k^t(\theta)] \tag{35}$$

where $\tilde{\Pi}_k^t(\theta) = \tilde{m}_k^t(\theta)\tilde{P}_{\mathscr{T}_{1t}}(\theta)$ and $\tilde{P}_{\mathscr{T}_{1t}}(\theta)$ denotes the denominator in (34).[23]

Asymptotic distribution: While $\bar{m}_k^t(\theta)$ is smooth in θ, it is a biased estimator for $m_k^t(\theta)$, for fixed h and S. While it may be possible to extend the bias correction to maintain consistency as S is fixed but h shrinks to zero, we do not pursue this here. Instead, we derive the asymptotic distribution for the minimizer of $\tilde{\tilde{Q}}_{S,T}(\theta)$ assuming that S diverges to infinity.

Following standard arguments in the literature on simulation estimation (cf. Pakes and Pollard, 1989; Gourieroux and Monfort, 1996) as $T \to \infty$, $S \to \infty$, and $S/T \to \infty$, $h \to 0$, the asymptotic distribution of $\hat{\theta} \equiv \operatorname{argmin} \tilde{\tilde{Q}}_{S,T}(\theta)$ is given by

$$\hat{\Sigma}^{-1/2}\sqrt{T}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathrm{N}(0, I) \tag{36}$$

[21] Unlike Laffont et al. (1995), the objective function $\tilde{Q}_{S,T}(\theta)$ is not smooth and differentiable; however, the tools in Pakes and Pollard (1989) could be used to derive the asymptotic distribution of $\tilde{\theta}$.

[22] See, for example, McFadden (1996) for more details.

[23] Alternatively, one could estimate θ via the Simulated Method of Moments, As noted in McFadden (1996), the SMM estimator also achieves consistency with fixed S, as $T \to \infty$. However, as with the bias-corrected SNLS estimator, if we employ smooth simulators to make the objective function better-bahaved in θ, the fixed-S consistency result may not obtain without additional assumptions.

where $\hat{\Sigma} = \hat{\mathscr{J}}^{-1}\hat{\mathscr{H}}\hat{\mathscr{J}}^{-1}$, and for $\tilde{\varepsilon}_t^k(\theta) \equiv [\tilde{P}_{\mathscr{T}_{1t}}(\theta)p_k^t - \tilde{\Pi}_k^t(\theta)]$:

$$\hat{\mathscr{J}} = \frac{1}{T}\sum_{t=1}^{T}\sum_{k=0}^{N_t-2}\frac{\partial}{\partial\theta}\left([\tilde{P}_{\mathscr{T}_{1t}}(\hat{\theta})p_k^t - \tilde{\Pi}_k^t(\hat{\theta})]\left[\frac{\partial\tilde{P}_{\mathscr{T}_{1t}}(\hat{\theta})}{\partial\theta}\,p_k^t - \frac{\partial\tilde{\Pi}_k^t(\hat{\theta})}{\partial\theta}\right]\right)$$

$$\hat{\mathscr{H}} = \frac{1}{T}\sum_{t=1}^{T}\left(\left[\sum_{k=0}^{N_t-2}\tilde{\varepsilon}_t^k(\hat{\theta})\left(\frac{\partial\tilde{P}_{\mathscr{T}_{1t}}(\hat{\theta})}{\partial\theta}\,p_k^t - \frac{\partial\tilde{\Pi}_k^t(\hat{\theta})}{\partial\theta}\right)\right]\right.$$
$$\left.\times\left[\sum_{k=0}^{N_t-2}\tilde{\varepsilon}_t^k(\hat{\theta})\left(\frac{\partial\tilde{P}_{\mathscr{T}_{1t}}(\hat{\theta})}{\partial\theta}\,p_k^t - \frac{\partial\tilde{\Pi}_k^t(\hat{\theta})}{\partial\theta}\right)\right]'\right)$$

where T denotes the total number of auctions in the dataset, and N_t the number of bidders in auction t. Both $\hat{\Sigma}$ and $\hat{\mathscr{H}}$ can be evaluated using numerical derivatives.

For our empirical illustration below, however, we compute standard errors using a parametric bootstrap resampling method.

4.3. Identification

While we pursue a parametric approach in this paper, nonparametric identification of the joint distribution $(V_1,\ldots,V_N,X_1,\ldots,X_N)$ in common value (or, more generally, affiliated values) models have been an important issue in the structural auction literature ever since the insightful result of Laffont and Vuong (1996) that, most generally, bids from a dataset of first-price auctions could be equally well rationalized by a common value as well as an affiliated private values model. While an ascending auction is a strategically richer model than the first-price auction in the presence of common values, and therefore imposes more restrictions on the data-generating process for the bids, it appears difficult to derive a direct proof that the joint distribution of $(V_1,\ldots,V_N,X_1,\ldots,X_N)$ is nonparametrically identified.

On the other hand, it is possible to formulate nonparametric tests for the presence of common value components by exploiting exogenous variation in the number of bidders (see, for example, Haile et al., 2000; Athey and Haile, 2000). Under the hypothesis of no common value components (and even allowing for affiliation between bidders' private values), bidders should drop out at their private value regardless of the number of competitors. In the symmetric framework of the Milgrom–Weber irrevocable dropout auction, one could formulate nonparametric tests of the private value hypothesis by testing whether the empirical marginal distributions of dropout prices are identical across auctions with different number of bidders. Furthermore, this testing approach could accommodate asymmetries if we observed the identities of the bidders and a given bidder participating in a large number of auctions. If we strengthen the assumption of exogeneity in the number of bidders to an assumption that a given bidder's marginal PV distribution remains constant across all auctions in which he participates, then the PV hypothesis would imply that a *given bidder's* empirical marginal distribution of dropout prices is identical across auctions with different numbers as well as identities of participants.

Obviously the ability to test nonparametrically for the existence of common value elements does not imply nonparametric identification of the entire joint distribution. In this paper, we restrict ourselves to the log-normal parametric specification. It appears that the parameters of this specification are (globally) parametrically identifiable from variation in our dataset.[24] The coefficients on bidder-specific covariates are identified off of across-bidder variation. The coefficients on auction-specific covariates are identified off across-auction variation. Both the distribution of bids and the distribution of the covariates, as well as the parametric assumptions, are useful for identifying the variance parameters of the information structure (s, t, r_0).

5. Empirical illustration

In this section, we illustrate the use of the econometric model and estimator described above using data from the FCC's recent auctions of licenses for Personal Communications Services (PCS) spectra. PCS spectra are suitable for transmitting signals for *digital* wireless communications services, including paging and cellular telephony. This digital technology was considered a marked improvement over the older analog wireless technology, most notably in terms of sound quality. Indeed, digital wireless services—many of them provided by the winners in these spectrum auctions—have become the dominant wireless medium across most of the United States today.

The licenses were allocated using a *simultaneous multiple round auction*. The main features of this auction format are multiple rounds and simultaneity. The multiple-round format, as explained above, "allows the bidders to react to information revealed in prior rounds, [thus] enabling the bidders to bid more aggressively" (Cramton, 1997, p. 497). Simultaneous auctioning of many objects allows bidders to realize cross license synergies, if any exist.

While the econometric model accommodates the multiple-round aspect of the FCC auctions, it does not include the simultaneity aspect. Furthermore, the eligibility rules in these auctions were more flexible than the irrevocable dropout assumptions made in the ascending auction model above. For these reasons, we would like to emphasize here that the main purpose of this example is to illustrate and suggest solutions to problems which arise in estimating this model in practice, rather than to provide robust empirical findings concerning equilibrium bidding behavior in the FCC auctions.

Each license covers a particular slice of the radio spectrum over a particular geographic area. Licenses were offered both at the MTA and BTA level (respectively, *major trading area* and *basic trading area*; the designations are from Rand McNally). The data used in this paper comes from the most important spectrum auction, the *MTA broadband PCS auction*, which began on December 5, 1994 and ended on March 13, 1995, after 112 rounds of bidding. 99 MTA licenses were offered—two 30 MHz licenses in most of the 51 MTAs which comprise the US and its territories

[24] Local identification in nonlinear parametric models typically obtains when the Jacobian matrix of the estimating equations is nonsingular at the true parameter values, which is verified in our application by the numerical convergence of the optimization algorithm and the nonsingular Jacobian matrix calculated at the estimated parameter values.

Table 1
Maximum likelihood estimation: results from Monte Carlo experiments

Coefficient[a]	Exp. 1: $S=100,\ h=0.01$			Exp. 2: $S=100,\ h=0.1$			Exp. 3: $S=50,\ h=0.01$			Exp. 4: $S=50,\ h=0.1$		
	25%	50%	75%	25%	50%	75%	25%	50%	75%	25%	50%	75%
Components of $\log s$												
Constant	0.0179	0.1717	0.2374	0.0108	0.1419	0.2715	0.0128	0.1694	0.2562	0.0111	0.1939	0.2552
Components of $\log t$												
Constant	0.0373	0.1756	0.2605	0.0379	0.1752	0.2690	0.0265	0.1575	0.2520	0.0470	0.1889	0.2656
Components of m												
Constant	0.0507	0.1576	0.2451	0.0308	0.1991	0.3029	0.0186	0.1454	0.2590	0.0397	0.1784	0.2510
POP (mills)	0.0896	0.2334	0.2824	0.0746	0.2114	0.3007	0.0211	0.1773	0.2544	0.0493	0.1947	0.2705
POP CHANGE (%)	0.0037	0.0153	0.0237	0.0028	0.0166	0.0279	0.0029	0.0145	0.0258	0.0063	0.0180	0.0255
Components of $\bar{a}$												
Constant[b]												
CEL PRES	0.0125	0.1617	0.2536	0.0107	0.1743	0.2660	0.0147	0.1528	0.2463	0.0126	0.1831	0.2569

Note: Each column contains the empirical median, 25th percentile, and 75th percentile absolute deviation for an experiment. Each of the four experiments consisted of 100 re-estimations on bids simulated for a 91-auction sample of auctions.

[a]The true values underlying the simulated were, respectively: 0.1, 0.2, 1.0, 1.5, 0.05, 0.1.

[b]Not separately identified from constant in m.

abroad. In this paper, we analyze the auctions of 91 of these licenses.[25] Thirty firms participated in this auction, and 19 of them eventually won licenses, yielding over $7 billion in government revenue. See Appendix B for details on data sources and variable definitions.

5.1. Monte Carlo experiments

Before presenting our estimation results, we consider findings from a series of Monte Carlo experiments which gauge the sensitivity of the estimation results to S (the number of draws used in simulating the truncation probability $\Pr(\mathscr{T}_{1t}(\theta))$) and h (the bandwidth which we employ in the kernel-smoother for the indicator functions $\mathbf{1}(\vec{x}\in\mathscr{T}_{1t}(\theta))$ which characterize the truncation region). Summary results for these Monte Carlo experiments are given in Tables 1 and 2.

For each experiment, we simulated 100 datasets of bids from 91 auctions, which is the same number of observations contained in our actual estimation dataset. Furthermore, in constructing the simulated datasets, we maintained the same firm identities and covariates as in our estimation dataset. The four experiments reported in Tables 1 and 2 differ in the values of S and h used to estimate the parameter values.

[25] We did not analyze the auctions for the licenses for Samoa, Guam, Puerto Rico, and Alaska.

Table 2
Simulated nonlinear least-squares estimator: results from Monte Carlo experiments

Coefficient[a]	Exp. 1: $S=100,\ h=0.01$			Exp. 2: $S=100,\ h=0.1$			Exp. 3: $S=50,\ h=0.01$			Exp. 4: $S=50,\ h=0.1$		
	25%	50%	75%	25%	50%	75%	25%	50%	75%	25%	50%	75%
Components of $\log s$												
Constant	0.0164	0.0588	0.1194	0.0127	0.0628	0.1398	0.0098	0.0340	0.1127	0.0161	0.0465	0.1194
Components of $\log t$												
Constant	0.0102	0.0474	0.1746	0.0112	0.0627	0.1660	0.0062	0.0312	0.1198	0.00083	0.0446	0.1343
Components of m												
Constant	0.0040	0.0174	0.1465	0.0054	0.0281	0.1247	0.0028	0.0258	0.1403	0.0023	0.0155	0.1204
POP (mills)	0.0023	0.0289	0.1350	0.0032	0.0255	0.1300	0.0019	0.0133	0.1500	0.0021	0.0149	0.1284
POP CHANGE (%)	0.0006	0.0051	0.0195	0.0011	0.0057	0.0181	0.0003	0.0024	0.0116	0.0005	0.0040	0.0113
Components of $\bar{a}$												
Constant[b]												
CEL PRES	0.0180	0.0786	0.1928	0.0246	0.0926	0.1636	0.0134	0.0463	0.1278	0.0260	0.0637	0.1300

Note: Each column contains the empirical median, 25th percentile, and 75th percentile absolute deviation for an experiment. Each of the four experiments consisted of 100 re-estimations on bids simulated for a 91-auction sample of auctions.

[a]The true values underlying the simulated were, respectively: 0.1, 0.2, 1.0, 1.5, 0.05, 0.1.

[b]Not separately identified from constant in m.

Each entry in the table reports the 25th, 50th (median), and 75th quantile of the empirical distribution (across the 100 replications of each experiment) of the *absolute deviation* $AD_i \equiv |\hat{\beta}_i - \beta^0|$ of each estimated parameter from its true value, where $\hat{\beta}_i$ denotes the estimated parameter for the ith simulated dataset, and β^0 denotes the true value of the parameter.

Encouragingly, the AD's are small for both the MLE as well as SNLS experiments, which indicate that the parameter estimates are quite stable, and not very sensitive to changes in the number of simulation draws and the smoothing bandwidth. However, notice that the AD's are uniformly smaller for the NLS experiments (in Table 2) than the MLE experiments (in Table 1). For this reason, we employ the simulated NLS estimator in our empirical illustration using actual data from the FCC auctions.

5.2. Estimation results

Table 3 shows the results for two specifications of the full model, estimated using the SNLS methodology described above. Section B.2 in the appendix discusses the parameterization choices that we made. Models A and B in Table 3 differ in the extent to which $\log s$—the log of the variance on the bidders' priors about the common value component—is parameterized.

Table 3
Simulated nonlinear least-squares estimates

Coefficient	Model A Estimate	Model A Std. error[a]	Model B Estimate	Model B Std. error
Components of $\log s$				
Constant	0.0665	0.0380	0.0643	0.0147
POP (mills)			−0.0023	0.0016
INCOME (per cap.,$'000)			−0.0228	0.0146
Components of $\log t$				
Constant	0.1627	0.3686	0.2146	0.1720
Log r_0[b]				
Constant	−0.0136	0.0248	−0.0189	0.0150
Components of m				
Constant	0.9676	0.3652	0.9557	0.1468
POP (mills)	1.4942	0.4015	1.4856	0.1757
POP CHANGE (%)	0.0475	0.0359	0.0449	0.0147
Components of $\bar{a}$				
Constant[c]				
CEL PRES	0.0707	0.0201	0.0693	0.0128
# auctions (T)	91		91	

Note: S (number of simulation draws): 100; h (bandwidth) = 0.01.
[a]Bootstrapped standard error, computed from empirical distribution of parameter estimates from 100 parametric bootstrap resamples.
[b]Variance of the prior distribution on common value component.
[c]Not separately identified from constant in m.

The bootstrapped standard errors indicate that the estimates are generally statistically significant from zero. The coefficients on POP and POP CHANGE are positive (1.4942 and 0.0475 for the Model A results): as expected, a larger population and higher growth rates increase a license's value. The magnitudes of the estimates for $\log s$, $\log t$ and $\log r_0$ indicate that the largest source of variation in bidders' signals is in their private value components.

Finally, the coefficient on CEL PRES (0.0707, with standard error 0.0201, for Model A), while small in magnitude, indicates some weak complementarities between offering PCS service in a given region, and existing cellular presence in another nearby region.[26] Note that, in these specifications, asymmetries across the bidders are captured only by the CEL PRES covariate in $\bar{a}$, the mean of the distribution from which bidders' private values are drawn. The small estimated coefficient indicates that bidders are largely *symmetric* given this specifications and our results. This finding has

[26] This confirms previous results in Moreton and Spiller (1998), which also detected the existence of PCS-Cellular complementarities in these auctions in reduced-form bid regressions.

implications on how the equilibrium bid functions for a given bidder changed during the course of the auction. Next, we explore the implications of our estimates on the equilibrium bidding strategies.

5.3. Estimated bid functions

Fig. 1 shows plots of the estimated (log) bid functions for the winning bidder ("bidder 1", using the indexing scheme employed earlier), in each of the rounds of four selected auctions. Here log-signal x_1 is plotted on the x-axis, while $b_1^k(x_1)$, her log bid functions for rounds $k=0,\dots,N-2$, are plotted on the y-axis. The units on the y-axis are log($mills).

First note that the log-bid functions are linear in the signals; this results from the log-normality assumption (cf. Eq. (3.12) above). Second, note that the bid functions decrease in slope as the auction progresses, implying that for any given valuation x_1 in the range in which bidder 1 would have won the auction, the targeted dropout price falls as bidders drop out. For example, for auction #30 (New Orleans block A, the lower-left hand corner graph), if $x_1 = 4$, then we can read off the graph that bidder 1's targeted log-dropout price falls from around $4 million in the opening round 0, to about $3.4 million in the final round.

This monotonic change in the slope of the bid functions is characteristic of *symmetric* ascending auctions. As noted above, the small point estimate of the CEL PRES coefficient suggests that bidders are essentially symmetric. Changes in the slope of the bid function occur because the conditioning events change as the auction progresses, as bidder 1 learns the private signals of the bidders who have dropped out.

In a symmetric ascending auction, where no differences exist among his competitors, bidder 1's expected valuation for the object is either increasing or decreasing in each and every one of her competitors' private signals. Furthermore, when bidder j remains in the auction, bidder 1 assumes in equilibrium that bidder j's private signal is equal to x_1.[27] Once bidder j drops out, bidder 1 learn x_j and, given symmetry, $x_j < x_1$. Essentially, bidder 1 "plugs" a smaller number x_j into her bid function. This is true for every bidder $j \neq 1$, since bidder 1 wins the auction.

This process of "plugging-in" smaller numbers (the x_j's, $j \neq 1$) in place of larger numbers (x_1) causes the slopes of the successive bid functions to change monotonically as the auction progresses. This change will be monotonically decreasing if bidder 1's expected valuation for the object is increasing in all her competitors' private signals. This is true for our additive log-normal model with a common value component, which induces positive correlation among all the bidders' signals.

For the asymmetric case, this monotonicity need not hold, even assuming, as in the log-normal model, that bidder 1's expected valuation is increasing in each and every private signal. This is because in every round, bidder 1 not only learns the private

[27] This is because, in equilibrium, bidder 1 bids a log-price $b_1(x_1)$ in which her expected log-revenue from winning is just equal to $b_1(x_1)$. If she in fact wins at the log-price $b_1(x_1)$, this must mean that bidder j has dropped out at that price, implying that bidder j's private log-signal $x_j = b_j^{-1}(b_1(x_1))$. For the symmetric case, $b_j(\cdot) = b(\cdot)$, $\forall\ j$, so that $x_j = b_j^{-1}(b_1(x_1)) = x_1$.

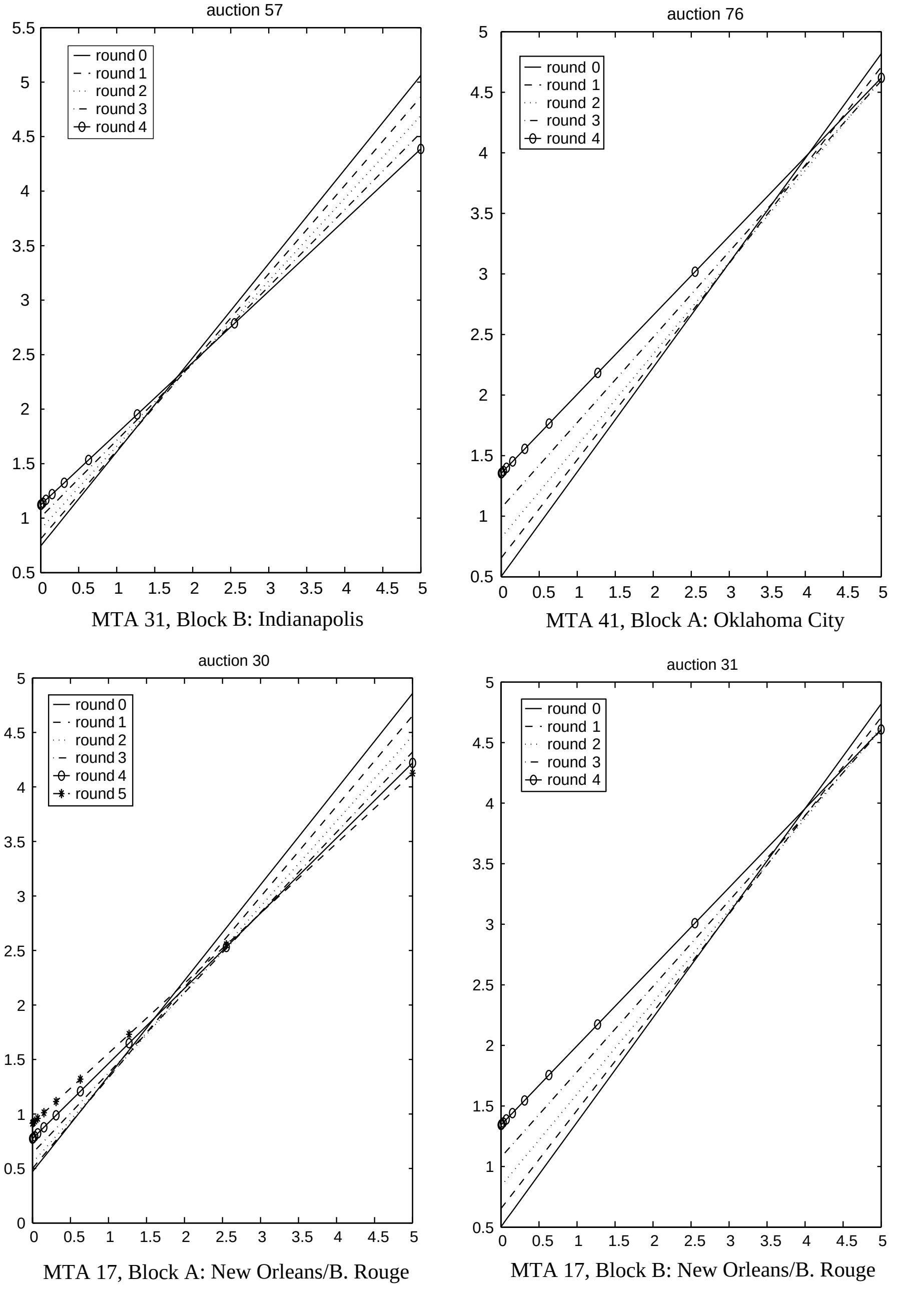

Fig. 1. Estimated bid functions: using model A results. *x*-axis: log-signals; *y*-axis: log-bid.

signal of the dropout bidder during that round, but also revises her beliefs about the remaining bidders' signals knowing that these bidders are also revising their beliefs upon observing dropout behavior. While the coefficients on all the private signals in bidder 1's expected valuation of the object are still positive, it is not clear whether the new values for the signals "plugged in" during each round are larger or smaller than the old values; therefore, it is unclear how this change in information affects the slope of her bid function.

In conclusion, therefore, while the log-linearity of the estimated bid functions results from the log-normality assumption, the monotonic decrease in slopes as the auction proceeds arises from our finding that the bidders were largely symmetric.

6. Conclusions

We have characterized an increasing-strategy Bayesian Nash equilibrium in asymmetric ascending (English) auctions. We showed that the equilibrium (inverse) bidding strategies in each round of the auction are defined implicitly via systems of nonlinear equations. This formed the basis of an algorithm we devised to calculate the likelihood function for an observed vector of bids. In the case that bidders' private signals are drawn from nonidentical log-normal distributions, we show that the vector of log-dropout prices observed in a given ascending auction is distributed as truncated multivariate normal. We illustrated the use of this model with data from the FCC spectrum auctions, and estimated examples of bid functions to demonstrate how equilibrium learning affects bidding behavior in ascending auctions.

An important extension to our current model is to relax the irrevocable dropout assumption. However, the result may be an "open call" auction which, as noted by Vickrey (1961), is strategically equivalent to a sealed bid second-price auction (since, essentially, without the irrevocable dropout requirement, no information can be credibly revealed during the course of the auction).

Nonetheless, there has been very little work to date on the structural estimation of sealed bid second-price auction models accommodating both common values and asymmetries.[28] This may be due in part to the difficulties involved in calculating equilibrium bidding strategies in these auctions. However, the empirical framework developed in this paper can be directly generalized to other auction formats, including the first- and second-price sealed bid auctions. The common element in all these auction models is that the equilibrium bid functions are described by systems of equations, which facilitates the numerical or computational algorithms required for empirical implementation of these models. We discuss these issues in more detail in Hong and Shum (1999), and we plan to apply this methodology to first- and second-price auction settings in future research.

[28] For example, empirical studies of open-call timber auctions in the structural vein (by, among others, Paarsch (1997), Baldwin et al. (1997), Haile and Tamer (2000)) have used the independent private values framework. Asymmetries are potentially important in these auctions, arising from both geographical locational differences among the firms as well as collusive behavior.

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Appendix A. Proofs

Proof of Proposition 1. Mimicking the proof of Theorem 10 in Milgrom and Weber (1982), we show that if all bidders $j \neq i$ follow their equilibrium strategies $\beta_j^k(\cdot)$, bidder i's best response is to play $\beta_i^k(\cdot)$ because this guarantees that bidder i will win the auction if and only if his expected net payoff is positive conditional on winning.

For any price P, (3) holds. If bidder i wins the auction in round k when all remaining bidders simultaneously exit a price of P, his ex post valuation is

$$\mathrm{E}[V_i|X_i; X_j = (\beta_j^k)^{-1}(P), j = 1, \ldots, n-k, j \neq i; \Omega_k]. \tag{A.1}$$

Since this conditional expectation is increasing in X_i (from Assumption A1), bidder i makes a positive expected profit from winning in round k by staying active in the auction at a price of P if and only if $X_i \geqslant (\beta_i^k)^{-1}(P) \Leftrightarrow \beta_i^k(X_i) \geqslant P$ (here we use Assumption A2, the monotonicity of equilibrium bid strategies). In other words, $\beta_i^k(X_i)$ specifies the price below which bidder i makes a positive expected profit by staying in the auction and above which bidder i makes a negative expected profit by staying in the auction. Therefore, for every realization of X_i, $\beta_i^k(X_i)$ specifies a best-response dropout price for bidder i in round k.[29] □

Proof of Lemma 1. Since $\mathscr{A}^k \equiv (\Lambda_k \Sigma_{k,1}^{*-1\prime})^{-1} l_k$, the lemma states that each row of $(\Lambda_k \Sigma_{k,1}^{*-1\prime})^{-1}$ sums to a positive number. Note that $(\Lambda_k \Sigma_{k,1}^{*-1\prime})$ is equal to the $(N-k)$ $(N-k)$ principal submatrix of $\Lambda_0 \Sigma^{*-1}$, indexed by $1, \ldots, N-k$. Since $v_i = a_i + v$,

[29] For the *symmetric* model of Milgrom and Weber (1982), Bikhchandani et al. (2000) described additional equilibria where the bidding strategies $\beta_i^k(X_i)$ for the bidders $i = 1, \ldots, k$ remaining in round k, and for $k = 0, \ldots, N-3$ (i.e., for all rounds except the last one), take the form of $\zeta * \mathrm{E}[V_i|X_i; X_j = (\beta_j^k)^{-1}(\beta_i^k(X_i)), j = 1, \ldots, n-k, j \neq i; \Omega_k]$, for $\zeta \in (0, 1]$. The asymmetric equilibrium we focus on is analogous to the "maximum" symmetric equilibrium discussed in Bikhchandani et al. (2000), which coincides with the construction of Milgrom and Weber (1982).

The estimation procedure described in this paper extends readily to accommodate any other value of ζ chosen a priori from $(0, 1]$. Indeed, ζ can be estimated simultaneously with the other parameters, if desired. See Bjorn and Vuong (1985) for a similar approach to dummy endogenous variable models. We do not take this approach in this paper.

we can write

$$\Lambda_0 = Cov(l_0\, v, x) + Cov(a, x) = l_0\, Cov(v, x) + Cov(a, x)$$

where we denote $a \equiv (a_1, \ldots, a_N)$. Hence $(\Lambda_k \Sigma_{k,1}^{*-1\prime})$ can be written as $l_k y' + D^*$, where y' is the first $N-k$ elements of $Cov(v,x)\Sigma^{*-1}$ (which are strictly positive due to the strict affiliation between v and x), and D^* is the first $(N-k)$ principal matrix of $D \equiv Cov(a,x)\Sigma^{*-1}$. As shown by Sarkar (1969), strict affiliation of x implies that Σ^{*-1} is a matrix with a *dominant diagonal*, as defined in McKenzie (1959, Theorem 4). Alternatively, one can write $\Sigma^* = (l y'' + \bar{D})$, for l a column of ones, y'' a row vector, and $\bar{D}$ a diagonal matrix. Then the same arguments as in Eq. (A.2) below show directly that Σ^{*-1} has a dominant diagonal, in the sense that up to multiplication by a diagonal matrix, each diagonal element is positive and each row sums to a positive number.

Since $Cov(a,x)$ is a diagonal matrix, D also has a dominant diagonal because the property of diagonal dominance is preserved under multiplication by a diagonal matrix. The property of diagonal dominance is also perserved by any principal submatrix of D, including D^*. Next we write (using, for example, Dhrymes, 1984, p. 39)

$$\begin{aligned}(l_k y' + D^*)^{-1} &= D^{*-1} - \frac{1}{1 + y' D^{*-1} l_k} D^{*-1} l_k y' D^{*-1} \\ &= D^{*-1}\left(I - \frac{1}{1 + y' D^{*-1} l_k} l_k y' D^{*-1}\right). \end{aligned} \tag{A.2}$$

Since D^{*-1} is a positive matrix (McKenzie, 1959, Theorem 4), $y^{*\prime} = y' D^{*-1}$ is a nonnegative vector. Next $1 + y' D^{*-1} l_k = 1 + \sum_{i=1}^{n-k} y^{*\prime}_i$, and each row of $l_k y' D^{*-1}$ is just the row vector $y^{*\prime}$. Therefore, the sum of each row of the second matrix in the last expression in (A.2) is $[1/(1 + \sum_{i=1}^{n-k} y_i^{*\prime})] > 0$. Since D^{*-1}'s elements are also nonnegative, it follows that the row sums of (A.2) must also be positive.

Alternatively, one could calculate $(\Lambda_k \Sigma_{k,1}^{*-1\prime})^{-1}$ explicitly using the information structure of the log-normal model, as we did in a previous version of this paper. This would show that $(\Lambda_k \Sigma_{k,1}^{*-1\prime})^{-1}$ can in fact be written in the form of $(l_k y'' + D^{**})^{-1}$, for some positive y'' and a positive diagonal matrix D^{**}. Then the same argument of (A.2) applies. □

Proof of Lemma 2. Consider the round j and round $j-1$ system of equations, evaluated at P_{j-1}, the dropout price for round $j-1$. In what follows, let $\tilde{\phi}_k^{N-k}(P) \equiv \exp(\phi_k^{N-k}(\log P))$, for each round $k = 1, \ldots, N-2$. In round $j-1$:

$$\begin{aligned} P_{j-1} = \mathrm{E}[V_i \mid & \tilde{\phi}_1^{j-1}(P_{j-1}), \tilde{\phi}_2^{j-1}(P_{j-1}), \ldots, \tilde{\phi}_{N-j+1}^{j-1}(P_{j-1}); \tilde{\phi}_{N-k}^{k}(P_k), \\ & k = 0, \ldots, j-2] \end{aligned} \tag{A.3}$$

for $i = 1, \ldots, N-j+1$. Let $[\tilde{\phi}^{j-1}](P_{j-1}) \equiv (\tilde{\phi}_1^{j-1}(P_{j-1}), \tilde{\phi}_2^{j-1}(P_{j-1}), \ldots, \tilde{\phi}_{N-j+1}^{j-1}(P_{j-1}))'$ denote the vector of inverse bid functions which solve this system of equations at the price P_{j-1}. Note that the $(j-1)$th element of this (that corresponding to bidder $N-j+1$) is X_{N-j+1}, which is this bidder's actual signal.

Let Ω^{j-1} denote $\{\tilde{\phi}^k_{N-k}(P_k), k=0,\ldots,j-2\}$, the information set in round $j-1$. Using this notation, we can write the round j system of equations as:

$$P_{j-1} = \mathrm{E}[V_1 \mid \tilde{\phi}^j_1(P_{j-1}), \tilde{\phi}^j_2(P_{j-1}), \ldots, \tilde{\phi}^j_{N-j}(P_{j-1}), \tilde{\phi}^{j-1}_{N-j+1}(P_{j-1}), \Omega^{j-1}],$$

$$P_{j-1} = \mathrm{E}[V_2 \mid \tilde{\phi}^j_1(P_{j-1}), \tilde{\phi}^j_2(P_{j-1}), \ldots, \tilde{\phi}^j_{N-j}(P_{j-1}), \tilde{\phi}^{j-1}_{N-j+1}(P_{j-1}), \Omega^{j-1}],$$

$$\vdots$$

$$P_{j-1} = \mathrm{E}[V_{N-j} \mid \tilde{\phi}^j_1(P_{j-1}), \tilde{\phi}^j_2(P_{j-1}), \ldots, \tilde{\phi}^j_{N-j}(P_{j-1}), \tilde{\phi}^{j-1}_{N-j+1}(P_{j-1}), \Omega^{j-1}]. \tag{A.4}$$

If we substitute in the first $(N-j)$ elements of $[\tilde{\phi}^{j-1}](P_{j-1})$ into the round j system, we get:

$$P_{j-1} = \mathrm{E}[V_1 \mid \tilde{\phi}^{j-1}_1(P_{j-1}), \tilde{\phi}^{j-1}_2(P_{j-1}), \ldots, \tilde{\phi}^{j-1}_{N-j}(P_{j-1}); \tilde{\phi}^{j-1}_{N-j+1}(P_{j-1}), \Omega^{j-1}],$$

$$P_{j-1} = \mathrm{E}[V_2 \mid \tilde{\phi}^{j-1}_1(P_{j-1}), \tilde{\phi}^{j-1}_2(P_{j-1}), \ldots, \tilde{\phi}^{j-1}_{N-j}(P_{j-1}); \tilde{\phi}^{j-1}_{N-j+1}(P_{j-1}), \Omega^{j-1}],$$

$$\vdots$$

$$P_{j-1} = \mathrm{E}[V_{N-j} \mid \tilde{\phi}^{j-1}_1(P_{j-1}), \tilde{\phi}^{j-1}_2(P_{j-1}), \ldots, \tilde{\phi}^{j-1}_{N-j}(P_{j-1}); \tilde{\phi}^{j-1}_{N-j+1}(P_{j-1}), \Omega^{j-1}]. \tag{A.5}$$

Note that Eqs. (A.5) exactly resembles the first $(N-j)$ equations in the round $j-1$ system (A.3): this immediately implies that $[\tilde{\phi}^{j-1}](P_{j-1})$ solves both the round j and $j-1$ systems of equations, at the price P_{j-1}. In other words, the bid functions for rounds j and $j-1$ must intersect at the point $([\tilde{\phi}^{j-1}](P_{j-1}), P_{j-1})$. □

Proof of Corollary 1. We break the proof into two steps. Throughout, we omit the conditioning arguments x^k_d and θ for brevity. First we show that (20) implies

$$b^{k+1}_{N-k-1}(x_{N-k-1}) \geqslant b^k_{N-k}(x_{N-k}), \forall k = 0, \ldots, N-3 \tag{A.6}$$

namely, that the constructed sequence of dropout prices are increasing. To see this, note that (20) implies

$$\begin{aligned} x_{N-k-1} &\geqslant \phi^k_{N-k-1}(b^k_{N-k}(x_{N-k})) \\ &\Rightarrow b^{k+1}_{N-k-1}(x_{N-k-1}) \geqslant b^{k+1}_{N-k-1}(\phi^k_{N-k-1}(b^k_{N-k}(x_{N-k}))) \\ &= b^k_{N-k}(x_{N-k}), \end{aligned}$$

where the inequality in the first line arises from (20), and the equality in the second line arises from Lemma 2. Clearly, this argument holds for all $k = 0, \ldots, N-3$.

Second, we use (20) and (A.6) to show (18). For a given round k, and bidder $i \leqslant N-k-1$:

$$
\begin{aligned}
x_i &\geqslant \phi_i^{N-i-1}(b_{i+1}^{N-i-1}(x_{i+1})) \\
&\geqslant \phi_i^{N-i-1}(b_{i+2}^{N-i-2}(x_{i+2})) = \phi_i^{N-i-2}(b_{i+2}^{N-i-2}(x_{i+2})) \\
&\geqslant \phi_i^{N-i-2}(b_{i+3}^{N-i-3}(x_{i+3})) = \phi_i^{N-i-3}(b_{i+3}^{N-i-3}(x_{i+3})) \\
&\geqslant \cdots \geqslant \phi_i^{k}(b_{N-k}^{k}(x_{N-k}))
\end{aligned}
$$

where the inequality in the first line arises from (20), the inequality in the second line arises from (A.6), and the equality in the second line arises from Lemma 2. Applying the $b_i^k(\cdots)$ transformation to the first and last terms in the above inequality yields $b_i^k(x_i) \geqslant b_{N-k}^k(x_{N-k})$. This argument applies $\forall k = 0, \ldots, N-2$, $\forall i \leqslant N-k-1$. □

Proof of Lemma 3. Note that

$$
\begin{aligned}
b_i^l(\phi_i^k(p_k)) &= b_i^l(\phi_i^{l+1}(b_i^{l+1}(\cdots \phi_i^{k-1}(b_i^{k-1}(\phi_i^k(p_k)))))) \\
&\geqslant b_i^l(\phi_i^{l+1}(b_i^{l+1}(\cdots \phi_i^{k-1}(b_i^{k-1}(\phi_i^k(p_{k-1})))))) \\
&= b_i^l(\phi_i^{l+1}(b_i^{l+1}(\cdots \phi_i^{k-1}(p_{k-1})))) \\
&\geqslant b_i^l(\phi_i^{l+1}(b_i^{l+1}(\cdots \phi_i^{k-1}(p_{k-2})))) = b_i^l(\phi_i^{l+1}(b_i^{l+1}(\cdots \phi_i^{k-2}(p_{k-2})))) \\
&\geqslant \cdots = b_i^l(\phi_i^{l+1}(p_{l+1})) \\
&\geqslant b_i^l(\phi_i^{l+1}(p_l)) = p_l,
\end{aligned}
$$

where the equality in the second and third lines use Lemma 2, and all the inequalities use the fact that the sequence of dropout prices $p_0, \ldots, p_{N-2}$ is nondecreasing. □

Appendix B. Data description

The data on the auction results from the MTA broadband auction is taken from the FCC's web site (`http://www.fcc.gov`). This data gives us information on the participants and the bids that they submitted during each round on the various licenses. We supplemented this data with market characteristics at the MTA level from the Rand–McNally guide. The cellular presence data came from the Cellular Telephone Industry Association's *Wireless Market Book* (Cellular Telephone Industry of America, 1996). We discuss how we created the dependent variable and the regressors in turn.

B.1. The dropout prices

In order to fit the model to the FCC data, we impose some assumptions about bidders' beliefs concerning the dropout behavior of the other bidders. We assign a

Table 4
Summary statistics for data variables

Variable	N	Mean	Std dev	Min	Max
Winning prices ($mill)	91[a]	75.87	89.71	4.39	493.5
Population (millions)	91	5.15	4.14	1.15	26.78
Pop'n change (1990–95,%)	91	6.00	3.53	0.40	12.80
Per capita income ('000)	91	15.86	3.71	11.96	20.70
Dropout prices ($mill)	423	53.18	69.28	0.89	493.5
Cell. pres	423	0.61	1.28	0	8

[a]We omitted the observations for: Puerto Rico, Guam, Samoa, and Alaska.

"dropout price" to bidder j which is the last price at which he was "active" (in a sense to be clarified below). We assume that all remaining bidders also believe this assigned price to be bidder j's dropout price.

Next we define how we classify a bidder as "active". The following example will be useful: suppose that there are four bidders (A, B, C, D) and we observe that the last submitted bids for A, B, and C were 10, 20, and 30, respectively. If the price goes up by increments of 5, then, D will win the object at a price of 35 (assuming that his valuation is greater than that).

One simple way would be to assign to each bidder a dropout price equal to his last submitted bid, and assume that the winner's dropout price was greater than or equal to the winning price, i.e., $P_A = 10, P_B = 20, P_C = 30, P_D \geqslant 35$. This method is inconsistent, because of the gap between the second-highest dropout price P_C and the lower bound on the highest dropout price P_D. As Milgrom and Weber (1982) note, their formulation of the ascending model model reduces to a second-price auction when there are only two bidders left– in this case, these would be C and D. One problem with the above assignment of dropout prices is that the winner—D—does not win the object at the "second-price", which is C's dropout price.

To address this problem, we assign a dropout price to a given bidder equal to the last submitted bid of the *next* bidder who drops out. In the example above: $P_A = 20, P_B = 30, P_C = 35, P_D \geqslant 35$. The reason this problem occurs is that the Milgrom and Weber (1982) (and Wilson, 1998) model assume continuously rising prices and instantaneous dropouts, whereas in the FCC auctions (and probably in most real-life situations) the price ascends by discrete intervals.

B.2. Specification details

Here we describe how the exogenous covariates enter the empirical model. First, m (the mean of the log common value distribution for a given license) should be a function of MTA-level demographic variables which capture the across-license variation in values. We use POP (population) and POP CHANGE (population change).

Second, $\bar{a}_i$ (the publicly known mean of the private value component of V_i) is a function of firm-and-object specific covariates. We only use CEL PRES, an indicator of cellular presence in the surrounding area. More precisely, this regressor is a tally of the total number of the BTA's surrounding [30] a particular MTA in which a given firm has cellular presence. [31]

Finally, (s, t, r_0) (the standard deviations for the noisiness of the signal, the private value component, and the common value component, respectively) are parameterized differently across the two specifications we estimated (Models A and B in Table 3). Both specifications restrict these quantities to be the same over all bidders; Model B, however, allows s to vary over objects as a function of POP and INCOME.

Table 4 presents summary statistics for all the variables we use in the analysis.

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[30] Firms with substantial cellular coverage in a given market were barred from bidding for PCS spectra in that market.

[31] We are grateful to P. Moreton for providing "neighboring county" tables which facilitated the construction of the CEL PRES variable for each (firm-MTA) combination.

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