Description of constrained optimization structural models

Equilibrium/optimality conditions: characterized by equations $h(\theta, \sigma) = 0$:

- θ: structural parameters of model (goal is to estimate these)
- σ: other endogenous parameters, implicitly defined by the equation above: that is, for given θ, $\sigma(\theta)$ satisfies $h(\theta, \sigma) = 0$. (assume this is unique for all θ)

Data: Y denotes observed action/choice, and X are other state variables. Observe $\{Y_i, X_i\}_{i=1}^n$.

Estimation:

A common estimation approach is what Su-Judd call the "nested fixed point" approach: estimate θ by optimizing objective function $Q_n(\theta, \sigma(\theta))$ with respect to θ. For each candidate parameter value θ, this requires solving the equilibrium conditions $h(\theta, \sigma) = 0$ to obtain $\sigma(\theta)$.

In contrast, MPEC approach is constrained optimization problem:

$$\max_{\theta,\sigma} Q_n(\theta, \sigma) \quad st. \quad h(\theta, \sigma) = 0.$$

Impose equilibrium restrictions between parameters of interest θ and nuisance or auxiliary parameters σ via constraints, rather than substituting the constraints directly into the problem.

Example: Rust dynamic engine replacement model

- $Y \in \{0, 1\}$, whether engine is replaced or not.
- X is mileage since last engine replacement. Assume that X is discrete, taking K values $x_{[1]}, \ldots, x_{[K]}$.
- Panel data: $\{y_t^i, x_t^i\}_{i=1,t=1}^{n,T}$
- The nuisance parameters σ are the expected value function $EV(x, y)$. Because y and x are both discrete, this function is finite dimensional.
- MPEC problem is constrained MLE. Log-likelihood function is:

$$Q_n(\theta, EV) = \sum_{i,t} \log \left(\frac{\exp[u(x_t^i, y_t^i; \theta) + \beta EV(x_t^i, y_t^i)]}{\sum_{y\in\{0,1\}} \exp[u(x_t^i, y; \theta) + \beta EV(x_t^i, y)]} \right) + \sum_{i,t} p_3(x_t^i | x_{t-1}^i, y_{t-1}^i; \theta)$$

where $u(\cdots;\theta)$ is the per-period utility function, and $p_3(\cdot|\cdots;\theta)$ is the finite-dimensional transition matrix for mileage, both of which are of known parametric form.

- The equilibrium restriction $h(\theta, \sigma)$ is the Bellman equation which implicitly defines the $EV(\cdots)$ function, given by:

$$0 = EV(x, y) - \sum_{x'} \log \left\{ \sum_{y' \in \{0,1\}} \exp[u(x', y'; \theta) + \beta EV(x', y')] \right\} \cdot p_3(x'|x, y)$$

for all $x \in \left\{x_{[1]}, \ldots, x_{[K]}\right\}$ and $y \in \{0, 1\}$. So there are a total of $2K$ restrictions.