

1 Introduction

An auction is a game of *private* and *incomplete* information:

- Bidders are competing to win an object.
- There are N players, or bidders, indexed by $i = 1, \dots, N$.
- Each bidder possesses a piece of information related to the valuation of the object: X_1, \dots, X_N . These information signals are *private*, in the sense that only bidder i knows X_i .
- If bidder i were to win the object, she would obtain a utility equal to $u_i(X_i, X_{-i})$, where $X_{-i} \equiv \{X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_N\}$, the vector of signals excluding bidder i 's signal. That is, bidder i 's utility depends not only on her information X_i , but also on her rivals' information X_{-i} .

Since signals are private, $V_i \equiv u_i(X_i, X_{-i})$, which we call bidder i 's *valuation*, is *incompletely* known to her.

- Example 1: internet auction for digital camera; $V_i = X_i$
- Example 2: Wallet game. X_i = money in bidder i 's wallet. $V_i = \sum_{j=1}^N X_j$, the sum of money in all bidders' wallets. Note that V_i is the *same* for all bidders.

Differing assumptions on the form of bidders' utility function lead to an important distinction:

- **Private value** model: $V_i = X_i, \forall i$. Each bidder knows his own valuation, but not that of his rivals. More generally, in a private value model, $u_i(X_i, X_{-i})$ is restricted to be a function *only* of X_i . Example 1 above.
- **Common value** model: When $u_i(X_i, X_{-i})$ is functionally dependent on X_{-i} , we have a common value model. In these models, rivals possess information which is valuable to bidder i in figuring out how much the object is worth.

Wallet game is example of *pure* common value model, because valuation is the same across all bidders.

Auction models also differ depending on the auction rules:

- **First-price** auction: the object is awarded to the highest bidder, at her bid.
- **Second-price** auction: awards the object to the highest bidder, but she pays a price equal to the bid of the second-highest bidder. (Second-price auctions are also called “Vickrey” auctions, after the Nobel laureate William Vickrey.)
- In an **English** or **ascending** auction, the price is raised continuously by the auctioneer, and the winner is the last bidder to remain, and he pays an amount equal to the price at which all of his rivals have dropped out of the auction.
- In a **Dutch** auction, the price is *lowered* continuously by the auctioneer, and the winner is the first bidder to agree to pay any price. (Flowers in Holland are sold using this mechanism.)
- Here we have considered only *single-object* auctions. There are also multi-object auctions. Example: treasury bill auctions, car auctions

2 Bidding behavior

Throughout, we will consider *strategic* behavior in auctions, taking a game-theoretic point of view. To motivate this, we first consider “naive” bidding.

Example (again): wallet game.

To formalize the intuition from the above example, we need to make some technical statistical assumptions:¹

- Symmetry: the joint distribution function $F(V_1, X_1, \dots, V_N, X_N)$ is symmetric (i.e., exchangeable) in the indices i so that, for example, $F(V_N, X_N, \dots, V_1, X_1) = F(V_1, X_1, \dots, V_N, X_N)$.

¹These assumptions are those for the so-called “general symmetric affiliated model”, used in the seminal paper of Milgrom and Weber (1982).

Essentially, this assumption is satisfied when all bidders are statistically identical, in the sense that bidders' valuations and information come from the same distribution. Rules out cases where bidders are intrinsically different: "toeholds" in takeover bids; informed vs. uninformed bidders in drilling rights auctions.

Reasonable to assume for wallet game.

- The random variables $V_1, \dots, V_N, X_1, \dots, X_N$ are *affiliated*, which means that large values for some of the variables make large values for the other variables most likely.

Formally, consider a joint distribution function $F(Z_1, \dots, Z_M)$, and let $\vec{Z} \equiv (Z_1, \dots, Z_M)$ and $\vec{Z}^* \equiv (Z_1^*, \dots, Z_M^*)$ denote two independent draws from this distribution. Let \bar{Z} and \underline{Z} denote, respectively, the component-wise maximum and minimum. Then we say that Z_1, \dots, Z_M are affiliated if $F(\bar{Z})F(\underline{Z}) \geq F(Z_1, \dots, Z_M)F(Z_1^*, \dots, Z_M^*)$.

Some important implications of affiliation:

- $E[Z_1|Z_2] \geq 0$.
- Let $Y_i \equiv \max_{j \neq i} X_j$, the highest of the signals observed by bidder i 's rivals. Given affiliation, the conditional expectation $E[V_i|X_i, Y_i]$ is increasing in both X_i and Y_i .
- Rules out negative correlation between bidders' valuations.
- satisfied for wallet game

Winner's curse As in the wallet game, consider the pure common value case, where $V_i = V$ for all bidders i . To begin with, consider a bidder i , who has information signal X_i . If she bids naively/unstrategically, it is reasonable to assume that, because she doesn't know V , that she should submit a bid equal to $E[V|X_i]$, her best guess of what V is, given her information X_i . By affiliation, note that $E[V|X = x]$ is increasing in x , implying that the winner is the bidder with the highest signal.

Focus on first-price auction. If bidder i wins the auction, she immediately learns that $X_i > Y_i$ (where Y_i is highest of bidder i 's rivals' signals, as defined before). Thus her

ex-post profits are

$$\underbrace{E[V|X_i, X_i > Y_i]}_{\text{ex-post valuation}} - \underbrace{E[V|X_i]}_{\text{price}}.$$

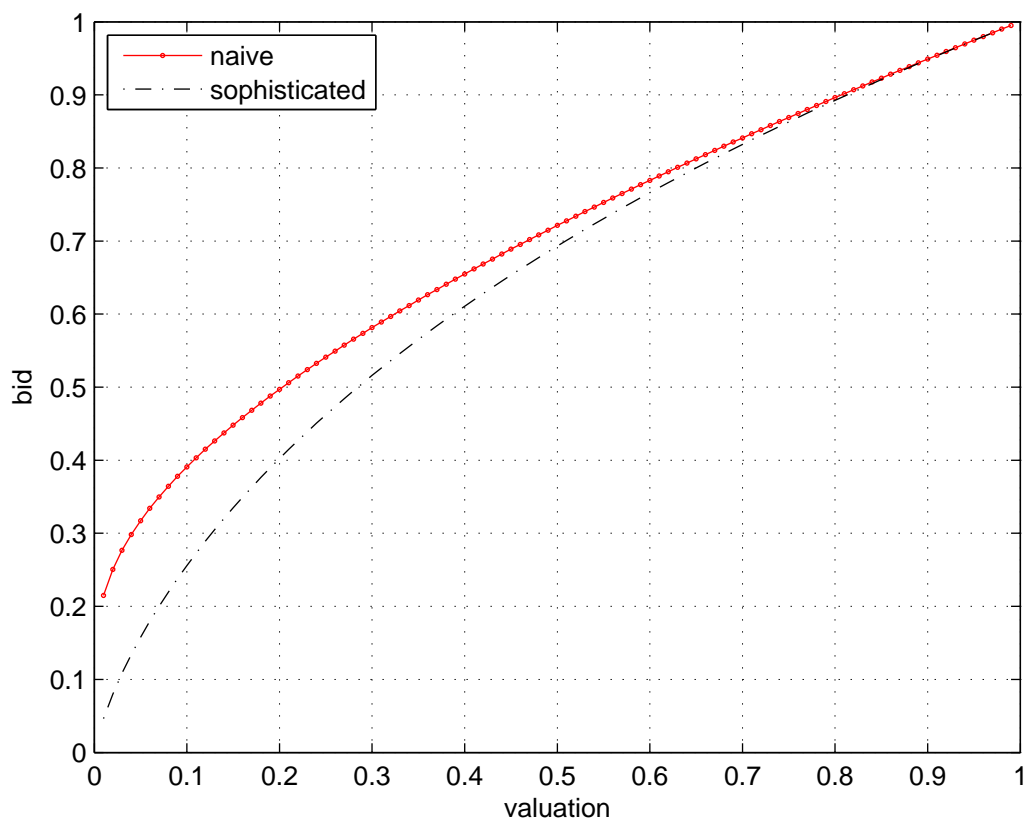
It turns out this is ≤ 0 . This follows from the affiliation assumption, as per the following:²

$$\begin{aligned} E[V|X_i] &= E_{X_{-i}|X_i} E[V|X_i; X_{-i}] \\ &= \underbrace{\int \cdots \int}_{N-1} E[V|X_i; X_{-i}] F(dX_1, \dots, dX_{i-1}, dX_{i+1}, \dots, dX_N|X_i) \\ &\geq \underbrace{\int^{X_i} \cdots \int^{X_i}}_{N-1} E[V|X_i; X_{-i}] F(dX_1, \dots, dX_{i-1}, dX_{i+1}, \dots, dX_N|X_i) \\ &= E[V|X_i, X_i > X_j, j \neq i] = E[V|X_i, X_i > Y_i]. \end{aligned}$$

In other words, if bidder i “naively” bids $E[V|X_i]$, her expected payoff from a first-price auction is negative for every X_i . This is called the “winner’s curse”.

²Law of iterated expectation:

$$\begin{aligned} Ex &= \int xf(x)dx \\ &= \int x \left[\int f(x, y)dy \right] dx \\ &= \int x \left[\int f(x|y)f(y)dy \right] dx \\ &= \int \left[\int xf(x|y)dx \right] f(y)dy \\ &= \int E[x|y]f(y)dy = E_y E[x|y]. \end{aligned}$$



In other words, what we've shown here, is that if all bidders choose their bids using the rule $b_i = E[V|X_i]$, for all bidders i , then each bidder will wish to deviate and bid something $\tilde{b}_i < E[V|X_i]$, to avoid the winner's curse. More formally, bidding $b_i = E[V|X_i]$ is not an *equilibrium strategy*. In the next section, we will derive the equilibrium strategies, for the first- and second-price auction models.

This winner's curse intuition arises in many non-auction settings also.

- For example, in two-sided markets where traders have private signals about unknown fundamental value of the asset, the ability to consummate a trade is “good news” for sellers, but “bad news” for buyers, implying that, without ex-ante gains from trade, traders may not be able to settle on a market-clearing price. This underlies “no-trade” theorems in finance. (Milgrom and Stokey (1982)).

- In used car markets, where sellers are better informed, can lead market to close. This is the famous “lemons” result by Akerlof (1970).
- Multi-object auctions: loser’s curse
- “Pivotal” jury voting.

3 Equilibrium bidding

Next, we cover the first- and second-price auctions in some detail. To motivate the analysis, first note that because each bidder i only observes her own information signal X_i , it's reasonable to restrict bidder i 's equilibrium strategy to be just a function of X_i ; that is, $b_i = b^*(X_i)$, with the form of $b^*(\cdot)$ to be determined. Second, as the above analysis suggested, equilibrium strategies should have the feature that, if bidders $j \neq i$ bid according to the equilibrium strategy $b_j = b^*(X_j)$, then bidder i should also choose to bid $b_i = b^*(X_i)$. That is, the equilibrium strategy $b^*(\cdot)$ should be a mutual *best response*. We first illustrate this idea for the second-price auction.

3.1 Second-price auctions

Let $b^*(x)$ denote the equilibrium bidding strategy (which maps each bidder's private information to his bid). Assume it is *monotonic*. Next we derive the functional form of this equilibrium strategy.

Given monotonicity, the price that bidder i will pay (if he wins) is $b^*(Y_i)$: the bid submitted by his closest rivals. He only wins when his bid $b < b^*(Y_i)$. Therefore, his

expected profit from participating in the auction with a bid b and a signal $X_i = x$ is:

$$\begin{aligned}
& E_{V_i, Y_i} [(V_i - b^*(Y_i)) \mathbf{1}(b^*(Y_i) < b) | X_i = x] \\
&= E_{V_i, Y_i} [(V_i - b^*(Y_i)) \mathbf{1}(Y_i < X_i) | X_i = x] \\
&= E_{Y_i | X_i} E_{V_i} [(V_i - b^*(Y_i)) \mathbf{1}(Y_i < X_i) | X_i = x, Y_i] \\
&= E_{Y_i | X_i} [(E(V_i | X_i, Y_i) - b^*(Y_i)) \mathbf{1}(Y_i < X_i)] \tag{1} \\
&\equiv E_{Y_i | X_i} [(v(X_i, Y_i) - b^*(Y_i)) \mathbf{1}(Y_i < X_i)] \\
&= \int_{-\infty}^{X_i} (v(x, Y_i) - b^*(Y_i)) f(Y_i | X_i = x) dY_i.
\end{aligned}$$

In equilibrium, bidder i also follows the equilibrium bidding strategy, so that $b_i = b^*(X_i)$. Hence, the upper bound of integration above is $X_i = (b^*)^{-1}(b)$, leading to

$$\int_{-\infty}^{(b^*)^{-1}(b)} (v(x, Y_i) - b^*(Y_i)) f(Y_i | X_i = x) dY_i. \tag{2}$$

Bidder i chooses his bid b to maximize his profits. The first-order conditions are (using Leibniz' rule):

$$\begin{aligned}
0 &= b^{*-1'}(b) * [v(x, b^{*-1}(b)) - b^*(b^{*-1}(b))] * f(b^{*-1}(b) | X_i) \Leftrightarrow \\
0 &= \frac{1}{b^{*'}(b)} [v(x, x) - b^*(x)] * f(b^{*-1}(b) | X_i) \Leftrightarrow \\
b^*(x) &= v(x, x) = E[V_i | X_i = x, Y_i = x].
\end{aligned}$$

Would bidder i ever *regret* winning the auction?

A special case In the private value case, with $V_i = X_i$, the equilibrium bidding strategy simplifies to

$$b^*(x) = v(x, x) = x.$$

- “Truth-telling” is equilibrium
- Relation between English and second-price auctions

3.2 First-price auctions

Next, we derive the symmetric monotonic equilibrium bidding strategy $b^*(\cdot)$ for first-price auctions. If bidder i wins the auction, he pays his bid $b^*(X_i)$. His expected profit is

$$\begin{aligned}
&= E_{V_i, Y_i} [(V_i - b) \mathbf{1}(b^*(Y_i) < b) | X_i = x] \\
&= E_{Y_i | X_i} E_{V_i} [(V_i - b) \mathbf{1}(Y_i < b^{*-1}(b)) | X_i = x, Y_i] \\
&= E_{Y_i | X_i} (E[V_i | X_i = x, Y_i] - b) \mathbf{1}(Y_i < b^{*-1}(b)) \\
&= E_{Y_i | X_i} (v(x, Y_i) - b) \mathbf{1}(Y_i < b^{*-1}(b)) \\
&= \int_{-\infty}^{b^{*-1}(b)} (v(x, Y_i) - b) f(Y_i | x) dY_i.
\end{aligned}$$

The first-order conditions are

$$\begin{aligned}
0 &= - \int_{-\infty}^{b^{*-1}(b)} f(Y_i | x) dY_i + \frac{1}{b^{*'}(x)} [(v(x, x) - b) * f_{Y_i | X_i}(x | x)] \Leftrightarrow \\
0 &= - F_{Y_i | X_i}(x | x) + \frac{1}{b^{*'}(x)} [(v(x, x) - b) * f_{Y_i | X_i}(x | x)] \Leftrightarrow \\
b^{*'}(x) &= (v(x, x) - b^*(x)) \left[\frac{f(x | x)}{F(x | x)} \right] \Rightarrow \\
b^*(x) &= \exp \left(- \int_{\underline{x}}^x \frac{f(s | s)}{F(s | s)} ds \right) b(\underline{x}) + \int_{\underline{x}}^x v(\alpha, \alpha) dL(\alpha | x)
\end{aligned}$$

where

$$L(\alpha | x) = \exp \left(- \int_{\alpha}^x \frac{f(s | s)}{F(s | s)} ds \right).$$

Initial condition: $b(\underline{x}) = v(\underline{x}, \underline{x})$.

Alternatively, for a more intuitive expression, integrate the above by parts to obtain:

$$b^*(x) = v(x, x) - \int_{\underline{x}}^{\bar{x}} L(\alpha | x) \frac{d}{d\alpha} v(\alpha, \alpha) d\alpha.$$

Roughly, the equilibrium bid is the winner's curse-adjusted valuation, $v(x, x)$, minus a "markdown term" $\int_{\underline{x}}^{\bar{x}} L(\alpha | x) \frac{d}{d\alpha} v(\alpha, \alpha) d\alpha \geq 0$.



For the IPV case:

$$\begin{aligned} V(\alpha, \alpha) &= \alpha \\ F(s|s) &= F(s)^{N-1} \\ f(s|s) &= (n-1)F(s)^{N-2}f(s) \end{aligned}$$

Then from above:

$$\begin{aligned} b^*(x) &= \int_{\underline{x}}^{\bar{x}} \alpha \exp\left(- (n-1) \int_{\alpha}^x \frac{f(s)}{F(s)} ds\right) \left((n-1) \frac{f(\alpha)}{F(\alpha)} \right) d\alpha \\ &= \int_{\underline{x}}^{\bar{x}} \alpha \left(\frac{F(\alpha)}{F(x)} \right)^{n-1} \left((n-1) \frac{f(\alpha)}{F(\alpha)} \right) d\alpha \\ &= \int_{\underline{x}}^{\bar{x}} \alpha \cdot \frac{\overbrace{(n-1)F(\alpha)^{n-2}f(\alpha)}^{\text{density of } Y_i}}{\underbrace{F(x)^{n-1}}_{\text{Pr}(Y_i \leq x)}} d\alpha \\ &= E[Y|Y < x] \end{aligned} \tag{3}$$

where Y denotes the maximum among bidder i 's rivals valuations. While this “looks” like the winner’s curse, note that this is the case of independent valuations, so the winner’s curse is absent.

An example $X_i \sim \mathcal{U}[0, 1]$, *i.i.d.* across bidders i . Then $F(s) = s$, $f(s) = 1$. Then

$$\begin{aligned} b^*(x) &= 0 + \int_0^x \alpha \exp\left(- \int_{\alpha}^x \frac{(n-1)f(s)}{sF(s)} ds\right) \frac{(n-1)f(\alpha)}{\alpha F(\alpha)} d\alpha \\ &= \int_0^x \exp\left(- (n-1) \log \frac{x}{\alpha}\right) (n-1) d\alpha \\ &= \int_0^x \left(\frac{\alpha}{x}\right)^{N-1} (N-1) d\alpha \\ &= \alpha \left(\frac{N-1}{N}\right) \left(\frac{\alpha}{x}\right)^N \Big|_0^x \\ &= \left(\frac{N-1}{N}\right) x. \end{aligned}$$

Contrast this with the result in second-price auction.

3.2.1 Reserve prices

A reserve price just changes the initial condition of the equilibrium bid function. With reserve price r , initial condition is now $b(x^*(r)) = r$. Here $x^*(r)$ denotes the screening value, defined as

$$x^*(r) \equiv \inf \{x : E[V_i | X_i = x, Y_i < x] \geq r\}. \quad (4)$$

Conditional expectation in brackets is value of winning to bidder i , who has signal x . Screening value is lowest signal such that bidder i is willing to pay at least the reserve price r .

(Note: in PV case, $x^*(r) = r$. In CV case, with affiliation, generally $x^*(r) > r$, due to winners curse.)

Equilibrium bidding strategy is now:

$$b^*(x) \begin{cases} = \exp\left(-\int_{x^*(r)}^x \frac{f(s|s)}{F(s|s)} ds\right) r + \int_{x^*(r)}^x V(\alpha, \alpha) dL(\alpha|x) & \text{for } x \geq x^*(r) \\ < r & \text{for } x < x^*(r) \end{cases}$$

For IPV, uniform example above:

$$b^*(x) = \left(\frac{N-1}{N}\right)x + \frac{1}{N}\left(\frac{r}{x}\right)^{N-1}r.$$

Role of reserve prices: limit entry by low-valuation bidders, to boost bids by high-valuation bidders. (Graph)

What about secret reserve prices??

References

- AKERLOF, G. (1970): "The Market for "Lemons": Quality Uncertainty and the Market Mechanism," *Quarterly Journal of Economics*, 84, 488–500.
- MILGROM, P., AND N. STOKEY (1982): "Information, Trade and Common Knowledge," *Journal of Economic Theory*, 26, 17–27.
- MILGROM, P., AND R. WEBER (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50, 1089–1122.