Notes on GHK

1 GHK simulator: get draws from truncated multivariate normal distribution

You want draws from

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \sim TN\left(\vec{\mu}, \Sigma; \vec{a}, \vec{b}\right) \equiv N\left(\vec{\mu}, \Sigma\right) \text{ s.t. } \vec{a} < \vec{x} < \vec{b} \tag{1}$$

where the difficulty is that Σ is not necessarily diagonal (i.e., elements of $\vec{x}$ are correlated).

Let $(u_1, \ldots, u_n)'$ denote an n-vector of independent multivariate standard normal random variables. Let $\Sigma^{1/2}$ denote the (lower-triangular) Cholesky factorization of Σ, with elements

$$\begin{bmatrix} s_{11} & 0 & \cdots & 0 & 0 \\ s_{21} & s_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & s_{ii} & 0 & 0 \\ s_{n1} & s_{n2} & \cdots & s_{nn-1} & s_{nn} \end{bmatrix}.$$

Then we can rewrite (1) as:

$$\vec{x} = \vec{\mu} + \Sigma^{1/2}\vec{u} \sim N\left(\vec{\mu}, \Sigma\right) \; s.t.$$

$$\begin{pmatrix} \frac{a_1-\mu_1}{s_{11}} \\ \frac{a_2-\mu_2-s_{21}u_1}{s_{22}} \\ \vdots \\ \frac{a_n-\mu_n-\sum_{i-1}^{n-1} s_{ni}u_i}{s_{nn}} \end{pmatrix} < \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} < \begin{pmatrix} \frac{b_1-\mu_1}{s_{11}} \\ \frac{b_2-\mu_2-s_{21}u_1}{s_{22}} \\ \vdots \\ \frac{b_n-\mu_n-\sum_{i-1}^{n-1} s_{ni}u_i}{s_{nn}} \end{pmatrix} \tag{2}$$

The above suggests that the answer is to draw $(u_1, \ldots, u_n)$ **recursively**. First draw u_1^s from $N\left(0, 1; \frac{a_1-\mu_1}{s_{11}}, \frac{b_1-\mu_1}{s_{11}}\right)$, then u_2^s from $N\left(0, 1; \frac{a_2-\mu_2-s_{21}u_1^s}{s_{22}}, \frac{b_2-\mu_2-s_{21}u_1^s}{s_{22}}\right)$, and so on.

Finally we can transform $(u_1^s, \ldots, u_n^s)$ to the desired $(x_1^s, \ldots, x_n^s)$ via the transformation

$$\vec{x}^s = \vec{\mu} + \Sigma^{1/2}\vec{u}^s.$$

Remark 1: It is easy to draw an n-dimensional vector $\vec{u}$ of independent truncated standard normal random variables with rectangular truncation conditions: $\vec{c} < \vec{u} < \vec{d}$. You draw a vector of independent uniform variables $\vec{\tilde{u}} \sim \mathcal{U}[\Phi(\vec{c}), \Phi(\vec{d})]$[1] and then transform $u_i = \Phi^{-1}(\tilde{u}_i)$.

Remark 2: Principle of importance sampling:

$$\int_{\mathcal{F}} s f(s) ds = \int_{\mathcal{G}} s \frac{f(s)}{g(s)} g(s) ds.$$

That is, sampling s from $f(s)$ distribution equivalent to sampling $s * w(s)$ from $g(s)$ distribution,. with importance sampling weight $w(s) \equiv \frac{f(s)}{g(s)}$. ($f$ and g should have the same support.)

The GHK simulator is an importance sampler. The importance sampling density is the multivariate normal density $N(\vec{\mu}, \Sigma)$ truncated to the region characterized in Eqs. (2). This is a recursively characterized truncation trgion, in that the range of, say, x_3 depends on the draw of x_1 and x_2. This is different than the multivariate normal density $N(\vec{\mu}, \Sigma)$ truncated to the region $(\vec{a} \leq \vec{x} \leq \vec{b})$.[2]

For the GHK simulator, the importance sampling weight attached to each draw $\vec{x}^s$ is given by:

$$w(\vec{x}^s) \equiv \left[\Phi\left(\frac{b_1 - \mu_1}{s_{11}}\right) - \Phi\left(\frac{a_1 - \mu_1}{s_{11}}\right)\right] \prod_{i=2}^{m} \left[\Phi\left(\frac{b_i - \mu_i - \sum_{j=1}^{i-1} s_{ij} u_j^s}{s_{ii}}\right) - \Phi\left(\frac{a_i - \mu_i - \sum_{j=1}^{i-1} s_{ij} u_j^s}{s_{ii}}\right)\right].$$

Hence, the GHK simulator for $\int_{\vec{a} \leq \vec{x} \leq \vec{b}} \vec{x} f(\vec{x}) d\vec{x}$, where $f(\vec{x})$ denote the $N(\vec{\mu}, \Sigma)$ density, is $\frac{1}{S} \sum_{s=1}^{S} \vec{x}^s w(x^s)$.

Remark 3: w^s (even just for one draw: cf. (Gourieroux and Monfort 1996, pg. 99)) is an unbiased estimator of the truncation probability Prob $\left(\vec{a} < \vec{x} < \vec{b}\right)$. But in general, we can get a more precise estimate by averaging over w^s:

$$T_{\vec{a}, \vec{b}} \equiv \text{Prob}\left(\vec{a} < \vec{x} < \vec{b}\right) \approx \frac{1}{S} \sum_{s} w^s$$

for (say) S simulation draws.

[1] Just draw $\hat{u}$ from $\mathcal{U}[0,1]$ and transform $\tilde{u} = \Phi(c) + (\Phi(d) - \Phi(c))\hat{u}$.

[2] See (Hajivassiliou and Ruud 1994), pg. 2005.

2 Monte Carlo Integration using the GHK Simulator

Clearly, if we can get draws from truncated multivariate distributions using the GHK simulator, we can use these draws to calculate integrals of functions of $\vec{x}$. There are two important cases here, which it is crucial not to confuse.

2.1 Integrating over untruncated distribution $\mathbf{F}(\vec{\mathbf{x}})$, but $\vec{\mathbf{a}} < \vec{\mathbf{x}} < \vec{\mathbf{b}}$ defines region of integration

If we want to calculate

$$\int_{\vec{a}<\vec{x}<\vec{b}} g(\vec{x}) f(\vec{x}) d\vec{x}$$

where f denotes the $N(\vec{\mu}, \Sigma)$ density, we can use the GHK draws to derive a Monte-Carlo estimate:

$$E_{\vec{a}<\vec{x}<\vec{b}} g(\vec{x}) \approx \frac{1}{S} \sum_s g(\vec{x}^s) * w^s.$$

The most widely-cited example of this is the likelihood function for the multinomial probit model (cf. (McFadden 1989)):

Multinomial probit with K choices, and utility from choice k $U_k = X\beta_k + \epsilon_k$. Probability that choice k is chosen is probability that $\nu_i \equiv \epsilon_i - \epsilon_k < X\beta_i - X\beta_k$, for all $i \neq k$. For each parameter vector β, use GHK to draw S (K-1)-dimensional vectors $\vec{\nu}^s$ subject to $\vec{\nu} < (\vec{x\beta})$. Likelihood function is

$$\begin{aligned} \text{Prob}(k) &= \int_{\vec{\nu}} \mathbf{1}\left(\vec{\nu} < (\vec{x\beta})\right) f(\vec{\nu}) d\vec{\nu} \\ &= \int_{\vec{\nu}<(\vec{x\beta})} f(\vec{\nu}) d\vec{\nu} \\ &\approx \frac{1}{S} \sum_s w^s. \end{aligned}$$

2.2 Integrating over truncated (conditional) distribution $\mathbf{F}(\vec{\mathbf{x}} | \vec{\mathbf{a}} < \vec{\mathbf{x}} < \vec{\mathbf{b}})$.

The most common case of this is calculating conditional expectations (note that the multinomial probit choice probability is *not* a conditional probability!)[3].

[3] This is a crucial point. The conditional probability of choice k conditional on choice k is trivially 1!

If we want to calculate

$$E_{\vec{a}<\vec{x}<\vec{b}}g(\vec{x}) = \int_{\vec{a}<\vec{x}<\vec{b}} g(\vec{x})f(\vec{x}|\vec{a}<\vec{x}<\vec{b})d\vec{x} = \frac{\int_{\vec{a}<\vec{x}<\vec{b}} g(\vec{x})f(\vec{x})d\vec{x}}{Prob(\vec{a}<\vec{x}<\vec{b})}.$$

As before, we can use the GHK draws to derive a Monte-Carlo estimate:

$$E_{\vec{a}<\vec{x}<\vec{b}}g(\vec{x}) \approx \frac{1}{T_{\vec{a},\vec{b}}}\frac{1}{S}\sum_s g(\vec{x}^s) * w^s.$$

The crucial difference between this case and the previous one is that we integrate over a conditional distribution by essentially integrating over the unconditional distribution over the restricted support, but then we need to divide through by the probability of the conditioning event (i.e., the truncation probability).

An example of this comes from structural common-value auction models, where:

$$\begin{aligned} v(x,x) \equiv \mathcal{E}\left(v|x_1 = x, \min_{j\neq 1} x_j = x\right) = \\ \underbrace{\int\cdots\int}_{x_k\geq x,\ \forall k=3,\dots,n} \mathcal{E}\left(v|x_1,\dots,x_n\right)\ dF\left(x_3,\dots,x_n|x_1=x,x_2=x,x_k\geq x,k=3,\dots,n;\theta\right) = \\ \frac{1}{T_x}\underbrace{\int\cdots\int}_{x_k\geq x,\ \forall k=3,\dots,n} \mathcal{E}\left(v|x_1,\dots,x_n\right)\ dF\left(x_3,\dots,x_n|x_1=x,x_2=x;\theta\right) \end{aligned} \tag{3}$$

where F here denotes the conditional distribution of the signals $x_3,\dots,x_n$, conditional on $x_1 = x_2 = x$, and T_x denotes the probability that $(x_k \geq x, k = 3,\dots,n|x_1 = x, x_2 = x;\theta)$.

If we assume that $\vec{x} \equiv (x_1,\dots,x_n)'$ are jointly log-normal, it turns out we can use the GHK simulator to get draws of $\tilde{x} \equiv \log \vec{x}$ from a multivariate normal distribution subject to the truncation conditions $\tilde{x}_1 = \tilde{x}, \tilde{x}_2 = \tilde{x}, \tilde{x}_j \geq \tilde{x}, \forall j = 3,\dots,n$. Let $\mathcal{A}(x)$ denote the truncation region, for each given x.

Then we approximate:

$$v(x,x) \approx \frac{1}{T_{\mathcal{A}(x)}}\frac{1}{S}\sum_s \mathcal{E}\left(v|\tilde{x}^s\right) * w^s$$

where $T_{\mathcal{A}(x)}$ is approximated by $\frac{1}{S}\sum_s w^s$.

References

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McFadden, D. (1989): "A Method of Simulated Moments for Estimation of Discrete Response Models without Numerical Integration," *Econometrica*, 57, 995–1026.