Goal of empirical work:

- We observe bids $b_1, \ldots, b_n$, and we want to recover valuations $v_1, \ldots, v_n$.
- Why? Analogously to demand estimation, we can evaluate the "market power" of bidders, as measured by the margin $v - p$.

  Interesting policy question: how fast does margin decrease as n (number of bidders) increases?
- Useful for the optimal design of auctions:
  1. What is auction format which would maximize seller revenue?
  2. What value for reserve price would maximize seller revenue?
- Methodology: identification, nonparametric estimation

1 Laffont-Ossard-Vuong (1995): "Econometrics of First-Price Auctions"

- Structural estimation of 1PA model, in IPV context.
- Example of a parametric approach to estimation.
- Another exercise in simulation estimation

MODEL

- I bidders
- Information structure is IPV: valuations $v^i,\ i = 1, \ldots, I$ are *i.i.d.* from $F(\cdot|z_l, \theta)$ where l indexes auctions, and z_l are characteristics of l-th auctions
- θ is parameter vector of interest, and goal of estimation
- p^0 denotes "reserve price": bid is rejected if $< p^0$.
- Dutch auction: strategically identical to first-price sealed bid auction.

Equilibrium bidding strategy is:

$$b^i = e\left(v^i, I, p^0, F\right) = \begin{cases} v^i - \frac{\int_{p^0}^{v^i} F(x)^{I-1} dx}{F(v^i)^{I-1}} & \text{if } v^i > p^0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Note: (1) $b^i(v^i = p^0) = p^0$; (2) strictly increasing in v^i.

$\blacksquare\blacksquare\blacksquare$

Dataset: only observe *winning bid* b_l^w for each auction l. Because bidders with lower bids never have a chance to bid in Dutch auction.

Given monotonicity, the winning bid $b^w = e\left(v_{(I)}, I, p^0, F\right)$, where $v_{(I)} \equiv \max_i v^i$ (the highest order statistic out of the I valuations).

Furthermore, the CDF of $v_{(I)}$ is $F(\cdot|z_l, \theta)^I$, with corresponding density $I \cdot F^{I-1} f$.

$\blacksquare\blacksquare\blacksquare$

Goal is to estimate θ by (roughly speaking) matching the winning bid in each auction l to its expectation.

Expected winning bid is (for simplicity, drop z_l and θ now)

$$\begin{aligned} E_{v_{(I)}>p^0}(b^w) &= \int_{p^0}^{\infty} e\left(v_{(I)}, I, p^0, F\right) I \cdot F(v|\theta)^{I-1} f(v|\theta) dv \\ &= I \int_{p^0}^{\infty} \left(v - \frac{\int_{p^0}^{v} F(x)^{I-1} dx}{F(v)^{I-1}}\right) F(v|\theta)^{I-1} f(v|\theta) dv \\ &= I \int_{p^0}^{\infty} \left(v \cdot F(v)^{I-1} - \int_{p^0}^{\infty} F(x)^{I-1} dx\right) f(v) dv. \qquad (*). \end{aligned}$$

$\blacksquare\blacksquare\blacksquare$

If we were to estimate by simulated nonlinear least squares, we would proceed by finding θ to minimize the sum-of-squares between the observed winning bids and the predicted winning bid, given by expression (*) above. Since (*) involves complicated integrals, we would simulate (*), for each parameter vector θ.

How would this be done:

- Draw valuations v^s, $s = 1, \ldots, S$ i.i.d. according to $f(v|\theta)$. This can be done by drawint $u_1, \ldots, u_S$ i.i.d. from the $U[0,1]$ distribution, then transform each draw:

  $$v_s = F^{-1}(u_s|\theta).$$

- For each simulated valuation v_s, compute integrand $\mathcal{V}_s = v_s F(v_s|\theta)^{I-1} - \int_{p^0}^{v_s} F(x|\theta)^{I-1} dx$. (Second term can also be simulated, but one-dimensional integral is that very hard to compute.)

- Approximate the expected winning bid as $\frac{1}{S}\sum_s \mathcal{V}_s$.

However, the authors do not do this— they propose a more elegant solution. In particular, they simplify the simulation procedure for the expected winning bid by appealing to the **Revenue-Equivalence Theorem**: an important result for auctions where bidders' signals are independent, and the model is symmetric. (See Myerson (1981); this statement is due to Klemperer (1999).)

Theorem 1 *(Revenue Equivalence) Assume each of N risk-neutral bidders has a privately-known signal X independently drawn from a common distribution F that is strictly increasing and atomless on its support $[\underline{X}, \bar{X}]$. Any auction mechanism which is (i)* efficient *in awarding the object to the bidder with the highest signal with probability one; and (ii) leaves any bidder with the lowest signal $\underline{X}$ with zero surplus yields the same expected revenue for the seller, and results in a bidder with signal x making the same expected payment.*

From a mechanism design point of view, auctions are complicated because they are multiple-agent problems, in which a given agent's payoff can depend on the reports of all the agents. However, in the independent signal case, there is no gain (in terms of stronger incentives) in making any given agent's payoff depend on her rivals' reports, so that a symmetric auction with independent signal essentially boils down to independent contracts offered to each of the agents individually.

Furthermore, in any efficient auction, the probability that a given agent with a signal x wins is the same (and, in fact, equals $F(x)^{N-1}$). This implies that each bidder's expected surplus function (as a function of his signal) is the same, and therefore that the expected payment schedule is the same.

$\blacksquare\blacksquare\blacksquare$

By RET:

- expected revenue in 1PA same as expected revenue in 2PA

- expected revenue in 2PA is $Ev^{(I-1)}$

- with reserve price, expected revenue in 2PA is $E\max(v^{(I-1)}, p^0)$. (Note: with IPV structure, reserve price r screens out same subset of valuations $v \leq r$ in both 1PA and 2PA.)

■■■

Hence, we have that

$$Eb^*(v_{(I)}) = E\left[\max\left(v_{(I-1)}, p^0\right)\right]$$

which is insanely easy to simulate:

For each parameter vector θ, and each auction l

- For each simulation draw $s = 1, \ldots, S$:
  - Draw $v_1^s, \ldots, v_{I_l}^s$: vector of simulated valuations for auction l (which had I_l participants)
  - Sort the draws in ascending order: $v_{1:I_l}^s < \cdots < v_{I_l:I_l}^s$
  - Set $b_l^{w,s} = v_{I-1_l:I_l}$ (ie. the second-highest valuation)
  - If $b_l^{w,s} < p_l^0$, set $b_l^{w,s} = p_l^0$. (ie. $b_l^{w,s} = \max\left(v_{I-1_l:I_l}^s, p_l^0\right)$)
- Approximate $E\left(b_l^w; \theta\right) = \frac{1}{S}\sum_s b_l^{w,s}$.

Estimate θ by simulated nonlinear least squares:

$$\min_\theta \frac{1}{L}\sum_{l=1}^{L}\left(b_l^w - E\left(b_l^w; \theta\right)\right)^2.$$

Results.

■■■

Remarks:

- Problem: bias when number of simulation draws S is fixed (as number of auctions $L \to \infty$). Propose bias correction estimator, which is consistent and asymptotic normal under these conditions.
- This clever methodology is useful for independent value models: works for all cases where revenue equivalence theorem holds.

- Does not work for affiliated value models (including common value models)

■■■

2 Guerre-Perrigne-Vuong (2000): Nonparametric Identification and Estimation in IPV First-price Auction Model

The recent emphasis in the empirical literature is on *nonparametric* identification and estimation of auction models. Motivation is to estimate bidders' unobserved valuations, while avoiding parametric assumption (as in the LOV paper).

- Start with first-order condition:

$$\begin{aligned} b'(x) &= (x - b(x)) \cdot (n-1)\frac{F(x)^{n-2}f(x)}{F(x)^{n-1}} \\ &= (x - b(x)) \cdot (n-1)\frac{f(x)}{F(x)}. \end{aligned} \tag{2}$$

- Now, note that because equilibrium bidding function $b(x)$ is just a monotone increasing function of the valuation x, the change of variables formulas yield that (take $b_i \equiv b(x_i)$)

  –
  $$G(b_i) = F(x_i)$$

  –
  $$g(b_i) = f(x_i) \cdot 1/b'(x_i)$$

  .

  Hence, substituting the above into Eq. (2):

$$\begin{aligned} &\frac{1}{g(b_i)} = (n-1)\frac{x_i - b_i}{G(b_i)} \\ &\Leftrightarrow x_i = b_i + \frac{G(b_i)}{(n-1)g(b_i)}. \end{aligned} \tag{3}$$

  Everything on the RHS of the preceding equation is observed: the equilibrium bid CDF G and density g can be estimated directly from the data *nonparametrically*. Assuming a dataset consisting of T n-bidder auctions:

$$\hat{g}(b) \approx \frac{1}{T \cdot n} \sum_{t=1}^{T} \sum_{i=1}^{n} \frac{1}{h} \mathcal{K}\left(\frac{b - b_{it}}{h}\right)$$
$$\hat{G}(b) \approx \frac{1}{T \cdot n} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbf{1}(b_{it} \leq b). \tag{4}$$

The first is a *kernel density estimate* of bid density. The second is the *empirical distribution function (EDF)*.

- In the above, $\mathcal{K}$ is a "kernel function". A kernel function is a function satisfying the following conditions:
  1. It is a probability density function, ie: $\int_{-\infty}^{+\infty} \mathcal{K}(d)du = 1$, and $\mathcal{K}(u) \geq 0$ for all u.
  2. It is symmetric around zero: $\mathcal{K}(u) = \mathcal{K}(-u)$.
  3. h is bandwidth: describe below
  4. Examples:
     (a) $\mathcal{K}(u) = \phi(u)$ (standard normal density function);
     (b) $\mathcal{K}(u) = \frac{1}{2}\mathbf{1}(|u| \leq 1)$ (uniform kernel);
     (c) $\mathcal{K}(u) = \frac{3}{4}(1 - u^2)\mathbf{1}(|u| \leq 1)$ (Epanechnikov kernel)

- To get some intuition for the kernel estimate of $\hat{g}(b)$, consider the histogram

$$h(b) = \frac{1}{Tn} \sum_{t} \sum_{i} \mathbf{1}(b_{it} \in [b - \epsilon, b + \epsilon])$$

  for some small $\epsilon > 0$. The histogram at b, $h(b)$ is the frequency with which the observed bids land within an ϵ-neighborhood of b.

- In comparison, the kernel estimate of $\hat{g}(b)$ replaces $\mathbf{1}(b_{it} \in [b-\epsilon, b+\epsilon])$ with $\frac{1}{h}\mathcal{K}\left(\frac{b-b_{it}}{h}\right)$. This is:
  - always ≥ 0
  - takes large values for b_{it} close to b; small values (or zero) for b_{it} far from b
  - takes values in $\mathbb{R}+$ (can be much larger than 1)
  - h is bandwidth, which blows up $\frac{1}{h}\mathcal{K}\left(\frac{b-b_{it}}{h}\right)$: when it is smaller, then this quantity becomes larger.

    Think of h as measuring the "neighborhood size" (like ϵ in the histogram). When $T \to \infty$, then we can make h smaller and smaller.

    Bias/variance tradeoff.

  - Roughly speaking, then, $\hat{g}(b)$ is a "smoothed" histogram,

- For $\hat{G}(b)$, recall definition of the CDF:

$$G(\tilde{b}) = Pr(b \leq \tilde{b}).$$

  The EDF measures these probabilities by the (within-sample) frequency of the events.

- Hence, the IPV first-price auction model is *nonparametrically identified.* For each observed bid b_i, the corresponding valuation $x_i = b^{-1}(b_i)$ can be recovered as:

$$\hat{x}_i = b_i + \frac{\hat{G}(b_i)}{(n-1)\hat{g}(b_i)}. \tag{5}$$

Hence, GPV recommend a two-step approach to estimating the valuation distribution $f(x)$:

1. In first step, estimate $G(b)$ and $g(b)$ nonparametrically, using Eqs. (4).

2. In second step, estimate $f(x)$ by using kernel density estimator of recovered valuations:

$$\hat{f}(x) \approx \frac{1}{T \cdot n} \sum_{t=1}^{T} \sum_{i=1}^{n} \frac{1}{h} \mathcal{K}\left(\frac{x - \hat{x}_{it}}{h}\right). \tag{6}$$

"Reduced-form" first-order conditions. The main estimating equation (2) was derived above formally using change of variables. More generally, consider a bidder choosing a bid to optimize a "reduced-form" profit expression:

$$\max_{b_i} (b_i - v_i) Pr(b_i \text{ wins}) \tag{7}$$

where $Pr(b_i \text{ wins}) = P(b_j \leq b_i, j \neq i) = G(b_i)^{n-1}$ is the probability that bidder i wins with a bid of b_i. Obviously, when we take the first-order condition for this reduced-form model, we get

$$G(b_i)^{n-1} + (b_i - v_i)(n-1)G(b_i)^{n-2} g(b_i) = 0 \tag{8}$$

which simplifies to Eq. (2). Such a "reduced-form" approach to generating empirical FOC's has been used in the structural estimation of auction models much more complicated than the single-object first-price auction considered here, including multi-unit auctions (Hortaçsu (2001)), combinatorial auctions (Cantillon and Pesendorfer (2006)), sponsored search auctions (Athey and Nekipelov (forthcoming)), among others.

Laffont (1997) offers a discussion of this idea, and some discussion of its applicability. He mentions that *independence* of the private information amongst bidders appears necessary for this to work. Furthermore, given that all bids are pooled together in estimating the equilibrium bid distributions G and g, an assumption that only one equilibrium is played in the data is required.

Partial observation of bids. Note that identification continues to hold, even when only the highest-bid in each auction is observed. Specifically, if only $b_{n:n}$ is observed, we can estimate $G_{n:n}$, the CDF of the maximum bid, from the data. Note that the relationship between the CDF of the maximum bid and the marginal CDF of an equilibrium bid is

$$G_{n:n}(b) = G(b)^n$$

implying that $G(b)$ can be recovered from knowledge of $G_{n:n}(b)$. Once $G(b)$ is recovered, the corresponding density $g(b)$ can also be recovered, and we could solve Eq. (5) for every b to obtain the inverse bid function.

Athey and Haile (2002) contains a comprehensive collection of nonparametric identification results for a variety of auction models (first-price, second-price) under a variety of assumption on the information structure (symmetry, asymmetry). One focus in their paper is on situations when only a subset of the bids submitted in an auction are available to a researcher.

3 Affiliated values models

Can this methodology be extended to affiliated values models (including common value models)?

However, Laffont and Vuong (1996) nonidentification result: from observation of bids in n-bidder auctions, the affiliated private value model (ie. a PV model where valuations are dependent across bidders) is indistinguishable from a CV model.

- Intuitively, all you identify from observed bid data is joint density of $b_1, \ldots, b_n$. In particular, can recover the correlation structure amongst the bids. But correlation of bids in an auction could be due to both affiliated PV, or to CV.

3.1 Affiliated private value models

Li, Perrigne, and Vuong (2002) proceed to consider nonparametric identification and estimation of the affiliated private values model. In this model, valuations $x_i, \ldots, x_n$ are drawn from some joint distribution (and there can be arbitrary correlation amongst them).

As before, start with the first-order condition for equilibrium bid in affiliated private values case:

$$b'(x) = (x - b(x)) \cdot \frac{f_{y_i|x_i}(x|x)}{F_{y_i|x_i}(x|x)}; \quad y_i \equiv \max_{j \neq i} x_i. \tag{9}$$

where $y_i \equiv \max_{j \neq i} x_i$ (highest among rivals' signals) and $b(\cdot)$ denotes the equilibrium bidding strategy.

Procedure similar to GPV can be used here to recover, for each bid b_i, the corresponding valuation $x_i = b^{-1}(b_i)$. Let b_i^* denote the maximum among bidder i's rivals bids: $b_i^* = \max_{j \neq i} b_j$. Then there is a monotonic transformation $b_i^* = b(y_i)$ so that, as before, we exploit the following change of variable formulas:

- $$G_{b^*|b}(b|b) = F_{y|x}(x|x)$$
- $$g_{b^*|b}(b|b) = f_{y|x}(x|x) \cdot 1/b'(x)$$

Note that the conditioning event $\{X = x\}$ (on right-hand side) is equivalent to $\{B = b\}$ (on left-hand side). Then we get

$$x = b + \frac{G_{b^*|b}(b|b)}{g_{b^*|b}(b|b)}. \tag{10}$$

Everything on the RHS is observed or estimable from the data.

[*] Note that the derivation of the "empirical" first-order condition (10) relies crucially on the monotonicity of the equilibrium bidding strategies, which enables the change in variables between b and x (and between b^* and y). Eq. (10) cannot be derived as the first-order condition of a "reduced-form" profit expression, as in Eqs. (7) and (8) above, due to the dependence between bidders' private information.

To prepare what follows, we introduce n subscript (so we index distributions according to the number of bidders in the auction).

Li, Perrigne, and Vuong (2000) suggest nonparametric estimates of the form

$$\begin{aligned}\hat{G}_n(b;b) &= \frac{1}{T_n \times h \times n}\sum_{t=1}^{T}\sum_{i=1}^{n} K\left(\frac{b-b_{it}}{h}\right)\mathbf{1}\left(b^*_{it} < b, n_t = n\right)\\ \hat{g}_n(b;b) &= \frac{1}{T_n \times h^2 \times n}\sum_{t=1}^{T}\sum_{i=1}^{n}\mathbf{1}(n_t = n) K\left(\frac{b-b_{it}}{h}\right) K\left(\frac{b-b^*_{it}}{h}\right).\end{aligned} \tag{11}$$

Here h and h are bandwidths and $K(\cdot)$ is a kernel. $\hat{G}_n(b;b)$ and $\hat{g}_n(b;b)$ are nonparametric estimates of

$$G_n(b;b) \equiv G_n(b|b) g_n(b) = \frac{\partial}{\partial b}\Pr(B^*_{it} \leq m, B_{it} \leq b)|_{m=b}$$

and

$$g_n(b;b) \equiv g_n(b|b) g_n(b) = \frac{\partial^2}{\partial m \partial b}\Pr(B^*_{it} \leq m, B_{it} \leq b)|_{m=b}$$

respectively, where $g_n(\cdot)$ is the marginal density of bids in equilibrium. Because

$$\frac{G_n(b;b)}{g_n(b;b)} = \frac{G_n(b|b)}{g_n(b|b)} \tag{12}$$

$\frac{\hat{G}_n(b;b)}{\hat{g}_n(b;b)}$ is a consistent estimator of $\frac{G_n(b|b)}{g_n(b|b)}$. Hence, by evaluating $\hat{G}_n(\cdot,\cdot)$ and $\hat{g}_n(\cdot,\cdot)$ at each observed bid, we can construct a pseudo-sample of consistent estimates of the realizations of each $x_{it} = b^{-1}(b_{it})$ using Eq. (9):

$$\hat{x}_{it} = \frac{\hat{G}_n(b_{it};b_{it})}{\hat{g}_n(b_{it};b_{it})} + b_{it}. \tag{13}$$

Subsequently, joint distribution of $x_1,\ldots,x_n$ can be recovered as sample joint distribution of $\hat{x}_1,\ldots,\hat{x}_n$.

3.2 Common value models: testing between CV and PV

Laffont-Vuong did not consider variation in n, the number of bidders.

In Haile, Hong, and Shum (2003), we explore how variation in n allows us to test for existence of CV.

Introduce notation:

$$v(x_i, x_i, n) = E[V_i | X_i = x_i, \max_{j \neq i} X_j = x_i, n].$$

This denotes the "value conditional on winning" (see theory notes, part 1). Recall the winner's curse: it implies that $v(x,x,n)$ is invariant to n for all x in a PV model but strictly decreasing in n for all x in a CV model (see theory notes, part 1).

Consider the first-order condition in the common value case:

$$b'(x,n) = (v(x,x,n) - b(x,n)) \cdot \frac{f_{y_i|x_i,n}(x|x)}{F_{y_i|x_i,n}(x|x)}; \quad y_i \equiv \max_{j \neq i} x_i.$$

Hence, the Li, Perrigne, and Vuong (2002) procedure from the previous section can be used to recover the "pseudovalue" $v(x_i,x_i,n)$ corresponding to each observed bid b_i. Note that we cannot recover $x_i = b^{-1}(b_i)$ itself from the first-order condition, but can recover $v(x_i,x_i,n)$. (This insight was also articulated in Hendricks, Pinkse, and Porter (2003).)

In Haile, Hong, and Shum (2003), we use this intuition to develop a test for CV:

$$H_0 \text{ (PV)}: \; E\left[v(X,X;\underline{n})\right] = E\left[v(X,X;\underline{n}+1)\right] = \cdots = E\left[v(X,X;\bar{n})\right]$$
$$H_1 \text{ (CV)}: \; E\left[v(X,X;\underline{n})\right] > E\left[v(X,X;\underline{n}+1)\right] > \cdots > E\left[v(X,X;\bar{n})\right]$$

Problem: bias at boundaries in kernel estimation of pseudo-values. The bid density $g(b,b)$ is estimated inaccurately for bids close to the boundary of the empirical support of bids.

Solution: use *quantile-trimmed means*: $\mu_{n,\tau} = E[v(X,X;n)\mathbf{1}\{x_\tau < X < x_{1-\tau}\}]$

above $\Rightarrow$

$$H_0 \text{ (PV)} \;:\; \mu_{\underline{n},\tau} = \mu_{\underline{n}+1,\tau} = \cdots = \mu_{\overline{n},\tau}$$
$$H_1 \text{ (CV)} \;:\; \mu_{\underline{n},\tau} > \mu_{\underline{n}+1,\tau} > \cdots > \mu_{\overline{n},\tau}.$$

Theorem 3 Let $\hat{\mu}_{n,\tau} = \frac{1}{n \times T_n} \sum_{t=1}^{T_n} \sum_{i=1}^{n} \hat{v}_{it}\, \mathbf{1}\{b_{\tau,n} \le b_{it} \le b_{1-\tau,n}\}$ and assume [...conditions for kernel estimation...]. Then
(i) $\hat{\mu}_{n,\tau} \xrightarrow{p} E[v(X,X,n)\;\mathbf{1}\{x_\tau < X < x_{1-\tau}\}]$;
(ii)$\sqrt{T_n h}\,(\hat{\mu}_{n,\tau} - \mu_{n,\tau}) \xrightarrow{d} N(0,\omega_n)$, where
$\omega_n = \left[\int \left(\int K(v)K(u+v)dv\right)^2 du\right] \left[\frac{1}{n}\int_{F_b^{-1}(\tau)}^{F_b^{-1}(1-\tau)} \frac{G_n(b;b)^2}{g_n(b;b)^3} g_n(b)^2 db\right].$

Test statistic Now use standard multivariate one-sided LR test (Bartholomew, 1959) for normally distributed parameters $\hat{\mu}_{n,\tau}$

- $a_n = \frac{T_n h}{\omega_n}$ (inverse variance weights)
- $\overline{\mu} = \frac{\sum_{n=\underline{n}}^{\bar{n}} a_n \hat{\mu}_{n,\tau}}{\sum_{n=\underline{n}}^{\bar{n}} a_n}$ (MLE under null)
- $\mu^*_{\underline{n}}, \ldots, \mu^*_{\bar{n}}$ solves

$$\min_{\mu_{\underline{n}},\ldots,\mu_{\bar{n}}} \sum_{n=\underline{n}}^{\bar{n}} a_n \left(\hat{\mu}_{n,\tau} - \mu_n\right)^2 \quad s.t. \quad \mu_{\underline{n}} \geq \mu_{\underline{n}+1} \geq \cdots \geq \mu_{\bar{n}}. \tag{13}$$

- $\bar{\chi}^2 = \sum_{n=\underline{n}}^{\bar{n}} a_n \left(\mu^*_{n,\tau} - \bar{\mu}\right)^2$
  - distributed as mixture of χ^2_k rv's, $k = 0, 1, \ldots, \overline{n} - \underline{n}$
  - mixing weights: $\Pr_{H_0}$ {soln to (13) has exactly k slack constraints}
    (obtain by simulation)
- estimate ω_n using asymptotic formula or with bootstrap

3.3 Endogenous participation

The validity of this test relies crucially on the assumption that variation in n, the number of bidders, across auction is exogenous. Next, we consider how this can be relaxed.

Idea: bidder participation determined by unobservable (to us) factors, denoted W, which are also correlated with bidder valuations.

Problems:

1. valuations varying with N ($\Longrightarrow$ *second-stage test may be invalid*). Extreme case: if N is decreasing in W, then $\mu_{N=2} > \mu_{N=3}$, even under PV. "Usual" problem that endogeneity can confound results.
2. to estimate pseudo-values using the FOC, we must condition on all info (both N and W, e.g.) bidders do ($\Longrightarrow$ *first stage estimation invalid too*!). We must estimate equilibrium bid distributions g and G conditional on both N and W.

IV approach: assume there is an instrument Z which satisfies

Assumption 1 $N = \phi(Z, W)$, with ϕ nonconstant in Z and strictly increasing in W. (Implies W uniquely determined given N and Z, and discrete.)

This assumption is strong, but we will see why we need it.

Assumption 2 Z is independent of $(U_1, \ldots, U_n, X_1, \ldots, X_n, W)$.

Assumption 3 The support of $N|Z$ consists of a set of contiguous integers.

With these assumptions, it turns out there is no loss in generality from taking $\phi(\cdots)$ to be additive, and equal to:

$$\phi(Z_t, W_t) = \texttt{int}[E(N|Z_t)] + W_t.$$

Hence, the unobserved factor in auction t, is essentially "observed" after we run a first-stage nonparametric regression of N_t on Z_t:

$$\hat{W}_t = N_i - \texttt{int}[\widehat{E(N|}Z_t)].$$

This suggests that we can adapt the test in the following way:

1. Estimate bid distributions $G(b, b|n, w)$ and $g(b, b|n, w)$ conditional on both n and w.

2. For bid b_{it} in auction t, we can recover the corresponding pseudovalue as:

$$\hat{v}(x_i, x_i|n_t, w_t) = b_{it} + \frac{\hat{G}(b_{it}, b_{it}|n_t, w_t)}{\hat{g}(b_{it}, b_{it}|n_t, w_t)}.$$

3. Now the winner's curse implies that under PV, the conditional expectation $E_x v(x, x|n, w)$ conditional on (n, w) is invariant in n, for all w. However, under CV, it is decreasing in n, for all w.

4 Haile-Tamer's "incomplete" model of English auctions

Haile and Tamer (2003)

Consider an ascending auction where the bidders have independent private valuations. The number of bidders n varies across the auctions in the dataset. The two behavioral assumptions are:

A1: Bidders do not bid more than they are willing to pay

A2: Bidder do not allow an opponent to win at a price they are willing to beat

From these two assumptions, they derive bounds on the CDF of bidders' valuations $F(v)$, as a function of the CDF of bids $G(b)$ which is observed from the data.

Upper bound for $F(v)$ Obviously, from A1, $b_i \leq v_i$ implies $F(v_i) \leq G(v_i)$. They derive a tighter bound.

From Lemma 1: A1 implies that $b_{i:n} \leq v_{i:n} \Rightarrow F_{i:n}(v) \leq G_{i:n}(v)$ where $i:n$ denotes the i-th highest order statistic out of n random draws.

This yields Theorem 1:

$$\begin{aligned} F(v) \leq & F_U(v) \equiv \min_{n,i} \underbrace{\phi(G_{i:n}(v), i, n)}_{\equiv \phi \in [0,1]:\ G_{i:n}(v) = \frac{n!}{(n-i)!(i-1)!} \int_0^\phi s^{i-1}(1-s)^{n-i} ds.} \end{aligned} \tag{14}$$

In the above, the function $\phi(H_{i:n}(v), i, n)$ is an "order statistic inversion" function which returns, for a given distribution of the $i:n$ order statistic $H_{i:n}(v)$, the corresponding "parent" CDF $H(v)$. $\phi(\cdot, i, n)$ is increasing in the first argument, implying that, for any i, n:

$$F(v) = \phi(F_{i:n}(v), i, n) \leq \phi(G_{i:n}(v), i, n)$$

which underlies Theorem 1. Eq. (14) is the main inequality for the upper bounds.

Lower bound Similar calculations lead to the lower bound. From A2, we know that

$$v_i \leq \begin{cases} \bar{v} & \text{if } b_i = b_{n:n} \\ b_{n:n} + \Delta & \text{if } b_i < b_{n:n}. \end{cases}$$

In the above, $\bar{v}$ denotes the upper bound of the valuation distribution; $b_{n:n}$ denotes the maximum bid observed in an auction with n bidders, and $\Delta \geq 0$ is a known bid increment. Because Δ is known, then also $G^{\Delta}_{n:n}(\cdot)$, the CDF of $b_{n:n} + \Delta$, is also known.

Lemma 3: $v_{n-1:n} \leq b_{n:n} + \Delta \ \Rightarrow\ F_{n-1:n}(v) \geq G^{\Delta}_{n:n}(v)$

Using the same $\phi(\cdots)$ function from before, this implies that

$$F(v) = \phi(F_{n-1:n}(v), n-1, n) \geq \phi(G^{\Delta}_{n:n}(v), n-1, n).$$

Hence we get Theorem 2:

$$F(v) \geq F_L(v) \equiv \max_n \phi(G^{\Delta}_{n:n}(v), n-1, n) \tag{15}$$

Eq. (15) is the main inequality for the lower bound.

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