

# 1 Auction design: some well-known results

How to design an auction to raise revenue for the seller?

## 1.1 Revenue Equivalence

A surprising result is that: when bidders have independent valuations, the revenue from a first-price and second-price auction is equal on average. This is the “revenue-equivalence principle”.

**Sidetrack: mechanism design** Consider the auction as an example of a *mechanism*, which is a general scheme whereby agents with private types  $v$  give a report  $\tilde{v}$  of their type (where possibly  $\tilde{v} \neq v$ ). The auction mechanism consists of two functions:

- $T(\tilde{v})$ : payment agent must make if his report is  $\tilde{v}$
- $P(\tilde{v})$ : probability of winning the object if his report is  $\tilde{v}$ .

The *surplus* of an agent with type  $v$ , who reports that his type is  $\tilde{v}$ , is “quasilinear”:

$$S(\tilde{v}; v) \equiv vP(\tilde{v}) - T(\tilde{v}).$$

The seller’s mechanism design problem is to choose the functions  $P(\cdot)$  and  $T(\cdot)$  to maximize his expected profits. By the “revelation principle” (for which the 2007 Nobel Laureate Roger Myerson is most well known for), we can restrict attention to mechanisms which satisfy an **incentive compatibility condition**, which requires that agents find it in their best interest to report the truth.

The revenue equivalence result falls out as an implication of quasilinearity and incentive compatibility. Assuming differentiability, incentive compatibility implies the following first-order condition:

$$\left. \frac{\partial S(\tilde{v}; v)}{\partial \tilde{v}} \right|_{\tilde{v}=v} = vP'(v) - T'(v) = 0. \quad (1)$$

Hence, under an IC mechanism, the surplus of an agent with type  $v$  is just equal to

$$\bar{S}(v) \equiv S(v; v) = vP(v) - T(v).$$

Given quasilinearity, instead of viewing the seller as setting  $P$  and  $T$  functions, can view him as setting  $P$  and  $\bar{S}$  functions, and  $T(v) = vP(v) - \bar{S}(v)$ .

By Eq. (1), we see that IC places restrictions on the  $\bar{S}$  function that seller can choose. Namely, the derivative of the  $\bar{S}$  function must obey:

$$\begin{aligned} \bar{S}'(v) &= P(v) + vP'(v) - T'(v) \\ &= P(v) + 0 \end{aligned} \tag{2}$$

where the second line follows from Eq. (1). This operation is the “envelope condition”. Hence, integrating up, the  $\bar{S}$  function obeys

$$\bar{S}(v) = \bar{S}(\underline{v}) + \int_{\underline{v}}^v P(v)dv \tag{3}$$

where  $\underline{v}$  denotes the lowest type. From the above, we see that any IC-auction mechanism with quasilinear preferences is completely characterized by (i) the quantity  $\bar{S}(\underline{v})$ , the surplus gained by the lowest type; and (ii)  $P(\cdot)$ , the probability of winning.

**Application to auctions** Now note that, under independence of types, the monotonic equilibrium in the standard auction forms (including first-price and second-price auctions) have the same  $P(v)$  function. This is because the object is always awarded to the bidder with the highest type, so that for any  $v$ ,  $P(v)$  is just the probability that a bidder with type  $v$  has the highest type; that is, with  $N$  bidders, and each type  $v$  distributed i.i.d. from distribution  $F(v)$ , we have  $P(v) = [F(v)]^{N-1}$ . Furthermore, the standard auction forms also have the condition that  $b(\underline{v}) = \underline{v}$ , so that  $\bar{S}(\underline{v}) = 0$ .

Hence, by quasilinearity, the standard auction forms also have the same  $T$  function, given by  $T(v) = S(v) - vP(v)$ . Since the  $T$  function is the revenue function for the seller, for each type, we conclude that the standard auction forms give the seller the same expected revenue.

## 1.2 Optimal auctions

Now consider the seller's optimal choice of mechanism. Let  $\vec{v} \equiv (v_1, \dots, v_N)$  denote the vector of all the bidders' valuations. As before, assume a "direct revelation mechanism". The seller wants to choose the functions  $Q_i(\vec{v}), \dots, Q_N(\vec{v})$  and  $M_1(\vec{v}), \dots, M_N(\vec{v})$  to maximize his revenues, where  $Q_i(\vec{v})$  denotes bidder  $i$ 's expected probability of winning for bidder  $i$ , as a function of all the reported valuations  $\vec{v}$  (so that  $\sum_i Q_i(\vec{v}) = 1$ ), and  $M_i(\vec{v})$  likewise denotes the expected payment, as a function of all the valuations. Seller wants to maximize his expected revenue across all the bidders, which is:

$$\sum_i \int \dots \int M_i(\vec{v}) f(v_1) \dots f(v_N) dv_1 \dots dv_N.$$

As shorthand, let  $\vec{v}_{-i} \equiv (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_N)$  and  $f(\vec{v}_{-i}) = \prod_{j \neq i} f(v_j)$ . The relationship between the  $Q_i$  and  $M_i$  functions and the  $P$  and  $T$  functions from the previous section are:

$$P(v_i) = \int Q_i(\vec{v}) f(\vec{v}_{-i}) dv_{-i}$$

$$T(v_i) = \int M_i(\vec{v}) f(\vec{v}_{-i}) dv_{-i}.$$

Given these equivalences, seller's expected revenue can be expressed as:

$$\begin{aligned}
& \sum_i \int \dots \int M_i(\vec{v}) f(v_1) \dots f(v_N) dv_1 \dots dv_N \\
&= \sum_i \int T(v_i) f(v_i) dv_i \\
&= \int_{\underline{v}}^{\bar{v}} [v_i P(v_i) - \bar{S}(v_i)] f(v_i) dv_i \\
&= \int_{\underline{v}}^{\bar{v}} v_i P(v_i) f(v_i) dv_i - \int_{\underline{v}}^{\bar{v}} f(v_i) \left[ \int_{\underline{v}}^v P(x) dx \right] dv_i - \int_{\underline{v}}^{\bar{v}} \bar{S}(v_i) f(v_i) dv_i \quad (4) \\
&= \int_{\underline{v}}^{\bar{v}} P(v_i) f(v_i) \left[ v_i - \frac{1 - F(v_i)}{f(v_i)} \right] dv_i - \bar{S}(\underline{v}) \\
&\equiv \int_{\underline{v}}^{\bar{v}} P(v_i) f(v_i) v_i^* dv_i \\
&= \int \dots \int Q_i(\vec{v}) f(\vec{v}) v_i^* d\vec{v}.
\end{aligned}$$

The third equality follows by substituting in the IC constraint (3). The fourth equality uses integration by parts to simplify the second term, with the substitution  $u = \int_{\underline{v}}^v P(x) dx$  and  $dv = f(v_i) dv_i$ . In the above,  $v_i^*$  is called the “virtual type” corresponding to a type  $v_i$ . The interpretation is that the revenue that the seller makes from a bidder with type  $v_i$  is  $v_i^* \leq v_i$ , so that the deviation  $\frac{1-F(v_i)}{f(v_i)}$  is the bidder's “information rent” which arises because only the bidder knows his type.

Under the additional assumption:

$$(*) \quad \frac{1 - F(v)}{f(v)} \text{ is decreasing in } v$$

then  $v^*$  is increasing in  $v$ .<sup>1</sup> In this case, it is clear from inspecting the above that, for all sets of valuations  $(v_1, v_2, \dots, v_N)$ , because  $v^*$  is increasing in  $v$ , the seller maximizes the above by awarding the item to the bidder with the highest valuation, ie. setting

$$Q_i(\vec{v}) = \begin{cases} 1 & v_i > v_j, j \neq i \\ 0 & \text{otherwise.} \end{cases}$$

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<sup>1</sup>In statistics, the quantity  $f(v)/[1 - F(v)]$  is called the hazard function, so that assumption (\*) is equivalently that the hazard function of  $v$  is increasing.

Correspondingly, he sets  $P(v_i) = Pr(v_i \text{ is highest out of } N \text{ signals}) = F(v_i)^{N-1}$ . This is just like in the standard auction forms.

However, because of the information rent, he will not be willing to sell to all bidders, but only those with non-negative  $v^*$ . Thus he will set a *reserve price*, such that all bidders with  $v^* \leq 0$  are excluded. Let  $r^*$  denote this optimal reserve price. Recall that, with a reserve price  $r$  and IPV, the lower bound on bids is given by the condition  $b(r) = r$ , that is, the reserve price  $r$  screens out bidders with valuations  $v \leq r$ . Hence, the optimal reserve price  $r^*$  satisfies

$$r^* - \frac{1 - F(r^*)}{f(r^*)} = 0.$$

$r^*$  depends on the distribution of types  $f(v)$ . For example, if  $v \sim U[0, 1]$ , then

$$0 = r^* - \frac{1 - F(r^*)}{f(r^*)} = r^* - \frac{1 - r^*}{1} = 2r^* - 1 \Rightarrow r^* = \frac{1}{2}.$$

The crucial assumption underlying the revenue equivalence property is independence of the bidders' valuations. In the affiliated case, the expected revenue in the second-price auction is usually larger than the revenue in the first-price auction: this is called the *linkage principle*.

### 1.3 Complete surplus extraction

When bidders have independent valuations, the revenue equivalence property implies that a bidder with valuation  $x$ , only needs to pay  $x^* = x - \frac{1-F(x)}{f(x)}$ , implying that he gets an “information rent” equal to the inverse hazard rate,  $\frac{1-F(x)}{f(x)}$ .<sup>2</sup>

When types are no longer independent (as in the general affiliated case of Milgrom and Weber (1982)), revenue equivalence of the standard auction forms no longer holds. However, a remarkable series of papers (Cremer and McLean (1985), Cremer and McLean (1988), McAfee and Reny (1992)) argued that the seller can potentially exploit dependence among bidder types to extract all the surplus in the auction.

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<sup>2</sup>The highest type  $\bar{x}$  gets zero information rent; this is the familiar “no distortion at the top” result from optimal mechanism design.

We consider this insight for a two-bidder correlated private values auction, as considered in McAfee and Reny (1992) (pp. 397–398). Bidders 1 and 2 have values  $v^1$  and  $v^2$ , which are both discrete-valued from the finite set  $\{v_1, \dots, v_n\}$ . Let  $P$  denote the  $n \times n$  matrix of conditional probabilities: the  $(i, j)$ -th element of  $P$  is the probability that (from bidder 1's point of view) bidder 2 has value  $v_j$ , given that bidder 1 has value  $v_i$ . Clearly, the rows of  $P$  should sum to one.

Cremer and McLean focus on simple two-part mechanisms. First, bidders find out their values, and then choose from among a menu of participation fee schedules offered by the seller. These schedules specify the fixed fee that each bidder must pay, as a function of *both bidders'* reports. Second, the two bidders participate in a Vickrey (second-price) auction. Truth-telling is assured in the Vickrey auction, so we focus on the optimal design of the participation fee schedules assuming that both bidders report the truth, i.e., report  $v^1$  and  $v^2$ .

In what follows, we focus on bidder 1 (argument for bidder 2 is symmetric). Intuitively, the seller wants to design the fee schedules so that bidder 1 pays an amount arbitrarily close to his expected profit  $\pi_1$  from participating which, given his value  $v^1$ , is  $v^1 - E[v^2|v^1]$  (due to second-stage Vickrey auction). Given the discreteness in values, without loss of generality we can consider fee schedules of the form where, for the  $m$ -th schedule, bidder 1 must pay an amount  $z_{mij}$  if he announces  $v^1 = v_i$ , but bidder 2 announces  $v^2 = v_j$ . Therefore bidder 1's expected profit from schedule  $m$ , given his type  $v^1 = v_i$ , is  $v_i - E[v^2|v^1 = v_i] - \sum_{j=1}^n p_{ij} z_{mij}$ .

Assume that the seller offers  $n$  different schedules, indexed  $m = 1, \dots, n$ . Schedule  $m$  is targeted to be chosen by a bidder who has valuation  $v_1 = v_m$ . Also assume that  $z_{mij} = v_m - E[v^2|v^1 = v_m] + \alpha_m * x_{mj}$ . Hence, the net expected payoff to bidder 1, who has valuation  $v_i$ , chooses schedule  $m$ , and then the bidders announce  $(v_i, v_j)$ , are:

$$v_i - E[v^2|v^1 = v_i] - (v_m - E[v^2|v^1 = v_m]) - \alpha_m \sum_{j=1}^n p_{ij} x_{mj}.$$

It turns out that the parameter  $\alpha$  will be the same across all schedules, and so we eliminate the  $m$  subscript from these quantities.

In order for the seller to design schedules which extract all of bidder 1's surplus, we require that:

1. For each  $i = 1, \dots, n$ , there exists a vector  $\vec{x}_i \equiv (x_{i1}, \dots, x_{in})'$  such that

$$\sum_{j=1}^n p_{ij}x_{ij} = 0, \quad \sum_{j=1}^n p_{i'j}x_{ij} \leq 0, \quad \forall i' \neq i :$$

that is, the vector  $\vec{x}_i$  “separates” the  $i$ -th row of  $P$  from the other rows, at 0.

2.  $\alpha$  must be set sufficiently large so that, for all  $i = 1, \dots, n$ , if bidder 1 has value  $v^1 = v_i$  he will choose schedule  $i$ . This implies that his expected profit from choosing schedule  $i$  exceeds that from choosing any schedule  $i' \neq i$ . These incentive compatibility conditions pin down  $\alpha$ :

$$\begin{aligned} & -\alpha * \sum_{j=1}^n p_{ij}x_{ij} > v_i - E[v^2|v^1 = v_i] - (v_{i'} - E[v^2|v^1 = v_{i'}]) - \alpha * \sum_{j=1}^n p_{ij}x_{i'j}, \quad \forall i' \neq i \\ \Leftrightarrow & 0 > v_i - E[v^2|v^1 = v_i] - (v_{i'} - E[v^2|v^1 = v_{i'}]) - \alpha_i * \sum_{j=1}^n p_{ij}x_{i'j}, \quad \forall i' \neq i \\ \Leftrightarrow & \alpha > \max_{i' \neq i} \frac{(v_i - E[v^2|v^1 = v_i]) - (v_{i'} - E[v^2|v^1 = v_{i'}])}{\sum_{j=1}^n p_{ij}x_{i'j}}. \end{aligned}$$

A sufficient condition for the first point above is that, for all  $i = 1, \dots, n$ , the  $i$ -th row of  $P$  is not in the ( $n$ -dimensional) convex set spanned by all the other rows of  $P$ .<sup>3</sup> Then we can apply the separating hyperplane theorem to say that there always exists a vector  $(x_{i1}, \dots, x_{in})'$  which separates the  $i$ -th row of  $P$  from the convex set composed of convex combinations of the other  $n - 1$  rows of  $P$ .

In order for the  $i$ -th row of  $P$  is not in the ( $n$ -dimensional) convex set spanned by all the other rows of  $P$ , we require that the  $i$ -th row of  $P$  cannot be written as a linear combination of the other  $n - 1$  rows, i.e., that all the rows of  $P$  are linear independent. As Cremer and McLean (1988) note, this is a *full rank* condition on  $P$ .

As an example, consider the case where the  $P$  matrix is diagonal (that is,  $p_{ii} = 1$  for all  $i = 1, \dots, n$ ). This is the case of “Stalin’s food taster”. In this case, the vector  $\vec{x}_i$  with element  $x_{ii} = 0$  and  $x_{ij} = -1$  for  $j \neq i$  satisfies point 1. The numerator of the

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<sup>3</sup>Or, using terminology in McAfee and Reny (1992), the  $i$ -th row of  $P$  is not in the convex hull of the other  $n - 1$  rows of  $P$ .

expression for  $\alpha$  is equal to 0. Thus any arbitrarily small participation fee will lead to full surplus extraction!

## References

- CREMER, J., AND R. MCLEAN (1985): “Optimal Selling Strategies under Uncertainty for a Discriminating Monopolist when Demands are Interdependent Strategy Auctions,” *Econometrica*, 53, 345–361.
- (1988): “Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions,” *Econometrica*, 56, 1247–1257.
- MCAFEE, P., AND P. RENY (1992): “Correlated Information and Mechanism Design,” *Econometrica*, 60, 395–421.
- MILGROM, P., AND R. WEBER (1982): “A Theory of Auctions and Competitive Bidding,” *Econometrica*, 50, 1089–1122.